TUFTS UNIVERSITY DEPARTMENT OF MATHEMATICS Midterm 1

Math 87, Duchin

Spring 2020

To start, read the fair division handout! This midterm will follow the notation and conventions from there. You will need to write your own Python code for whatever you need to do this midterm, but you should have ample scaffolded material to draw from the HW notebooks. Your writeup should aim to be more of a report than a problem set, which means that there's an emphasis on providing all the necessary information for the reader to understand what is important. Include pseudo-code (simplified explanations of your code) where appropriate. Include plots generated with python where appropriate.

The midterm is expected to be conducted on your own. Please work independently and write up original solutions in your own words. Cite sources if you use them.

Part I: Sharing a park

The handout has an example problem about three groups sharing park resources, particularly considering how ponds and trees should be allocated. Here, let's focus on the 2×2 case, so we'll ignore the teens and study how to partition the resources among kids and seniors. The condensed taste/preference matrix is

$$T = \begin{bmatrix} .92 & .08\\ .5 & .5 \end{bmatrix}.$$

- 1. Come up with a way of generating a random division matrix $D \in \mathcal{D}$. Since *T* is given, for each random *D*, you can plot (v_{11}, v_{22}) . Show a plot with enough points that you start to see the shape of \mathcal{D} in that plane.
- Explain why D corresponds to a convex polygon in the value plane and plot it exactly for this example. Where in your plot is the equal division? What part of D corresponds to fair divisions? What part of D corresponds to equitable divisions? Where is the Pareto frontier? (You may want to hand-draw these pictures.)
- 3. Find the optimal allocation of ponds and trees to kids and seniors by solving for the "best" division matrix *D* (in the sense of Theorem 4) using linear programming. Your answer must include the linear program that you solved and you must show how you found a solution either by hand or by computer. Explain in plain language in what sense it is best.
- 4. Suppose that a grumpy utilitarian came along and insisted that the "best" solution was instead the one that maximizes the "total utility" $v_{11} + v_{22}$. Solve that version of the problem and confirm that it has a different answer than ours. Now discuss the pros and cons of the two formulations.

Part II: Negotiation

This is taken from a real study of Israeli–Palestinian conflict in East Jerusalem. Out of respect for the massive human toll of the Israeli–Palestinian conflict, I'll make sure to put all of our fairness words in quotes below, to emphasize how inadequate they are to really provide fairness in the sense of *justice*.

A researcher identified six contentious sites for the Israeli and Palestinian people, then conducted interviews with leaders on both sides to estimate relative weight on the importance of controlling the different zones.

Palestinian preference weights, thought of as percent of total value:

- Palestinian neighborhoods outside the Old City: 19% of the total.
- Muslim and Christian Quarters in the Old City: 22%.
- Al-Haram al-Sharif/Temple Mount¹: 48%.
- Jewish and Armenian Quarters in the Old City: 6%.
- Western Wall: 4%.
- Israeli neighborhoods outside the Old City: 1%.

Israeli weights:

- Palestinian neighborhoods outside the Old City: 0%.
- Muslim and Christian Quarters in the Old City: 9.5%.
- Al-Haram al-Sharif/Temple Mount: 22%.
- Jewish and Armenian Quarters in the Old City: 18%.
- Western Wall: 31%.
- Israeli neighborhoods outside the Old City: 19.5%.

Of course, many would say that these sites are not realistically divisible. Ignoring that for a moment, find a division that would be optimal for the associated linear program, as in Theorem 4. Next, consider all binary allocations (where each site is allocated wholly to one side or the other). Which of those are "fair"? Find a "fair" and "equitable" solution that only divides one of the six sites.

Briefly discuss your interpretation of this modeling problem in terms of real-world negotiating insights. (Or explain why it is not useful.)

¹This site is religiously sacred to both Jews and Muslims. Although all of Jerusalem is under Israeli control and considered part of Israel by the Israeli government, the al-Haram al-Sharif is administered by the Muslim authorities, and the Chief Rabbinate has ruled that Jews should not enter the site.