Psychology and the Search for Certainty in Everyday Life

Edited by

Daphne Halkias

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Middle School Mathematics Teachers' Implicit Conceptions about Multiple Representations for Functions

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The present study belongs to a broader educational/research project that seeks to improve the teaching and learning of mathematics in middle school (Grades 5-9) within the geographical area of New England (United States). More specifically, our project aims to promote students mathematical learning by developing teachers' expertise regarding the mathematics of algebra and functions, their multiple systems of representation, and their use for modeling situations involving quantities in scientific and everyday contexts. In the second week of our educational program, we presented the following algebraic situation to the first cohort of middle school teachers: *"The Candy Box Problem." Show two identical boxes of candies to your students. One is John's box and the other is Mary's box. Put 3 candies on the top of Mary's box and explain that: (1) John and Mary's boxes have exactly the same number of candies, and (2) Mary's box has 3 extra candies on top of it.* Right after the presentation of the algebraic situation, the teachers were given a set of open-ended questions intended to get them to articulate some aspects of their mathematical knowledge for teaching and promote their reflection regarding different mathematical and pedagogical issues. The first question we asked them was: *1) Please show what you know about the number of candies John and Mary each have. Feel free to use any helpful means to represent the problem.* As explained below, the teachers provided their answers to this and other questions using an online

system. To introduce and frame the topic covered by this paper, let's look at the answers provided by two of our participants.

<u>Teacher #1</u>
John = x.
Mary = x + 3.

<u>Teacher #2</u>
This problem could be represented in many different ways. Not only are there graphs, tables, ordered pairs, and functions we could use to represent this relationship, but we could also look at the problem from a few different perspectives. Let's see some examples:
- *Words: Mary's box of candy has 3 more than John's box.*
- *Table:*

John	Mary
0	3
1	4
2	5
3	6

- *Formula: M=J+3 or John's =x, Mary's=x+3*
- *Graph: I would graph the above values but not connect the dots because I consider the whole pieces of candy. I would also have some constraints on how many candies they logically can have in the size box that is given.*

In addition to the examples above there are some nice visuals we could use to demonstrate the relationship between Mary and John's candies, such as arrow diagrams or even drawings!

It is interesting to notice the differences between these two answers. Whereas Teacher #1 provided exclusively an algebraic representation of the scenario (John = x; Mary = $x + 3$), Teacher #2 provided us with a broader range of representations, including algebraic expressions. Although both responses can be considered as *correct* from a mathematical viewpoint, it cannot be denied that the second one is more comprehensive and sophisticated than the first one, especially if we adopt an educational perspective, as justified below. Considering the specific formulation of the question (i.e., *"use any helpful means to represent the problem"*), one could argue that these two teachers implicitly hold a very different view about the usefulness of representational systems for mathematical thinking. Thus, the responses to that question —together with other similar ones— inform us about teachers' implicit

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conceptions about multiple representations for functional relations. Broadly speaking, this is the topic addressed in the present paper.

Multiple Representations of Functional Relations

Algebra can be defined "as the study of functions, relations, and joint variation" (Kaput, 1999, p. 146). In recent years, a considerable deal of attention has been given to the teaching of algebra in the early ages, especially because of its importance for mathematics learning in higher educational levels (high school, college level, etc.) as well as the gatekeeping role it has come to play for access to higher education (e.g., Chazan, 2000). The US National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) suggests that students from pre-kindergarten through grade 12 should "understand patterns, relations, and functions, represent and analyze mathematical situations and structures using algebraic symbols" (p. 37).

Algebra plays a particularly important role in middle school mathematics, as students are required to move to progressively more abstract forms of mathematical reasoning. However, many studies have shown that middle school students have considerable difficulties with algebra. For example, students seem to experience difficulties in translating ordered pairs into algebraic representations, as well as in producing and interpreting algebraic and graphical representations (e.g., Leinhardt, Zaslavsky & Stein, 1990). One of the factors underlying students' struggles may be that representations are often times taught and learned as if they were "contents" themselves, rather than as learning and teaching tools. According to NCTM's standards (2000), we think that treating representations as contents considerably limits their potential and utility as tools for mathematical thinking and learning.

