Inverse dynamic problem for the 1-D Dirac system on finite metric tree graphs. Leaf-peeling method

OLHA SUS

Tufts University

Department of Mathematics/Department of Education

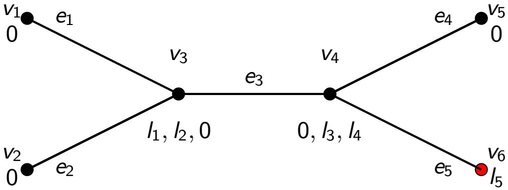
Graduate Student Seminar

September 26, 2022

Introduction

Let

- $\Omega = (V, E)$, $E = \{e_1, \cdots, e_N\}$, $V = \{v_1, \cdots, v_M\}$, where $e_i \in E$ is identified with $(0, l_k)$ for $i = 1, \cdots, N$
- $\Gamma = \{\gamma_1, \cdots, \gamma_L\}$, γ_L is a root of Ω
- $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be a Pauli matrix
- $Q = \begin{pmatrix} p_k & q_k \\ q_k & -p_k \end{pmatrix}$, $p_k, q_k \in C^1(e_k; \mathbb{R})$, $k = 1, \cdots, N$
- $U := (u^1, u^2)^\top \big|_{e_k} = \{(u_k^1, u_k^2)^\top\}_{k=1}^N$, $u_k^1, u_k^2 \in L^2(e_k; \mathbb{C})$



We associate the following IVBP for the 1–D Dirac system to the graph Ω:

$$iU_t + JU_x + QU = 0, \quad t \geq 0, \quad x \in e_k \tag{1}$$

$$u_k^1(v,t) = u_j^1(v,t), \quad e_k \sim v, \quad e_j \sim v, \quad v \in V\backslash\Gamma \quad t \geq 0, \quad k \neq j \tag{2}$$

$$\sum_{k|e_k \sim v} u_k^{2,\pm}(v,t) = 0, \quad v \in V\backslash\Gamma, \quad t \geq 0, \tag{3}$$

$$U^1\big|_\Gamma = F, \quad \text{on } \Gamma \times [0,T], \tag{4}$$

$$U(\cdot,0) = 0 \tag{5}$$

where

- $F = (f^1(t), \cdots, f^{L-1}(t))^\top \in L^2(0,T;\mathbb{C}^{L-1})$

The Inverse Problem: for the given controls $f^k(t)$, $k = 1, \cdots, L-1$, we want to recover the connectivity of the edges, their lengths l_k and potential functions p_k, q_k on each edge e_k, $k = 1, \cdots, N$ from the dynamic response data.

$$R^T\{F\}(t) := u^2(\cdot, t)\Big|_{\Gamma \setminus \{\gamma_L\}}, \quad t \in [0, T]. \tag{6}$$

$$(R^T F)(t) = (\mathcal{R} * F)(t) = \int_0^t \mathcal{R}(t-s)F(s)\, ds,$$

where $\mathcal{R}(t) = \{R_{kj}\}_{k,j=1}^{L-1}$ is a response matrix.

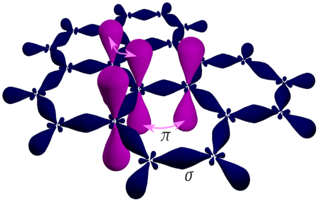
Motivation, background and novelty of the research

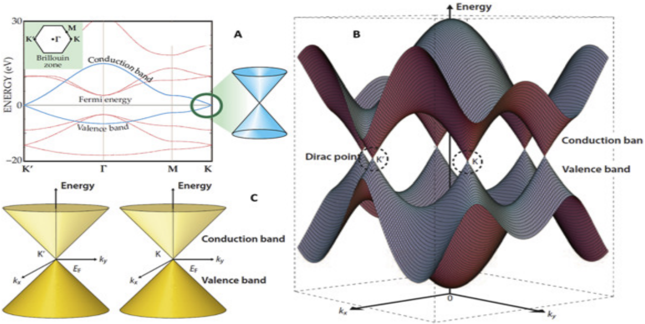
Applications (electronic properties of graphene):

- Electrons propagating through graphene's honeycomb lattice effectively lose their mass, producing quasi-particles that are described by a 2D analogue of the Dirac equation

$$\left(\beta mc^2 + c\sum_{n=1}^{3}\alpha_n p_n\right)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

where $\psi(x,t)$ is the wave function for the electron of rest mass m with spacetime coordinates x, t. The p_1, p_2, p_3 are the components of the momentum, understood to be the momentum operator in the Schrödinger equation.





