NOTES ON CURVATURE IN COMPLEX HYPERBOLIC SPACE

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1. Introduction

The goal of these notes is to introduce the reader to complex hyperbolic space $\mathbb{CH}^n$ and explore its geometry and curvature. We will do so while making connections when possible to what is hopefully a more familiar example, real hyperbolic space, $\mathbb{H}^n$. After defining $\mathbb{CH}^n$, we will take a look at two kinds of special subspaces, *totally real subspaces* and $\mathbb{C}$-*affine lines*. This study will lead us to question and explore curvature and triangles that live "in between" the two special cases.

Complex hyperbolic space is a homogeneous space which can be given both complex and Riemannian structures. These notes will focus on the Riemannian case. One can think of $\mathbb{C}^n$ as the open unit ball in $\mathbb{CH}^n$ with a Riemannian metric. Under this metric, $\mathbb{CH}^n$ has nonconstant sectional curvature, and it is this aspect of complex hyperbolic space that we aim to explore.

I would like to thank my advisor Moon Duchin for her guidance. I'd also like to thank Rosemary Guzman, Daniel Waite, and David Polletta for their helping me piece together the big picture of these notes.

2. Hyperboloid and projective models of $\mathbb{H}^n$

Before we define complex hyperbolic space, it can be helpful to first look at real hyperbolic space $\mathbb{H}^n$. In these notes, we will explore the *hyperboloid* and *projective* models of $\mathbb{H}^n$.

2.1. **Hyperboloid model.** The first step in defining the hyperboloid model is to consider the symmetric bilinear form $\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ which for $x = (x_1, \ldots, x_{n+1})$, and $y = (y_1, \ldots, y_{n+1})$, gives

$$\langle x, y\rangle = \left(\sum_{i=1}^{n} x_i y_i\right) - x_{n+1}y_{n+1}.$$

Note that this form does not define an inner product. In fact, we have *positive*, *null*, and *negative vectors*, i.e. vectors x such that $\langle x, x\rangle$ is positive, zero, or negative, respectively. For example, observe that

$$\langle x, x\rangle = 0 \quad \Leftrightarrow \quad x_1^2 + \ldots + x_n^2 = x_{n+1}^2.$$

The solution set is a cone in $\mathbb{R}^{n+1}$, often referred to as the *light cone* (see Figure 1). This light cone of null vectors will later serve to define the boundary of hyperbolic space.

The hyperboloid model of real hyperbolic space is the top sheet of the n-dimensional hyperboloid defined by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = -1\}.$$

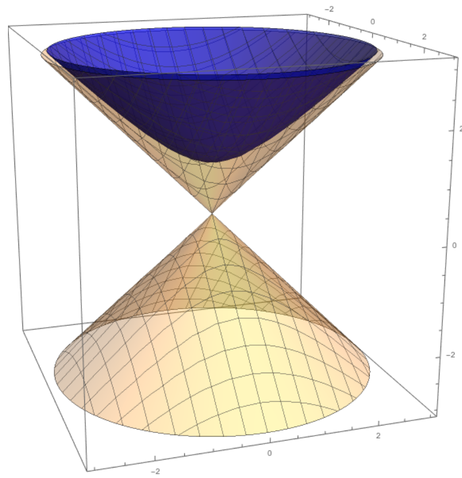

FIGURE 1. Hyperboloid model of $\mathbb{H}^2$ (blue) with light cone (yellow).

2.2. **The metric.** Given x and y in the hyperboloid model of $\mathbb{H}^n$, the distance between the points is defined by

$$\cosh d(x, y) = -\langle x, y\rangle.$$

It turns out, however, that when $\langle\ ,\ \rangle$ is restricted to the tangent space at any point $x \in \mathbb{H}^n$, it gives us an inner product, and thus the Riemannian metric.

