TUFTS UNIVERSITY DEPARTMENT OF MATHEMATICS
Midterm 1 Solutions

Math 87, Duchin **Spring 2020**

Part I: Sharing a park

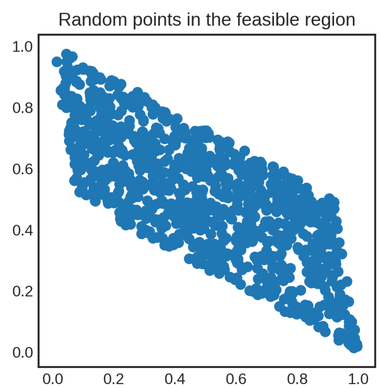
The handout has an example problem about three groups sharing park resources, particularly considering how ponds and trees should be allocated. Here, let's focus on the 2×2 case, so we'll ignore the teens and study how to partition the resources among kids and seniors. The condensed taste/preference matrix is

$$T = \begin{bmatrix} .92 & .08 \\ .5 & .5 \end{bmatrix}.$$

1. Come up with a way of generating a random division matrix $D \in \mathcal{D}$. Since T is given, for each random D, you can plot (v_{11}, v_{22}). Show a plot with enough points that you start to see the shape of $\mathcal{D}$ in that plane.

   In this case, we have $\mathcal{D} = \left\{ \begin{bmatrix} x & y \\ 1-x & 1-y \end{bmatrix} \right\}$, where $0 \leqslant x, y \leqslant 1$. I can get away with just these two decision variables and these bounds because the other requirement, that the columns sum to 1, is achieved by the way I've set up the matrices.

   I just computed $v_{11} = .92x + .08y$ and $v_{22} = 1 - \frac{x+y}{2}$, then plotted 1000 points with random $x, y \in [0, 1]$ and here's what I got:

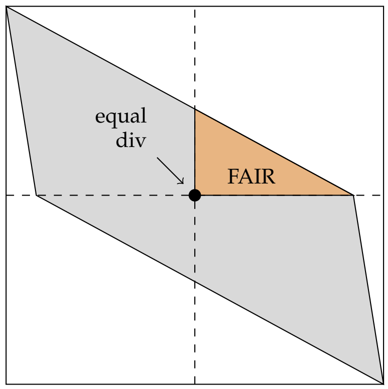


   Plausibility: looks reasonable because each variable is between 0 and 1, and I see tradeoffs between the value of one party and the other. (It shouldn't be possible to make both parties perceive full value at the same time.)

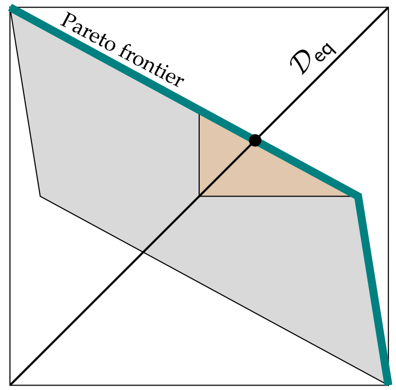
2. Explain why $\mathcal{D}$ corresponds to a convex polygon in the value plane and plot it exactly for this example. Where in your plot is the **equal division**? What part of $\mathcal{D}$ corresponds to **fair** divisions? What part of $\mathcal{D}$ corresponds to **equitable** divisions? Where is the **Pareto frontier**? (You may want to hand-draw these pictures.)

   We saw that the only constraints on x, y are that each varies between 0 and 1. That means that in the xy–plane, the feasible region is just the unit square, which is a convex polygon. When I compute $v_{11} = \vec{t}_1 \cdot \vec{d}_1 = .92x + .08y$ and $v_{22} = 1 - \frac{x+y}{2}$, I'm applying a linear map to that square, and shifting it. That means parallel lines map to parallel lines, so the region in the "value plane" is a parallelogram, which is a convex polygon. (This checks out with my random-points plot above.)

To get the extreme values, I just plug in $x = 0, 1$ and $y = 0, 1$ to get $(0, 1)$, $(.92, .5)$, $(1, 0)$, and $(.08, .5)$.



Feasible, equal division, fair



Equitable, Pareto-optimal

Explanations: equal division means both parties perceive 0.5 share of the full value of the park, so it's plotted at $(.5, .5)$. Fair means each party perceives at least that much value, so it's defined by lying in the quadrant where v_{11} and v_{22} are both $\geqslant .5$. Equitable means $v_{11} = v_{22}$, which appears as a diagonal line. And finally, Pareto-optimal points can't be improved in one v_{ii} without worsening the other; geometrically, this means that the "northeast" quadrant based at a Pareto-optimal point is totally outside the picture. This picks out the top and right-hand edges of the feasible polyhedron.

3. Find the optimal allocation of ponds and trees to kids and seniors by solving for the "best" division matrix D (in the sense of Theorem 4) using linear programming. Your answer must include the linear program that you solved and you must show how you found a solution either by hand or by computer. Explain in plain language in what sense it is best.