- The energy spectrum of an undoped graphene sheet in the presence of a magnetic field and in the vicinity of the Dirac cones can be obtained by solving eigenvalue equation

$$[v_F\vec{\sigma}(\vec{\mathcal{P}} + e\vec{A}) + \tau\Delta\sigma_z]\Psi(x,y) = E\Psi(x,y),$$

where v_F is the Fermi velocity, e is the electron charge, $\vec{A}$ is the electromagnetic vector potential, $\vec{\sigma}$ denotes Pauli matrix, Δ is the on-site potential induced by the substrate on the sublattice (mass term within the continuum model), and τ is a valley index ($\tau = 1$ for the K Dirac point, and -1 for the K' Dirac point).

Here the eigenstates $\Psi = [\Psi_1, \Psi_2]$ are the two-component spinors associated with the probability amplitudes of sublattice.

Background:

- Inverse problem for a one-dimensional dynamical Dirac system (BC-method): M.I. Belishev and V.S. Mikhailov (2010)
- Inverse spectral problems for one-dimensional Dirac system on a finite tree: A. Mikhaylov, V. Mikhaylov, and G. Murzabekova (2015)
- Dirac type systems: spectral and Weyl-Titchmarsh matrix functions, direct and inverse problems: B. Fritzsche and others (2012), A.L. Sakhnovich (2002)
- Leaf peeling method for the wave equation on metric tree graphs: S.A. Avdonin, Y. Zhao (2021)
- On inverse dynamical and spectral problems for the wave and Schrödinger equations on finite trees. The leaf-peeling method: S.A. Avdonin, V.S. Mikhaylov, and K.B. Nurtazina (2015)
- Inverse problems for quantum trees: S.A. Avdonin, P.B. Kurasov (2008)
- A note on the Dirac operator with Kirchoff-type vertex conditions on metric graphs: W. Borrelli, R. Carlone, and L. Tentarelli (2020)

Novelty:

- We solve the dynamic inverse problem for 1-D Dirac system on finite metric tree graphs by applying a leaf-peeling method. Also, we present a new dynamic algorithm to solve the forward problem for the 1-D Dirac system on general finite metric tree graphs.

The forward problem for the 1-D Dirac system on a finite interval

$$iU_t + JU_x + QU = 0, \quad 0 < x < l, \quad 0 < t < T, \tag{7}$$

$$u^1(0,t) = f(t), \quad u^1(l,t) = 0, \quad 0 < t < T, \tag{8}$$

$$U(x,0) = 0, \quad 0 < x < l. \tag{9}$$

$$U^{f,+}(x,t) = \begin{cases} 0, & 0 < t < x, \\ \begin{pmatrix} 1 \\ i \end{pmatrix} f(t-x) + \int\limits_x^t W^+(x,s) f(t-s)\, ds, & 0 < x < t, \end{cases} \tag{10}$$

where $W^+ = (w^{1,+}, w^{2,+})^\top$ is a vector-kernel such that $W^+|_{t<x} = 0$, $W^+|_{\Delta^T} \in C^1(\Delta^T; \mathbb{C}^2)$, $w^1(0,\cdot) = 0$ $\left(\Delta^T := \{(x,t) \mid x \geq 0,\ 0 \leq t \leq T,\ x \leq t\}\right)$.

By setting

$$p(2nl \pm x) = p(x), \quad q(2nl \pm x) = q(x), \quad n \in \mathbb{N},$$

we obtain

$$U^{f,+}(x,t) = \sum_{n=0}^{\lfloor \frac{t-x}{2l} \rfloor} \left(\begin{pmatrix} 1 \\ i \end{pmatrix} f(t-2nl-x) + \int_{2nl+x}^{t} W^{+}(2nl+x,s) f(t-s)\, ds \right)$$
$$+ \sum_{n=1}^{\lfloor \frac{t+x}{2l} \rfloor} \left(\begin{pmatrix} -1 \\ i \end{pmatrix} f(t-2nl+x) + \int_{2nl-x}^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} W^{+}(2nl-x,s) f(t-s)\, ds \right), \qquad (11)$$

where $\lfloor \cdot \rfloor$ is a floor-function.