Representations are essential components of mathematical activity, as well as crucial vehicles for capturing mathematical ideas (Cai, 2005). Psycho-educational researchers within the field of mathematics education have suggested that using multiple representations (e.g., natural language, algebraic notations, graphs, tables, pictorial representations) is extremely powerful to foster students' mathematical learning. According to recent studies (Brizuela & Earnest, 2008; Goldin, 1998), using multiple representations in a flexible and inter-related manner

allows students to deepen their understanding of mathematical concepts and enhance their problem solving skills. This is so because the combined use of representations allows students to externalize and visualize different and complementary aspects of the same mathematical ideas in meaningful ways (Cuoco, 2001).

Regarding the topic of functional relations, Williams (1993) highlighted the importance of three different representations: algebraic expressions, graphs, and tables. Drawing on Williams' (1993) work, Brenner and collaborators (1997) "added verbal representations as another primary way in which students should be expected to understand functions" (p. 667). Certainly, being able to represent functional relations verbally and convert them into graphs, tables, and algebraic expressions has the potential to contribute to students' conceptual understanding of functions, and improve their problem solving representational capabilities (Brenner et al., 1997).

Research on Teachers' Conceptions about Multiple Representations

Numerous research studies have shown that the conceptions held by teachers mediate and condition their actual educational approaches, which consequently can have a crucial impact in students' learning (e.g., Entwistle, 2007). In the context of our project, it is reasonable to think that students' ability to use multiple representations might be influenced by their teachers' conceptions and pedagogical approaches regarding the teaching of representations. This study is focused on teachers' conceptions, based on the assumption that in order to help mathematics teachers to improve their teaching approaches, we also need to help them change their conceptions. Adopting a constructivist viewpoint, we think that knowing what teachers' conceptions are is essential to foster the development of more sophisticated conceptions and teaching practices (Entwistle, 2007).

We still know little about how teachers conceive of the role of representations in the teaching and learning of functions. In a recent interview study, Stylianou (2010) investigated the conceptions held by 18 middle school mathematics teachers. Results showed that teachers do not consider representations as a central issue throughout middle school mathematics. For many of them, representations constitute a topic of study rather than a learning/teaching tool. In other words, teachers primarily conceive of representations as a product of mathematical activity

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rather than as a process. Moreover, many teachers think that representations constitute a learning goal for the high-performing students only. In consequence, the degree of attention paid to representation in their teaching practice tends to be quite peripheral.

The purpose of this paper is to investigate the implicit conceptions that middle school mathematics teachers hold regarding multiple representations in problems that deal with functional relations. Our study is based on the same problem with which we opened this manuscript (the "*Candy Box Problem*"). Drawing on teachers' responses to two open-ended tasks, we aim to answer the following questions:

a. Which and how many representations for functions do middle school mathematics teachers use and/or refer to when representing the number of candies for themselves?

b. Which and how many representations do they expect their students to use? Are there any differences across students from different grade levels?

c. Are there any differences in the implicit conceptions of teachers with different educational backgrounds?

Given that the data analyzed in this article were collected in the beginning of the educational program program for middle school teachers, middle school teachers, the results presented below need to be interpreted as part of the "base-line" in the study of their conceptions.

Method

Participants

Participants were 57 grade 5 to 9 mathematics teachers from nine school districts in the geographical area of New England (US). When data were collected, teachers were participating in the first of three graduate-level online courses focused on algebra and the mathematics of functions as part of a larger professional development and research project. Regarding the independent variable for this study, "Educational Background," we established four modalities:

a. "*Mathematics,*" when the teachers earned their major or master's degree in mathematics (12 teachers);

b. *"Mathematics Education,"* when their major or master's degree was in mathematics education (8 teachers);

c. *"Science-Related Disciplines,"* when their major or master's degree was in disciplines such as Physics, Engineering, or Chemistry (9 teachers);

d. *"Others,"* such as History, English, and Literature (28 teachers).