First note that I can solve this without linear programming by Thm 4(a), since the figure above shows that there's a unique point that is equitable and Pareto-optimal. I just have to find the point on the line from $(0, 1)$ to $(.92, .5)$ that satisfies $x = y$.

But to solve it as a linear program, I should use 4(c). Then it looks like this:

Maximize $z = v_{11} = .92x + .08y$

subject to $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and $.92x + .08y = 1 - \frac{x+y}{2}$.

The last equation can be rewritten $1.42x + .58y = 1$.

If I code this up in Python via

```
x = cp.Variable()
y = cp.Variable()
objective = cp.Maximize((.92 * x) + (.08 * y))
constraints = [
    1.42*x+.58*y == 1,
```

```
    x >= 0,
    x <= 1,
    y >= 0,
    y <= 1
]
prob = cp.Problem(objective, constraints=constraints)
list_solution = prob.solve()
print('Optimal value: {:.4f}'.format(list_solution))
print('x: {:.4f}, y: {:.4f}'.format(x.value, y.value))
```

Then I get

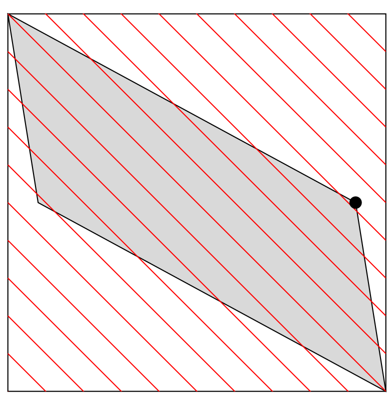
```
Optimal value: 0.6479
x: 0.7042, y: 0.0000
```

So in other words, both kids and seniors feel that got about 64.79% of the full value of the park, and certainly more than equal division, if I split the park to give the kids about 70.42% of the ponds and no trees at all.

This is "best" because it's the only solution that is both equitable (the parties value their shares equally) and Pareto-optimal (there's no other division that improves value for both of the parties). It's also "best" because it's the way to get the highest value for less-advantaged party. And finally, it's "best" because it's the way to get the highest value for each party while maintaining equity. (As a bonus, it's envy-free!)

4. Suppose that a grumpy utilitarian came along and insisted that the "best" solution was instead the one that maximizes the "total utility" $v_{11} + v_{22}$. Solve that version of the problem and confirm that it has a different answer than ours. Now discuss the pros and cons of the two formulations.

   To our grumpy utilitarian, the measure of total utility is $v_{11} + v_{22}$. The level curves of this are the diagonals drawn in the picture below. Clearly when those sweep across the feasible polygon, the last point that they touch is the vertex $(.92, .5)$, which corresponds to a division that gives all the ponds to the kids and all the trees to the seniors.



   Indeed, this has $v_{11} + v_{22} = 1.42$, which is greater than our previous solution, which had $v_{11} + v_{22} = 1.2958$. Which is the superior solution? It's a matter of opinion, because neither solution dominates the other, but I'm persuaded that our solution is better. First of all, one party receives far more value than the other in the new solution: kids feel they got 92% of the full park's value, while seniors only feel they got 50%. This might make this unstable as a negotiation.

But I have a more serious critique. We should not add things that are in different units and expect that to be meaningful! Since I only know the relative value that the groups place on the resources, I can't be sure about whether the kids don't really care about the park while the seniors are passionately invested in the park. The values v_{11} and v_{22} are on a different scale, so adding them is probably not reasonable.

Part II: Negotiation

Palestinian preference weights, thought of as percent of total value:

- Palestinian neighborhoods outside the Old City: 19% of the total.
- Muslim and Christian Quarters in the Old City: 22%.
- Al-Haram al-Sharif/Temple Mount[1]: 48%.
- Jewish and Armenian Quarters in the Old City: 6%.
- Western Wall: 4%.
- Israeli neighborhoods outside the Old City: 1%.

Israeli weights:

- Palestinian neighborhoods outside the Old City: 0%.
- Muslim and Christian Quarters in the Old City: 9.5%.
- Al-Haram al-Sharif/Temple Mount: 22%.
- Jewish and Armenian Quarters in the Old City: 18%.
- Western Wall: 31%.
- Israeli neighborhoods outside the Old City: 19.5%.

Of course, many would say that these sites are not realistically divisible. Ignoring that for a moment, find a division that would be optimal for the associated linear program, as in Theorem 4.

Even though there is a zero in the preference/taste matrix, I can still use 4(c) to set up a linear program. I'll write my division matrix as $D = \begin{bmatrix} x_1 & x_2 & \dots & x_6 \\ 1-x_1 & 1-x_2 & \dots & 1-x_6 \end{bmatrix}$ with six decision variables.

[1]This site is religiously sacred to both Jews and Muslims. Although all of Jerusalem is under Israeli control and considered part of Israel by the Israeli government, the al-Haram al-Sharif is administered by the Muslim authorities, and the Chief Rabbinate has ruled that Jews should not enter the site.