Proposition 1. For potential functions $p, q \in C^1(e)$ and control function $f \in L^2(0,T;\mathbb{C})$, the IBVP (7)-(9) has the unique generalized solution $U^{f,+} \in C([0,T]; L^2(0,l;\mathbb{C}^2))$. Furthermore, for $0 \le t \le x$ we have that $U^{f,+} = 0$, and for $t > x$ it is defined by formula (11).

The forward problem for the 1-D Dirac system on a three-star graph

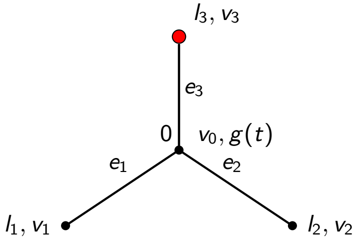
$$iU_t + JU_x + QU = 0, \quad t \geq 0, \quad x \in e_k, \quad k = 1, 2, 3, \tag{12}$$

$$u_k^1(0, t) = u_j^1(0, t), \quad k \neq j, \quad t \geq 0, \quad k, j = 1, 2, 3, \tag{13}$$

$$\sum_{k=1}^{3} u_k^{2,\pm}(0, t) = 0, \quad t \geq 0, \tag{14}$$

$$u_k^1(l_k, t) = f^k(t), \quad k = 1, 2, \quad u_3^1(l_3, t) = 0, \quad 0 \leq t \leq T, \tag{15}$$

$$U(\cdot, 0) = 0. \tag{16}$$



Let

$$g(t) := u_k^1(0,t) \text{ for } k = 1,2,3,\ g(t) = 0 \text{ for } t < l_1.$$

Then

$$\begin{gathered} U_k(x,t) = U_k^{f^k,-}(x,t) + U_k^{g,+}(x,t), \quad k = 1,2, \\ U_3(x,t) = U_3^{g,+}(x,t). \end{gathered} \tag{17}$$

By substituting (17) into (24)-(25), we obtain

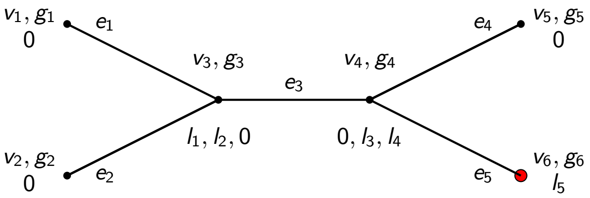
$$3ig(t) + \int_0^t \left(\sum_{k=1}^{3} w_k^{2,+}(0,s) \right) g(t-s)\, ds = G(t) \tag{18}$$

The forward problem for the 1-D Dirac system on general graphs

Let $\Omega = (E, V)$, $E = \{e_1, \cdots, e_5\}$, $V = \{v_1, \cdots, v_6\}$, $e_k = (v_i, v_j)$ is identified with $(0, l_k)$ for $k = 1, \cdots, 5$.

Let

$$g_i(t) = u^1(v_i, t) \text{ for each } v_i \in V,\ i = 1, \cdots, 6.$$



We define $W_k^{\pm} : L^2(0, T) \mapsto C([0, T]; L^2(0, l_k))$ by the rule:

$$(W_k^- g^k)(x,t) = U_k^{g^k,-}(x,t) = (u_k^{1,g^k,-}, u_k^{2,-})^\top,$$
$$(W_k^+ g^k)(x,t) = U_k^{g^k,+}(x,t) = (u_k^{1,g^k,+}, u_k^{2,+})^\top.$$

Let

$$A = \begin{pmatrix} W_1^+ & 0 & W_1^- & 0 & 0 & 0 \\ 0 & W_2^+ & W_2^- & 0 & 0 & 0 \\ 0 & 0 & W_3^+ & W_3^- & 0 & 0 \\ 0 & 0 & 0 & W_4^- & W_4^+ & 0 \\ 0 & 0 & 0 & W_5^+ & 0 & W_5^- \end{pmatrix}$$

We define $\mathcal{O}_k^{\pm}$ on $C([0, T]; L^2(0, l_k; \mathbb{C}^2))$ by the rule:

$$\mathcal{O}_k^+ U_k = u_k^2(0, \cdot),$$
$$\mathcal{O}_k^- U_k = -u_k^2(l_k, \cdot).$$