We define the orthogonal complement of a vector x to be

$$x^\perp = \{u \in \mathbb{R}^{n+1} \mid \langle x, u\rangle = 0\}.$$

Proposition 1. *If x is a negative vector, i.e. $\langle x, x\rangle < 0$, then the restriction of the form $\langle\ ,\ \rangle$ to the orthogonal complement $x^\perp$ of x is positive definite.*

Proof. Suppose $u \neq 0$, $u \in x^{\perp}$, and $\langle x, x\rangle = -c$ for some $c > 0$. Since $u \in x^{\perp}$, we know

$$\langle x, u\rangle = x_1u_1 + \ldots + x_nu_n - x_{n+1}u_{n+1} = 0.$$

We also know that $x_{n+1} \neq 0$ since x is a negative vector. Therefore we can rewrite this line as

$$u_{n+1} = \frac{x_1u_1 + \ldots + x_nu_n}{x_{n+1}}.$$

Now using the Cauchy-Schwarz inequality for the Euclidean inner product on $\mathbb{R}^n$, we calculate:

$$\begin{aligned}
\langle u, u\rangle &= u_1^2 + \ldots u_n^2 - \left(\frac{x_1u_1 + \ldots + x_nu_n}{x_{n+1}}\right)^2 \\
&= \left(\sum_{i=1}^n u_i^2\right) - \left(\frac{1}{x_{n+1}}\right)^2 \left(\sum_{i=1}^n x_iu_i\right)^2 \\
&\geq \left(\sum_{i=1}^n u_i^2\right) - \left(\frac{1}{x_{n+1}}\right)^2 \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n u_i^2\right) \\
&= \left(1 - \frac{1}{x_{n+1}^2}\left(\sum_{i=1}^n x_i^2\right)\right)\left(\sum_{i=1}^n u_i^2\right) \\
&= \left(1 - \frac{1}{x_{n+1}^2}\left(x_{n+1}^2 - c\right)\right)\left(\sum_{i=1}^n u_i^2\right) \\
&= \frac{c}{x_{n+1}^2}\left(\sum_{i=1}^n u_i^2\right) > 0.
\end{aligned}$$

□

Thus the form when restricted to the orthogonal complement at each point $x \in \mathbb{H}^n$ gives us an inner product. For these inner products to define a Riemannian metric, we would have to identify the tangent space at each point with its orthogonal complement. We omit the proof that such an isomorphism exists but will include the proof later of the isomorphism in the complex case. Therefore, this family of inner products $\langle\ ,\ \rangle\big|_{x^{\perp}}$ defines a Riemannian metric on $\mathbb{H}^n$.

2.3. **Projective model of $\mathbb{H}^n$.** Note that $\mathbb{H}^n$ maps bijectively onto its image in $\mathbb{R}P^n$. We'll call the projection map π. Then if we give the image $\pi(\mathbb{H}^n) \subset \mathbb{R}P^n$ the metric defined by the pullback of the metric through π^{-1}, making π an isometry, then this is a new projective model of $\mathbb{H}^n$. We'll use $[x]$ to denote the equivalence class of x in $\mathbb{R}P^n$. This projective model is defined by

$$\mathbb{H}^n = \{[x] \in \mathbb{R}P^n \mid \langle [x], [x] \rangle < 0\}.$$

It is worth mentioning that after projectivizing, we no longer can pick out a level set, but whether the equivalence class $[x]$ of a vector x is positive, null, or negative is well-defined.

3. Complex hyperbolic space $\mathbb{CH}^n$

In the real case, hyperbolic space is defined by its constant negative sectional curvature. In the complex case, we will seek constant negative holomorphic curvature. Just like real hyperbolic space, complex hyperbolic space has several models, each of which has its advantages. Here we will define the projective and ball models of $\mathbb{CH}^n$.