General Curriculum and Research Tasks

The courses that teachers participated in were presented through an online system. Course 1, on which this study focuses, was divided into 14 weeks, each of which was divided into two parts. Four of these 14 weeks were educational in nature. The *"Candy Box Problem"* was presented to teachers as part of the second week. In Part 1 of the four educational weeks, once the reading/viewing of certain materials was completed, teachers were asked to answer a set of open-ended questions aimed at promoting their exploration of relevant mathematical topics. Teachers received feedback on their responses and were given the opportunity to review and discuss the answers from other teachers. In Part 2 of the four educational weeks, new materials were provided (generally videotapes of classroom episodes, samples of students' work, readings, and notes containing the views of specialists, research articles, etc.). Based on these materials, a new set of open-ended questions was asked. In order to analyze the development of teachers' mathematical thinking, certain questions formulated in Part 1 were oftentimes re-formulated in Part 2.

The questions analyzed in this study belong to Part 1 of the second week in the course, in which the teachers were asked about the *"Candy Box Problem"* (see Introduction). In particular, we focus on two questions:

Question #1: *"Please show what you know about the number of candies John and Mary each have. Feel free to use any helpful means to represent the problem."*

Question #3: *"What drawings, words, and numbers would you expect students to use to show the number of candies John and Mary have? Indicate the grade level for students you were thinking about."*

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Data Analysis

The data coding process began by qualitatively examining the answers collected from all the participants. During our initial exploration of teachers' answers, we realized that the categories suggested by Williams (1993) and Brenner et al. (1997) were not sufficient for us to capture the complexity of certain responses. Therefore, we decided to create some additional categories. A total of 9 categories were defined to analyze the answers to both Question #1 and #3: *"In the answer, the teacher presents and/or makes reference to the use of...*

1. *... natural language"* (i.e., written words).
2. *... table/s."*
3. *... ordered pairs of numbers."*
4. *... pictorial representation/s."*
5. *... graph/s."*
6. *... arrow diagram/s."*
7. *... algebraic expression (no form of function)"* (e.g., "x" or "x + 3")
8. *... algebraic expression in the form of a function"* (e.g., f(x) = x + 3).
9. *... algebraic expression in the form of an equation/s."*

Thus, each individual answer was coded nine times as "Yes" or "No," depending on whether or not it presented, mentioned, and/or referred to each of these nine representations for the *"Candy Box Problem."* As will be seen below, the analyses we conducted were descriptive and non-parametric in nature. In order to provide a graphic representation of the results for the variable "Educational Background," two Simple Correspondence Factorial Analyses (SCFA) were conducted, one for Question #1 and another for Question #3. SCFA is a multivariate analysis technique that allows the exploration of the relations of association and opposition between two variables, by projecting their categories on a factorial plane (for a complete overview, see Lebart, Morineau & Warwick, 1984). The interpretation of the factorial plane is customarily based on those categories whose contribution to one or both factorial axes is higher than the average value (i.e., 100/number of categories). Those categories have been underlined and capitalized in the two factorial planes below (see Figure 2 and Figure 4). Both figures show a two-dimensional representation of a multi-dimensional space. The intersection of the axis shows the location of the *centroid*, that is, the central point of gravity among the categories. The closer to

this point the location of a category is, the more common is its presence among all the participants. In both figures, the sizes of the elements projected on the plane (squares, triangles, etc.) provide an analogical representation of their contribution to the factorial axis. Statistical software SPAD.N (version 5.5; manufacturer: Décisia) was used to conduct the two SCFA presented below.