Then, on e_k for $k = 1, \cdots, N$:

$$(\mathcal{O}_k^- W_k^+)(g^k) = \mathcal{O}_k^- U_k^{g^k,+}(x,t) = -u_k^{2,+}(l_k, t) \tag{19}$$

$$(\mathcal{O}_k^+ W_k^+)(g^k) = \mathcal{O}_k^+ U_k^{g^k,+}(x,t) = u_k^{2,+}(0, t) \tag{20}$$

$$(\mathcal{O}_k^- W_k^-)(g^k) = \mathcal{O}_k^- U_k^{g^k,-}(x,t) = -u_k^{2,-}(l_k, t) \tag{21}$$

$$(\mathcal{O}_k^+ W_k^-)(g^k) = \mathcal{O}_k^+ U_k^{g^k,-}(x,t) = u_k^{2,-}(0, t) \tag{22}$$

Let

$$\widetilde{A} = \begin{pmatrix} \mathcal{O}_1^+ & 0 & 0 & 0 & 0 \\ 0 & \mathcal{O}_2^+ & 0 & 0 & 0 \\ \mathcal{O}_1^- & \mathcal{O}_2^- & \mathcal{O}_3^+ & 0 & 0 \\ 0 & 0 & \mathcal{O}_3^- & \mathcal{O}_4^- & \mathcal{O}_5^+ \\ 0 & 0 & 0 & \mathcal{O}_4^+ & 0 \\ 0 & 0 & 0 & 0 & \mathcal{O}_5^- \end{pmatrix}$$

Let

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, the Kirchhoff-type conditions at v_3, v_4 can be represented by

$$D\widetilde{A}Ag = 0$$

or

$$\begin{cases} 3ig_3(t) + \int\limits_0^t K_3(0,s)g_3(t-s)\,ds + 2ig_4(t-l_3) + 2\int\limits_{l_3}^t w_3^{2,-}(l_3,s)g_4(t-s)\,ds = F_1(t) \\ 3ig_4(t) + \int\limits_0^t K_4(0,s)g_4(t-s)\,ds - ig_3(t-l_3) - \int\limits_{l_4}^t w_3^{2,+}(l_3,s)g_3(t-s)\,ds = F_2(t) \end{cases}$$

The inverse problem for the 1-D Dirac system on a three-star graph. Leaf-peeling algorithm

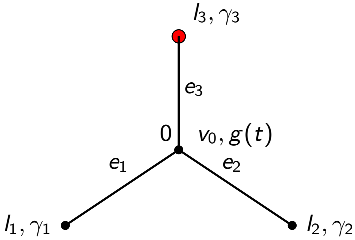
$$iU_t + JU_x + QU = 0, \quad t \geq 0, \quad x \in e_k, \quad k = 1, 2, 3, \tag{23}$$

$$u_k^1(0,t) = u_j^1(0,t), \quad k \neq j, \quad t \geq 0, \quad k, j = 1, 2, 3, \tag{24}$$

$$\sum_{k=1}^{3} u_k^{2,\pm}(0,t) = 0, \quad t \geq 0, \tag{25}$$

$$u_k^1(l_k, t) = f^k(t), \quad k = 1, 2, \quad u_3^1(l_3, t) = 0, \quad 0 \leq t \leq T, \tag{26}$$

$$U(\cdot, 0) = 0. \tag{27}$$



Proposition 3. For every fixed $k = 1, 2, 3$ and $j = 0, 1, 2, 3$, let $\{\xi_{k,j,n}\}_{n=1}^{\infty}$ be the set of all distinct lengths of all walks on Ω from γ_k to v_j. Then all discontinuities of $g(t)$ are located on $\{\xi_{k,j,n}\}_{n=1}^{\infty}$, and

$$g(t) = \sum_{n=1}^{\infty} \alpha_{k,j,n} \delta(t - \xi_{k,j,n}) + \theta_{k,j}(t), \tag{28}$$

where $\theta_{k,j}(t)$ is a piece-wise continuous function with possible jump discontinuities at $\{\xi_{k,j,n}\}_{n=1}^{\infty}$ and $\alpha_{k,j,n} \in \mathbb{C}$.