3.1. **Projective model of $\mathbb{CH}^n$.** As in the real case, we will start by defining a form, in this case a Hermitian form $\langle\ ,\ \rangle : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$. We will use the same notation $\langle\ ,\ \rangle$ for the form, which we justify by noting it generalizes the one defined above. For $z = (z_1, \ldots, z_{n+1})$ and $w = (w_1, \ldots, w_{n_1})$, we have

$$\langle z, w \rangle = \left(\sum_{i=1}^{n} \bar{z}_i w_i \right) - \bar{z}_{n+1} w_{n+1}.$$

This is also an indefinite form with positive, null, and negative vectors. Again, the negative vectors will give the points of our space. Our space $\mathbb{CH}^n$ will have complex dimension n, and we will obtain it by a projectivization of the $(2n+1)$ real dimensional locus $\{\langle z, z \rangle = 1\}$. As above we will projectivize via the standard projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}P^n$, and we will use $[z]$ to denote the equivalence class of z in $\mathbb{C}P^n$. We can then define the projective model.

The *projective model* of complex hyperbolic space is the set

$$\mathbb{CH}^n = \{[z] \in \mathbb{C}P^n \mid \langle [z], [z] \rangle < 0\}.$$

3.2. **The metric.** Given two points $[z], [w] \in \mathbb{CH}^n$, we will define the distance between the points by

$$\cosh^2 d([x], [y]) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.$$

One can check, or refer to [BH13] to check, that this does indeed define a metric.

3.3. **Tangent vectors.** For a point $z \in \mathbb{C}^{n+1}$, let $z^\perp = \{w \in \mathbb{C}^{n+1} : \langle z, w\rangle = 0\}$.

Now we want to think about the tangent space at a point $[z] \in \mathbb{CH}^n$. Note that $\mathbb{CH}^n$ is an open subset of $\mathbb{C}P^n$. In particular, $\mathbb{CH}^n$ is an open subset of the open set $U_{n+1} = \{[w] \in \mathbb{C}P^n : w_{n+1} \neq 0\}$. Since U_{n+1} is diffeomorphic to $\mathbb{C}^n$, the tangent space at $[w]$ in U_{n+1} is isomorphic to $\mathbb{C}^n$ as a vector space.

Let $\phi : T_{[z]}\mathbb{CH}^n \to z^\perp$ be the map described by

$$\phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = \begin{bmatrix} a_1 z_{n+1} + \frac{z_1\lambda}{z_{n+1}} \\ \vdots \\ a_n z_{n+1} + \frac{z_n\lambda}{z_{n+1}} \\ \lambda \end{bmatrix},$$

where $\lambda = \frac{-z_{n+1}^2}{\langle z,z\rangle}\left(\sum_{i=1}^n \bar{z}_i a_i\right)$. This map can be found using the differential of the natural projection map π.

Proposition 2. *The map $\phi : T_{[z]}\mathbb{CH}^n \to z^\perp$, is an isomorphism of $\mathbb{C}$ vector spaces.*

Proof: It is not hard to see that this map is linear. Also, since $\mathbb{C}^n$ and $z^\perp$ both have real dimension $2n$, it is enough to show that ϕ is injective. Suppose the vector $a \in T_{[z]}\mathbb{CH}^n$ is nonzero. Recall that since $[z] \in \mathbb{CH}^n$, we know $\langle z, z\rangle \neq 0$ and $z_{n+1} \neq 0$. If $\left(\sum_{i=1}^n \bar{z}_i a_i\right) \neq 0$, then $\lambda \neq 0$, and so $\phi(a) \neq 0$. If $\left(\sum_{i=1}^n \bar{z}_i a_i\right) = 0$, then

$$\phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = \begin{bmatrix} a_1 z_{n+1} \\ \vdots \\ a_n z_{n+1} \\ 0 \end{bmatrix},$$

and since one of the a_i must be nonzero, then the image is nonzero as well. Thus ϕ is injective and, therefore, an isomorphism.

□

So from now, we will identify $z^\perp$ with the tangent space at $[z]$ in $\mathbb{CH}^n$. There can be some ambiguity when talking about tangent vectors since for $\lambda \in \mathbb{C}^\times$, $(\lambda z)^\perp = z^\perp$. To resolve this, we will talk about pairs under the equivalence relation $(z, v) \sim (\lambda z, \lambda v)$. Note that under this equivalence relation, $(z, \lambda v)$ represents a *different* tangent vector from (z, v).