Results

Question #1: Helpful "Means" to represent the Problem (for oneself)

Our first analysis of Question #1 exclusively focused on the *number* of representational means presented and/or referred to in teachers' responses. The results show that, on average, teachers made reference to 3.24 representations for the "*Candy Box Problem*" (standard deviation was 1.62). The maximum number of representations indentified among the responses was 7 (four teachers), whereas the minimum was 1 representation (three teachers). In fact, as shown in Figure 1, most of the teachers provided algebraic expressions within their verbal description of the problem. In other words, practically all the teachers answered this question making use of —at least— two representational systems (most frequently Natural Language and Algebraic Expression [no form of function], as reported below).

Regarding the average number of representational means across different categories of the variable "Educational Background," the results show that teachers from *Mathematics* presented and/or referred to the highest number of representations for functions (4.33 representations). In contrast, teachers from *Mathematics Education* and *Others* presented and/or referred to the least numbers of representations (2.87 and 2.67 representations, respectively). Teachers with backgrounds in *Science-Related Disciplines* presented and/or referred to an intermediate number of representations (3.88 representations).

In terms of the *type* of representations used by these middle school mathematics teachers, Figure 1 illustrates that the most common means to represent the "*Candy Box Problem*" was Natural Language. That is, teachers used "words" to describe the relation between the number of candies that each John and Mary have (e.g., *"I know that they each have a box of candy which has the same number of candies. I also know that Mary has three extra pieces of candy. It could be written Mary's candy = John's candy plus three or John's candy = Mary's candy minus*

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three"). In fact, 51 out of the 57 answers were coded "Yes" in the Natural Language category. However, in most of those cases (specifically 48 teachers), other types of representations were used and/or referred to within the answers, thus coexisting with the verbal description of the problem. The category Algebraic Expression (no form of function) was the most frequent in this respect (e.g., "*Since I do not know the number of candies in each box, I would say that each box has X candies in it. So X equals the number of candies in each box and Mary has x + 3 since she has three more candies*"). The categories Table/s and Algebraic Expression in the Form of an Equation/s were identified in the answers provided by 22 teachers, followed by the categories Algebraic Expression (in the form of a function) (15 teachers) and Graph/s (11 teachers). Less than 10 teachers used and/or referred to the categories Ordered Pairs of Numbers, Pictorial Representations, and Arrow Diagram (see Figure 1). Due to the low frequency of the category Arrow Diagram (3 cases out of 57), we decided to eliminate it from subsequent statistical analyses.

Figure 1. *Frequencies of the Nine Representations for Functions Identified in Question #1 (N = 57)*
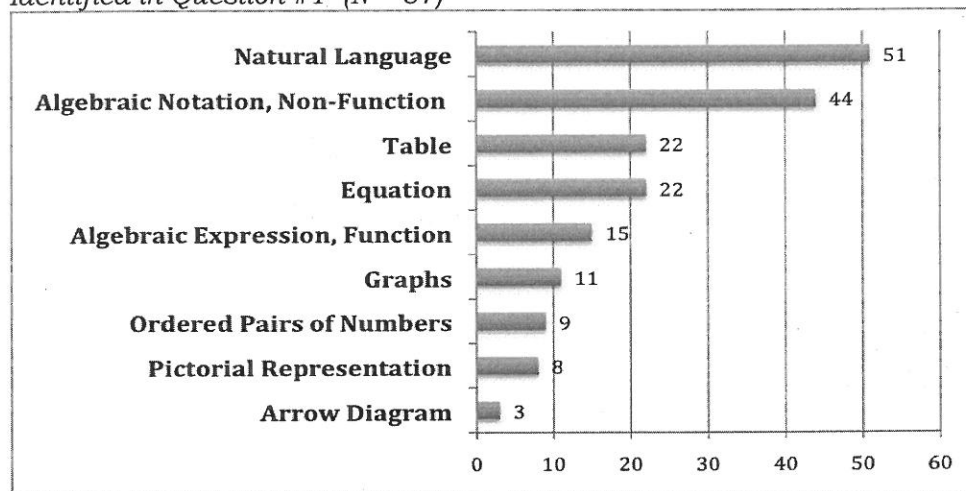
A Simple Correspondence Factorial Analysis (SCFA) was conducted to explore the relations of association/opposition among the nine categories of representations for functions (i.e., Natural Language, Table/s, Graphs, etc.) and the four modalities of the variable "Educational Background" (i.e., *Mathematics, Mathematics Education, Science-Related Disciplines,* and *Others*). The resulting factorial plane is presented in Figure 2. Results