$$R^T\{F\}(t) := u^2(\cdot, t)\big|_{\Gamma \setminus \gamma_3}, \quad t \in [0, T]$$

$$\big(R^T F\big)(t) = \big(\mathcal{R} * F\big)(t) = \int_0^t \mathcal{R}(t-s) F(s)\, ds,$$

where $\mathcal{R}(t) = \{R_{kj}\}_{k,j=1}^{m-1}$ is a response matrix.

$$R_{kj} = \sum_{n=1}^{\infty} \beta_{k,j,n}\delta(t - \eta_{k,j,n}) + \chi_{k,j}(t), \tag{29}$$

where $\chi_{k,j}(t)$ is a piece-wise continuous function, $\beta_{k,j,n} \in \mathbb{C}$ and $\{\eta_{k,j,n}\}_{n=1}^{\infty}$ is the set of distinct lengths of all walks from γ_k to γ_j.

The Inverse Problem: for the problem (23)-(27) we want to recover the length l_k of the edge e_k for $k = 1, 2, 3$, and the matrix potential function $Q(x)$ on e_k from the response data $R^T(t)$ for $t > 0$.

Leaf-peeling algorithm

STEP 1. We recover l_1, l_2 and $Q(x)$ on $e_1, e_2 \in E(\Omega)$.

Proposition 4. For the known entry R_{11}, one can determine l_1, p_1, q_1.

Indeed, for $t < 3l_1$, one obtains

$$ig(t) = \frac{2}{3} i\delta(t - l_1) + \theta(t).$$

Now, for $t < 3l_1$, we get

$$(R_{11} f^1)(t) = u_1^{f^1,2}(l_1, t) = u_1^{f^1,2,-}(l_1, t) + u_1^{g,2,+}(l_1, t).$$

Hence, with $f^1(t) = \delta(t)$

$$R_{11}(t) = u_1^{\delta,2}(l_1, t) = \frac{1}{3} i\delta(t - 2l_1) + i\delta(t) + \chi(t).$$

That $R_{11}^{2l_1}$ determines p_1 and q_1 on e_1 is proven in [1].

STEP 2. By knowing $R_{22}^{2l_2}(t)$, one can determine p_2, q_2 on the edge e_2.

STEP 3. We recover l_3 and p_3, q_3 on the edge e_3.

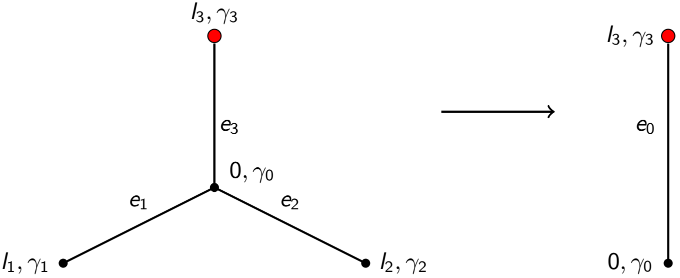


Figure: Three-star graph Ω and a peeled graph $\widetilde{\Omega}$

M. I. Belishev, V. S. Mikhailov, *Inverse problem for a one-dimensional dynamical Dirac system (BC-method)*, Inverse Problems **30** (2014), no. 12, Article ID 125013.

Let

- $\widetilde{\Omega} = (\widetilde{V}, \widetilde{E})$, where $\widetilde{V} = \{\gamma_0, \gamma_3\}$, $\widetilde{E} = \{e_0\}$.
- $\widetilde{\mathcal{R}}(t)$ be the reduced response operator on $\widetilde{\Omega}$ with an entry $\widetilde{R}_{00}(t)$.
- $g(t) := u_1^1(\gamma_0, t)$, $A(t) := u_3^2(\gamma_0, t)$.

Lemma. One can compute $g(t)$ and $A(t)$ from $\mathcal{R}(t)$ on Ω.

Indeed, by superposition principle

$$R_{11}(t) = u_1^{2,\delta,-}(l_1, t) + u_1^{2,g,+}(l_1, t).$$

Taking into account Kirchhoff-type conditions at γ_0, one obtains

$$A(t) = -u_1^{2,\delta,-}(0, t) - u_1^{2,g,+}(0, t) - u_2^{2,g,+}(0, t).$$

By Duhamel's principle

$$A(t) = g(t) * \widetilde{R}_{00}(t), \tag{30}$$

where

$$\widetilde{R}_{00}(t) = \sum_{n=1}^{\infty} c_{0,n}\delta(t - \lambda_{0,n}) + \varphi(t) \tag{31}$$

By equating the singular parts in (30), one obtains

$$\sum_{n=1}^{N}\sum_{m=1}^{N} a_n c_{0,m}\delta(t - \mu_n - \lambda_{0,m}) = \sum_{n=1}^{N} b_{0,n}\delta(t - \nu_{0,n}) \tag{32}$$

where $\{\mu_n\}_{n\geq 1}$, $\{\nu_{0,n}\}_{n\geq 1}$ are sets of distinct lengths of all walks on Ω from γ_1 to γ_0, $\{\lambda_{0,m}\}_{m\geq 1}$ is the set of distinct lengths of all closed walks on Ω from γ_0 to itself.