3.4. **Ball model of $\mathbb{CH}^n$.** We can get a different, although similar, model of $\mathbb{CH}^n$ by taking a canonical set of lifts of the projective model to $\mathbb{C}^{n+1}$. By the negativity condition, just as in the real case, if $[z] \in \mathbb{CH}^n$ then we know $z_{n+1} \neq 0$. Thus we can choose the representative of $[z]$ with $z_{n+1} = 1$, i.e., $[\frac{z_1}{z_{n+1}} : \ldots : \frac{z_n}{z_{n+1}} : 1]$. In the set of representatives $\{z \in \mathbb{C}^{n+1} \mid [z] \in \mathbb{CH}^n, z_{n+1} = 1\}$, the condition that $\langle z, z \rangle < 0$ turns into

$$|z_1|^2 + \ldots |z_n|^2 < 1.$$

Therefore $\mathbb{CH}^n$ can be parametrized as the open unit ball in $\mathbb{C}^n$. Each point $w = (w_1, \ldots, w_n)$ in this unit ball corresponds to a unique point $[w_1 : \ldots : w_n : 1]$ in the projective model. This allows us to define the metric on the open unit ball by pulling the metric back through this bijection.

3.5. **Hyperbolic segments and real hyperbolic angle.** Given $[z] \in \mathbb{CH}^n$, we can choose a representative z of $[z]$ such that $\langle z, z \rangle = -1$. Choose a representative$u \in z^{\perp}$ of (z, u) with $\langle u, u \rangle = 1$. This is possible since we know Then the *geodesic segment* from $[z]$ in the direction of u is the curve $\gamma : \mathbb{R} \to \mathbb{CH}^n$,

$$\gamma(t) = [z \cosh(t) + u \sinh(t)].$$

These turn out to be all the geodesics in $\mathbb{CH}^n$. See [BH13] for more details. For any $[z] \in \mathbb{CH}^n$ with two nonzero tangent vectors u, $v \in T_{[z]}\mathbb{CH}^n$, we define the *real hyperbolic angle* $\angle_z$ between u and v by

$$\cos \angle_z(u, v) := \frac{\Re\langle u, v \rangle}{\sqrt{\langle u, u \rangle}\sqrt{\langle v, v \rangle}},$$

where $\Re(w)$ is the real part of the complex number w.

3.6. **Alexandrov angle.** In this section, we define the Alexandrov angle between two geodesics issuing from a point p in an arbirtrary metric space, as it is presented in [BH13]. Suppose X is a metric space, and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesics where $p = c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, we can consider the comparison triangle $\bar{\Delta}(c(0), c(t), c'(t'))$ in $\mathbb{E}^2$. We also have the comparison angle $\overline{\angle}_p(c(t), c'(t'))$. The *Alexandrov angle* or *upper angle* between the geodesic paths c and c' is the number $\angle(c, c') \in [0, \pi]$ given by

$$\angle(c, c') := \limsup_{t, t' \to 0} \overline{\angle}_p(c(t), c'(t')) = \lim_{\epsilon \to 0} \sup_{0 < t, t' < \epsilon} \overline{\angle}_p(c(t), c'(t')).$$

Proposition 3. *The real hyperbolic angle $\angle_z$ defined in Section 3.5 agrees with the Alexandrov angle in $\mathbb{CH}^n$.*

4. Curvature and Special subspaces

4.1. **Curvature.** When talking about curvature (specifically negative curvature here), we will need to reference model spaces of various curvatures. The model spaces for negative curvature will be obtained from real hyperbolic space $\mathbb{H}^n$ with its hyperbolic metric $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ as defined above. Tor $\kappa < 0$, we define the *n-dimensional model space of curvature κ, M^n_κ,* to be the metric space $(\mathbb{H}^n, d_\kappa)$, where $d_\kappa = \frac{d}{\sqrt{-\kappa}}$. This scaling of the hyperbolic metric provides us with constant negative sectional curvature κ. Later we will make use of the law of cosines for the model space M^n_κ, so we will cite it here.