Then I compute
$$v_{11} = .19x_1 + .22x_2 + .48x_3 + .06x_4 + .04x_5 + .01x_6,$$
$$v_{22} = .095x_2 + .22x_3 + .18x_4 + .31x_5 + .195x_6.$$

I'll maximize v_{11} subject to $0 \leqslant x_i \leqslant 1$ and $v_{11} = v_{22}$. I get

```
Optimal value: 0.7494
x1: 1.0000, x2: 1.0000, x3: 0.7071, x4: 0.0000, x5: 0.0000, x6: 0.0000
```

Is this reasonable? Well, it allocates the sites whose relative value is much higher for one side than the other to the side you would expect. And it divides the only site that is of very high (relative) value to both sides. This seems plausible.

Next, consider all binary allocations (where each site is allocated wholly to one side or the other). Which of those are "fair"?

There are 64 binary allocations, because there are 6 sites that can be assigned 0 (if allocated to Israelis) or 1 (if allocated to Palestinians). So I have to make a 2-way choice six times, which can be done $2^6 = 64$ ways in all.

Here is a brute force check of which ones are "fair":

```
bin = [0,1]
allperms = [ (x,y,z,a,b,c) for x in bin for y in bin for z in bin \
    for a in bin for b in bin for c in bin]
Pvalues = [.19*perm[0]+.22*perm[1]+.48*perm[2]+.06*perm[3]+.04*perm[4]+ \
    .01*perm[5] for perm in allperms]
Ivalues = [.095*(1-perm[1])+.22*(1-perm[2])+.18*(1-perm[3])+.31*(1-perm[4])+ \
    .195*(1-perm[5]) for perm in allperms]
for t in range(63):
    if Pvalues[t] >= .5 and Ivalues[t] >=.5:
        print (allperms[t],"{:.4f}".format(Pvalues[t]),"{:.4f}".format(Ivalues[t]))
```

This yields

```
(0, 0, 1, 1, 0, 0) 0.5400 0.6000
(0, 1, 1, 0, 0, 0) 0.7000 0.6850
(0, 1, 1, 1, 0, 0) 0.7600 0.5050
(1, 0, 1, 0, 0, 0) 0.6700 0.7800
(1, 0, 1, 0, 0, 1) 0.6800 0.5850
(1, 0, 1, 1, 0, 0) 0.7300 0.6000
(1, 1, 1, 0, 0, 0) 0.8900 0.6850
(1, 1, 1, 1, 0, 0) 0.9500 0.5050
```

so there are eight out of 64 binary allocations that are "fair."

Is this plausible? Well, it might be regarded as surprising that all of these divisions give the Temple Mount fully to the Palestinian side, while it is very highly valued by both sides. This is partly explained by the fact that it is very hard to achieve Palestinian value of .5 without the Temple Mount (which has a value of .48). It does check out that all of our fair binary solutions allocate the Western Wall fully to the Israeli side.

Find a "fair" and "equitable" solution that only divides one of the six sites.

In fact, my linear program already found one:

```
Optimal value: 0.7494
x1: 1.0000, x2: 1.0000, x3: 0.7071, x4: 0.0000, x5: 0.0000, x6: 0.0000
```

This is clearly not the only "equitable" and "fair" solution, because I can notice the following in my binary allocation list:

```
(0, 0, 1, 1, 0, 0) 0.5400 0.6000
(1, 0, 1, 1, 0, 0) 0.7300 0.6000
```

Clearly since Israelis are indifferent to site 1, I can adjust its allocation fractionally until $v_{11} = 0.6000$, which gives me an "equitable" and "fair" solution. (But dominated by the earlier one, so not Pareto-optimal!)

Briefly discuss your interpretation of this modeling problem in terms of real-world negotiating insights. (Or explain why it is not useful.)

The initial optimal solution found above tells me to allocate control of sites 1 and 2 to the Palestinian side, sites 4, 5, and 6 to the Israeli side, and to subdivide site 3, the Temple Mount.

How interpretable is this? Can we subdivide the Temple Mount? If we're being charitable to the model, we might say that we can alter the rules of control, such as who polices the site, who enters it, who maintains it. Another view is that we should add a seventh negotiable item so that if the Temple Mount is allocated in a binary fashion to one side, it can be balanced with a suitable extra item allocated to the other side.

But a reasonable person is just as likely to conclude that actual matters of satisfaction in high stakes negotiations are unlikely to turn out the way an exercise like this dictates. In other words, ALL MODELS ARE WRONG, BUT SOME MODELS ARE USEFUL... UP TO A POINT and this model may have encountered its limits.

What are the strengths of this model? It does force the negotiating parties to be explicit about *relative preferences* rather than staking out a position of demands.

What are the weaknesses of this model? It was formed by conducting interviews with selected "experts" and is unlikely to reflect a broad consensus among everyday people. It assumes that relative preference weights are capable of capturing a negotiating position, and that $v_{ii} \geqslant .5$ is a baseline of satisfaction.

P.S. I learned about this Israel-Palestine negotiation problem from my colleague Christoph Börgers's excellent book *Mathematics of Social Choice*. He credits the setup to a manuscript titled, "Adjusted Winner and the Future Negotiations over East Jerusalem" by Moshe Hirsch.