One can show that (32) can be written in the matrix form

$$A\mathfrak{c} = \mathfrak{b} \tag{33}$$

where $\mathfrak{c} = (c_{0,1}, \cdots, c_{0,N})^\top$, $\mathfrak{b} = (b_{0,1}, \cdots, b_{0,N})^\top$, and A is a lower triangular matrix.

By equating the regular parts in (30), we obtain the integral equation with respect to unknown function $\varphi(t)$.

The inverse problem for the 1-D Dirac system on finite metric tree graphs

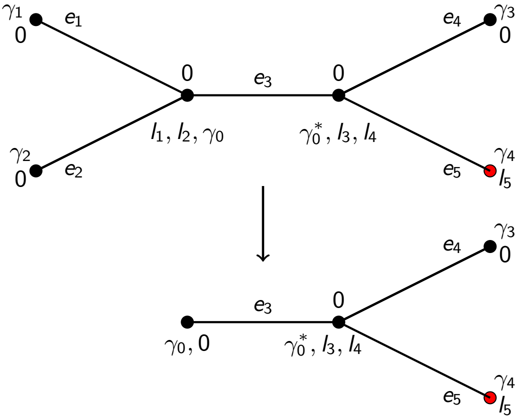


Figure: Graph Ω and peeled graph $\widetilde{\Omega}$

Definition. The star shaped subgraph Ω' of a tree graph Ω is a **sheaf** if it contains all edges of the graph Ω incident to some internal vertex and if all but one of its edges are **leaf** edges. The one edge that is not a leaf edge is called the **stem** edge of a sheaf.

Let

$$g(t) := u_1^1(\gamma_0, t), \quad A(t) := u_3^2(\gamma_0, t), \quad k = 1, 2, 3.$$

Lemma. One can compute $g(t)$ and $A(t)$ from $\mathcal{R}(t)$ on Ω.

GOAL: We want to compute $\widetilde{R}_{00}$, $\widetilde{R}_{03}$, $\widetilde{R}_{30}$, and $\widetilde{R}_{33}$ for the peeled tree graph $\widetilde{\Omega}$.

STEP 1. For known $g(t)$, $f^1(t)$, and $R_{13}(t)$, due to the Duhamel's principle, one can compute $\widetilde{R}_{03}$ from

$$\int_0^t R_{13}(s) f^1(t-s)\, ds = \int_0^t \widetilde{R}_{03}(s)(g * f^1)(t-s)\, ds.$$

STEP 2. For known $g(t)$, $f^1(t)$, and $A(t)$, one can compute $\widetilde{R}_{00}$ from

$$\int_0^t \widetilde{R}_{00}(s)(g * f^1)(t-s)\,ds = (A * f^1)(t).$$

STEP 3. For known functions $f^3(t)$, $A(t)$, $\widetilde{R}_{00}(t)$, and $g(t)$, one can compute $\widetilde{R}_{30}$ from

$$\int_0^t \widetilde{R}_{30}(s) f^3(t-s)\,ds = A(t) - (\widetilde{R}_{00} * g)(t).$$

STEP 4. For known functions $g(t)$, $R_{33}(t)$, $f^3(t)$, one can compute $\widetilde{R}_{33}$ from

$$\int_0^t \widetilde{R}_{33}(s) f^3(t-s)\,ds = R_{33}(t) - (\widetilde{R}_{03} * g)(t)$$

Corollary. One can recover the lengths of edges, the tree topology, and the matrix potential $Q(x)$ on a finite metric tree graph Ω from $\widetilde{\mathcal{R}}(t)$ for $t \in (0, T)$, where $T > 2l^*$ and l^* is the longest path from the leaf vertex to the root vertex of Ω.

Thank you for your attention!