Proposition 4. *Given a geodesic triangle in M^n_κ, with sides of positive length a, b, and c and Alexandrov angle γ at the vertex opposite the side of length c, for $\kappa < 0$,*

$$\cosh(\sqrt{-\kappa} \cdot c) = \cosh(\sqrt{-\kappa} \cdot a) \cosh(\sqrt{-\kappa} \cdot b) - \sinh(\sqrt{-\kappa} \cdot a) \sinh(\sqrt{-\kappa} \cdot b) \cos(\gamma).$$

Note that when $\kappa = -1$, we get back the law of cosines for real hyperbolic space $\mathbb{H}^n$.

When we say that a metric space (X, d) is $\mathrm{CAT}(\kappa)$, what we mean is that "geodesic triangles in X aren't any fatter than their comparison triangles in (M^2_κ, d_κ)". Suppose we have a triangle $\Delta = \Delta ABC$ in our space X with side lengths a, b, and c. A *comparison triangle* in M^2_κ is a triangle $\Delta' = \Delta A'B'C'$ such that the side lengths are still a, b, and c. It's not hard to show that any two such comparison triangles in M^n_κ are isometric. If x is a point on Δ, we can find its comparison point x' on Δ' by parametrizing the geodesic edges of the triangles by arc length. The triangle Δ' satisfies the *CAT(κ) inequality* if for any points x, $y \in \Delta$ and comparison points $x', y' \in \Delta'$, we have

$$d(x, y) \leq d_\kappa(x', y').$$

A geodesic space X is said to be a *$CAT(\kappa)$ space* if all of its geodesic triangles satisfy the $\mathrm{CAT}(\kappa)$ inequality.

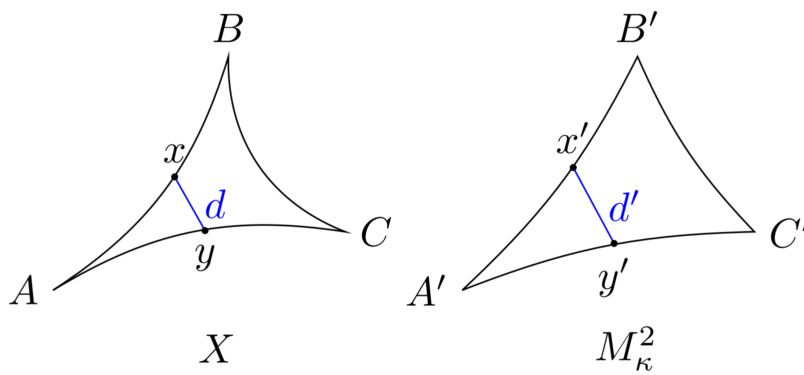

FIGURE 2. Comparison triangles and CAT(κ).

Proposition 5. $\mathbb{CH}^n$ *is a CAT(-1) space.*

See Chapter II.10 of [BH13] for a proof.

4.2. **Special subspaces.** Note that we can view $\mathbb{C}^{n+1}$ as an $\mathbb{R}$-vector space. An $\mathbb{R}$-vector subspace $V \subset \mathbb{C}^{n+1}$ is said to be *totally real* in $\mathbb{C}^{n+1}$ with respect to $\langle\ ,\ \rangle$ if $\langle u, v\rangle \in \mathbb{R}$ for all u, $v \in V$. Suppose V is a totally real subspace of dimension $k+1$ and that there exists a negative vector $z \in V$. Then we call

$$\pi(V \backslash \{0\}) \cap \mathbb{CH}^n$$

a *totally real subspace of dimension* k *in* $\mathbb{CH}^n$.