show that the modality *Mathematics* and three of the categories (Graph/s, Algebraic Expression [no form of function], and Algebraic Expression in the Form of an Equation) did not contribute to the constitution of the factorial axes. Notice that these categories and modality are not underlined/capitalized, and also that they are located relatively close to the *centroid* of the plane. These findings seem to indicate that the way in which the teachers belonging to the *Mathematics* group answered Question #1 was not clearly "distinctive" from other groups. Besides, the results show that teachers from all groups made use of the categories Graph/s, Algebraic Expression (no form of function), and Algebraic Expression in the Form of an Equation in similar proportions.

Figure 2. *Factorial Plane resulting from the SCFA for the Variable "Educational Background" in Question #1*
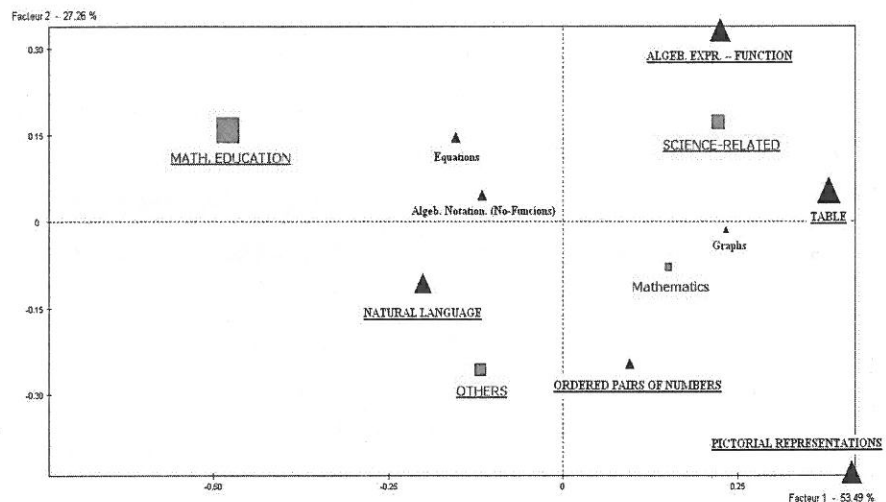
On the right-hand side of the plane, it can be observed that the modality *Science-Related Disciplines* is closely associated with the categories Algebraic Expression (in the form of a function) and Table/s, as well as with Pictorial Representations to a lesser extent. Even though most of the teachers described the *"Candy Box Problem"* by means of Natural Language, as reported above, it seems that those teachers belonging to *Science-Related Disciplines* expressed the problem in words less frequently. Responses of teachers from the modality *Others* were associated with the categories Ordered Pairs of Numbers and Natural Language, and to a lesser extent with Pictorial Representations, like teachers from the modality *Science-Related*

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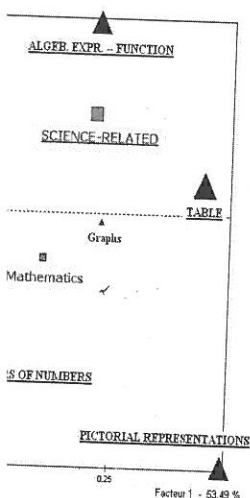
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Disciplines. Responses of teachers belonging to the group *Mathematics Education* were associated with the category Natural Language only.

*Question #3: Representations that Students are expected to use*

Let's focus first on the *number* of representational means that teachers expect their students to use. On average, teachers presented and/or referred to 3.05 representations for the "*Candy Box Problem*," a number slightly lower than in Question #1 (in which the average was 3.21 representations). Among responses to Question #3, the maximum number of representations indentified was 8 (one teacher only), whereas the minimum was 1 representation (seven teachers). The standard deviation was 1.48.