Example: Consider $V = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$. Obviously this is a totally real vector subspace in $\mathbb{C}^3$, and the vector $z = (0, 0, 1)$ is a negative vector in V. Thus if we project and intersect with complex hyperbolic space, we will get a totally real subspace of (real) dimension 2 in $\mathbb{CH}^2$. Picking lifts with $x_3 = 1$, in the ball model of $\mathbb{CH}^2$, this totally real subspace is parametrized by

$$\{(x_1, x_2, 1) : x_1^2 + x_2^2 < 1\}.$$

We get a copy of the real open unit disk living in $\mathbb{CH}^n$.

Suppose $U \subset \mathbb{C}^{n+1}$ is a $\mathbb{C}$-vector subspace of complex dimension $k+1$ and that U contains a negative vector w. Then we call

$$\pi(U \backslash \{0\}) \cap \mathbb{CH}^n$$

a $\mathbb{C}$-*affine subspace of dimension* k *in* $\mathbb{CH}^n$.

Example: Consider $U = \{(z_1, 0, z_3) : z_1, z_3 \in \mathbb{C}\}$. Again the vector $z = (0, 0, 1)$ satisfies the negativity condition. Thus if we project and intersect with complex hyperbolic space, we will get a $\mathbb{C}$-affine subspace of (complex) dimension 1, i.e., a $\mathbb{C}$-affine line. In the ball model, it is parametrized by

$$\{(z_1, 0, 1) : |z_1|^2 < 1\}.$$

We get a copy of the complex open unit disk living in $\mathbb{CH}^n$.

Note that in each example, we got a 2 (real) dimensional open unit disk. One has real coordinates, though, while the other has complex coordinates. It turns out that this difference will be very important when we consider the curvatures of these subspaces.

4.3. **Curvature of special subspaces.** Let $\Delta \subset \mathbb{CH}^n$ be a geodesic triangle with vertices A, B, and C, where $C = [z]$ and $\langle z, z\rangle = -1$. Let $u, v \in z^\perp$ be the initial unit vectors at z of the geodesic segments $\overline{CA}$ and $\overline{CB}$. See Figure 3. One could ask about the "curvature of this triangle." We know that for all nonpositively curved model spaces M^2_κ there exists a comparison triangle Δ' in M^2_κ. This embedding preserves side lengths, but we can ask for more. What if we want Alexandrov angles to be preserved as well? We can also ask if the convex hull of the triangle embeds isometrically into M^2_κ for some κ, necessarily preserving both side lengths and Alexandrov angles.

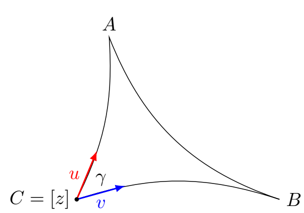

FIGURE 3. Picture for Proposition 6.

The motivation for the special subspaces above is revealed by the following proposition cited from [BH13], whose proof can be found in section II.10.12.

Proposition 6. *Let Δ be a triangle in $\mathbb{CH}^n$ with A, B, C, u, and v as above. Suppose that the real hyperbolic angle $\angle_C(A, B)$ is not equal to 0 or π. Then:*

(1) *The convex hull of Δ is isometric to the convex hull of its comparison triangle in $M^2_{-1} = \mathbb{H}^2$ if and only if $\langle u, v\rangle$ is real.*

(2) *The convex hull of Δ is isometric to the convex hull of its comparison triangle in M^2_{-4} if and only if u and v span a $\mathbb{C}$-affine line.*

So this proposition answers our question for two cases. If u and v are independent vectors in a totally real subspace V, then Δ looks like a triangle of curvature -1. And if u and v are part of a $\mathbb{C}$-affine line, then Δ looks like a triangle of curvature -4. The following theorem from [BH13] section II.10.16 generalizes the proposition.