With respect to the average number of representations in relation to the variable "Educational Background," the results indicate that teachers from *Science-Related Disciplines* and *Mathematics* presented and/or referred to the highest number of representations for functions (3.66 and 3.25 representations, respectively). In contrast, teachers from the categories *Mathematics Education* and *Others* presented and/or referred to the lowest number of representations for functions (2.75 and 2.65 representations, respectively). Interestingly, among teachers from all four modalities, frequencies in Question #3 resulted lower than in Question #1.

As illustrated in Figure 3, the types of representations most commonly referred to among these middle school mathematics teachers were Table/s, Pictorial Representations, and Algebraic Expression (no form of function) (37, 31, and 31 teachers, respectively). It is interesting to notice that, unlike in Question #1, the category Natural Language was not the most utilized form of representation. In this case, only 20 teachers made reference to the importance for students to represent the problem "in words." It might be the case that our participants assumed and/or interpreted that students would be given the "Candy Box" problem in written or verbal form. The categories Graph/s and Algebraic Expression in the Form of an Equation were identified in the answers provided by 17 teachers, followed by the categories Ordered Pairs of Numbers (11 teachers), Algebraic Expression (in the form of a function) (6 teachers), and Arrow Diagrams (5 teachers).

Figure 3. *Frequencies of the nine representations for functions identified in Question #3 (N = 57)*
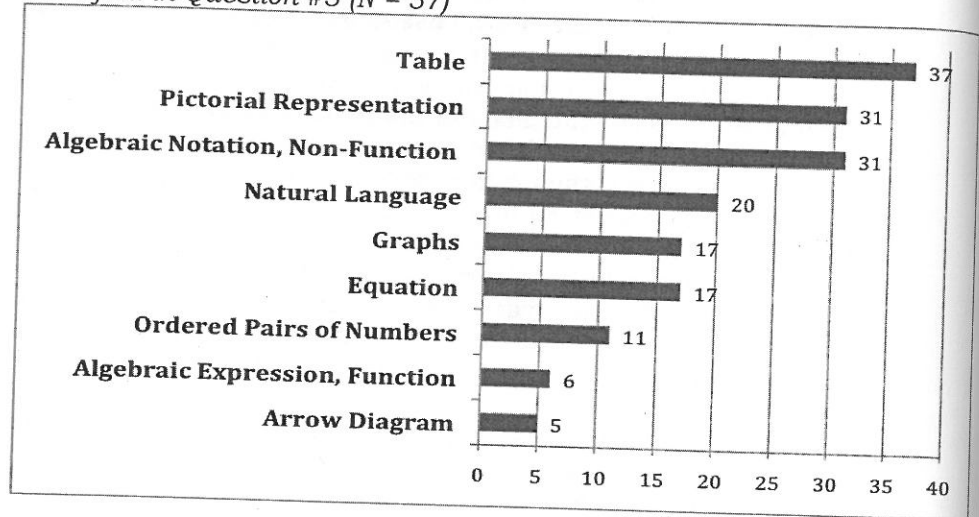
Question #3 explicitly asked the teachers to indicate what grade level they were thinking of when answering the question. Thus, it seems relevant to analyze the existence of possible differences across educational levels. Table 1 presents several descriptive statistics. The column "Average" is especially interesting to us. As can be observed, from Grade 5 to Grade 8 there is a progressive and consistent increase in the number of representations for functions expected for students to produce, ranging from 2.37 to 3.78 representations. However, this tendency stops in Grade 9, when the average drops to 3.00 representations. According to our data, teachers thinking of Grade 9 students mostly focused on Algebraic Expression in the Form of an Equation, Graphs, and Pictorial Representations. Several possible explanations for this finding will be presented in the Discussion section.