Theorem 1. (1) *The totally real subspaces of dimension k in $\mathbb{CH}^n$ are precisely those subsets which are isometric to the real hyperbolic space $\mathbb{H}^k$.*

(2) *The $\mathbb{C}$-affine lines in $\mathbb{CH}^n$ are precisely those subsets which are isometric to the model space M^2_{-4}. Moreover, $\mathbb{CH}^n$ does not contain any subsets isometric to M^m_{-4} for $m > 2$.*

5. Explorations

Now we will explore curvature and triangles that live neither in totally real nor in $\mathbb{C}$-affine subspaces. We still would like to embed arbitrary triangles from $\mathbb{CH}^2$ into model spaces of negative curvature, but in general we won't be able to preserve side lengths *and* all three Alexandrov angles in these embeddings. Instead, we will aim to preserve only the side lengths and the Alexandrov angle at vertex $[z]$. Therefore, from this point on we can forget some information define *triangles* to be 4-tuples Tri(a, b, c, γ), where a, b, and c are the side lengths and γ is the Alexandrov (or real hyperbolic) angle at $[z]$. See Figure REFERENCE. If Tri(a, b, c, γ) can be embedded into the model space M^2_κ in a way that preserves the Alexandrov angle γ, then we will say *the triangle has curvature κ.*

NEW FIGURE

Now we'll look at a family of triangles in $\mathbb{CH}^2$. Let $z = (0, 0, 1) \in \mathbb{CH}^2$, so that $\langle z, z\rangle = -1$. Let $u = (1, 0, 0) \in z^\perp$, and let $v_\alpha = (i \sin\alpha, \cos\alpha, 0) \in z^\perp$ be the family of tangent vectors at $[z]$ parametrized by $0 \le \alpha \le \pi/2$. Consider the geodesic triangle that is formed by traveling from z in the directions of u and v_α for time $t = 1$ and then connecting those two endpoints by a geodesic. Using the notation from above, these are the triangles Tri(1, 1, c_α, γ_α). See Figure 4.

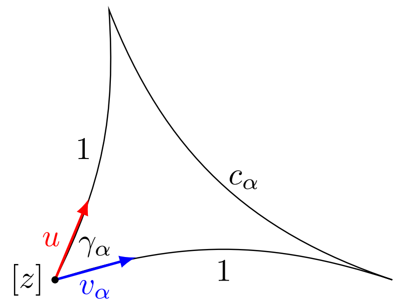

FIGURE 4. Setup for the first exploration.

Note that α does not describe the real hyperbolic angle between u and v_α, but rather serves as a parameter. The real hyperbolic angle between u and v_α is

$$\gamma_\alpha = \angle_z(u, v_\alpha) = \cos^{-1}\left(\frac{\Re\langle u, v_\alpha\rangle}{\sqrt{\langle u, u\rangle}\sqrt{\langle v_\alpha, v_\alpha\rangle}}\right) = \cos^{-1}(\Re(-i\sin\alpha)) = \pi/2.$$

So our family of triangles looks like Tri(1, 1, c_α, $\frac{\pi}{2}$), where

$$c_\alpha = \cosh^{-1}\left(\sqrt{\sin^2\alpha\cdot\sinh^4 1 + \cosh^4 1}\right).$$

As we try to determine the curvature of these triangles, the previous section gives us information about the two extremal cases.

Case 1: If $\alpha = 0$, the triangle lives in a totally real subspace: $\langle u, v\rangle \in \mathbb{R}$, and hence the triangle has curvature -1.

Case 2: If $\alpha = \frac{\pi}{2}$, the triangle lives in a $\mathbb{C}$-affine line: u and v span a $\mathbb{C}$-affine line, and the triangle has curvature -4.

For each α, we would be able to find a comparison triangle with side lengths 1, 1, and c_α in any model space of negative curvature. If we ask that the comparison triangle has Alexandrov angle $\frac{\pi}{2}$, Proposition 4 (the law of cosines) tells us that κ is uniquely determined by α. Therefore, there is a unique model space M^2_κ in which we can realize the triangle $\mathrm{Tri}(1, 1, c_\alpha, \frac{\pi}{2})$. See Figure 5 to see the relationship between α and κ.

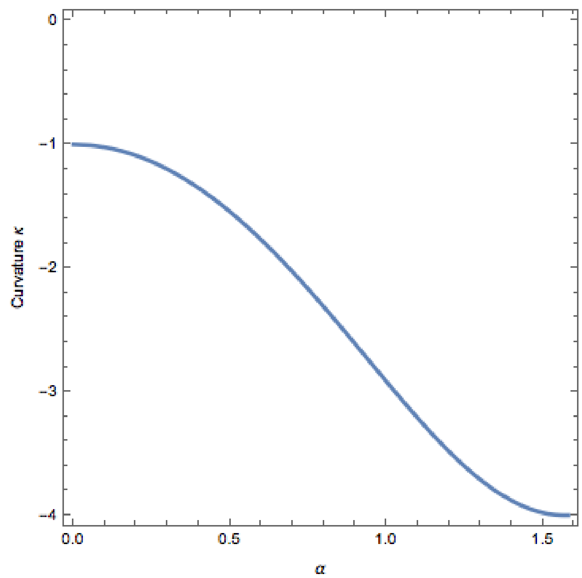

FIGURE 5. Curvature of triangles varying α, computed numerically.

It is worth mentioning that the embeddings of $\mathrm{Tri}(1, 1, c_\alpha, \frac{\pi}{2})$ do *not* preserve the other two angles of the triangle $0 < \alpha < \frac{\pi}{2}$. In general, γ_α is the only angle to be preserved. But the calculation does allow us to see some aspects of the curvature as we look around in different directions from $[z]$.

Now we ask what happens if our triangles have side lengths not equal to 1. For this experiment, we'll fix α. Instead of traveling for time 1, we'll travel along the geodesics in the directions of u and v_α for time t. Then we can use the law of cosines again to determine κ.

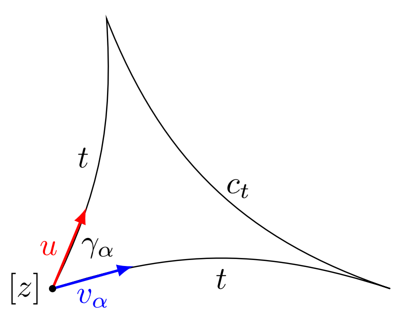

FIGURE 6. Setup for the second exploration.

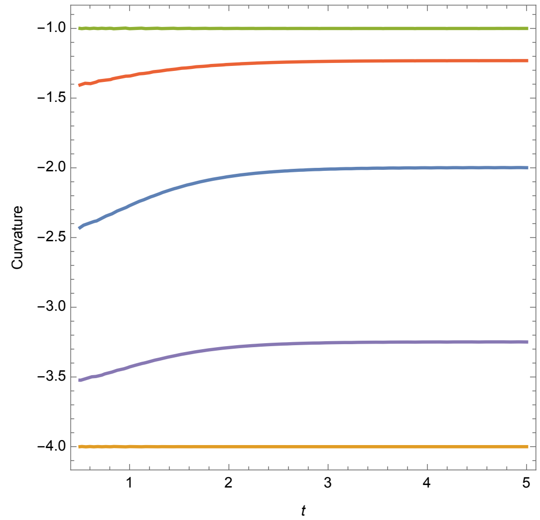

FIGURE 7. Fixing α as 0 (green), $\frac{\pi}{8}$ (red), $\frac{\pi}{4}$ (blue), $\frac{3\pi}{8}$ (purple), and $\frac{\pi}{2}$ (yellow).

Figure 7 shows κ as a function of t for several fixed values of α.

What have we learned? In both explorations, we are standing at $[z] = (0, 0, 1)$ in complex hyperbolic space. We fix one axis u, and then swivel around as we vary v_α. As we do so, we are getting a sense of what the curvature is in these different directions. In the second exploration, varying t instead of α, it is as if we're zooming in and out in these particular directions to see what happens to curvature. Figure 7 would imply that curvature converges at a larger scale in these directions.

NEED CONCLUSION

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