Table 1. *Descriptive Statistics for the Variable "Students' Grade Level" in Question #3*

	Maximum	Minimum	Average	Standard Deviation	N
Grade 5	4	1	2.37	1.06	8
Grade 6	6	1	2.72	1.34	11
Grade 7	8	1	2.91	1.92	12
Grade 8	7	1	3.78	1.36	14
Grade 9	5	1	3.00	1.41	6
					Total = 51 (Missing Values = 6)

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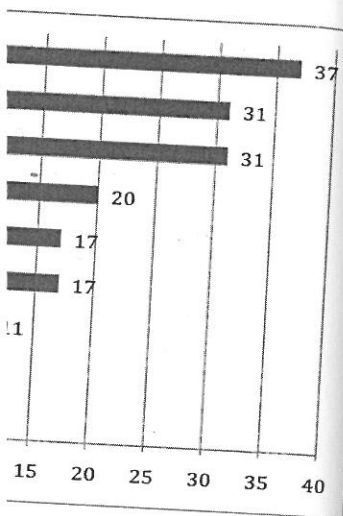
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Another SCFA was conducted to explore the relations among the four modalities of "Educational Background" and the nine representations for functions categories. The category Arrow Diagram was considered as *illustrative* due to its low frequency (5 out of 57 teachers). The resulting factorial plane is shown in Figure 4. The results indicate that the modality *Science-Related Disciplines* and the categories Natural Language and Algebraic Expression (no form of function) did not contribute to the constitution of the factorial axes (for that reason, they have not been underlined/capitalized on the plane). As can be observed, these categories and modality are located very close to the factorial plane's *centroid*. This suggests that the way in which teachers belonging to the group *Science-Related Disciplines* answered Question #3 was not distinctive regarding other groups. Further, these findings indicate that teachers from all four modalities referred to the categories Natural Language and Algebraic Expression (no form of function) in similar proportions.

Figure 4. *Factorial Plane resulting from the SCFA for the Variable "Educational Background" in Question #3*
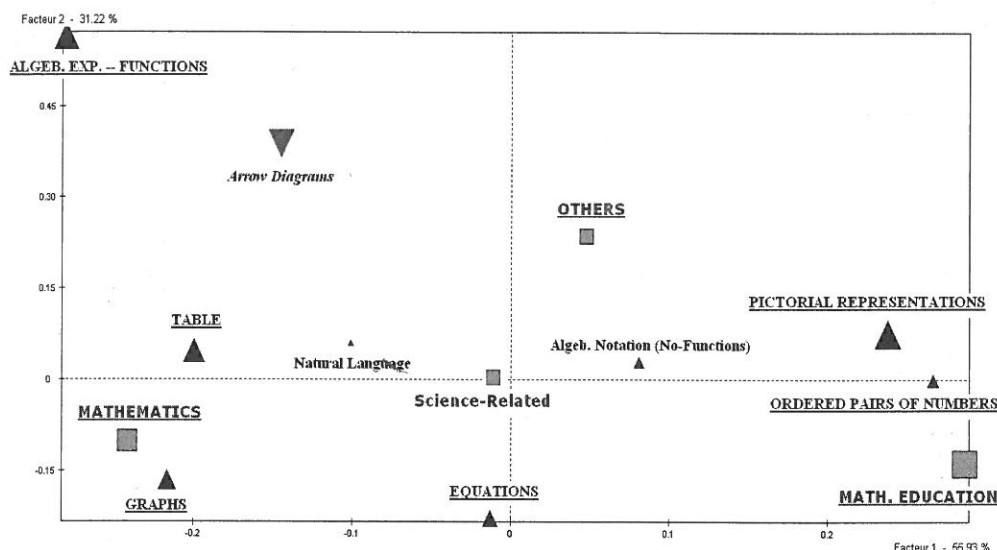
The modality *Mathematics* (left hand-side of the plane) is closely associated with the categories Graph/s, Table, Algebraic Expression in the Form of an Equation, and to a lesser extent with Algebraic Expression (in the form of a function). At the opposite side of the plane (right hand-side), we find the modality

Mathematics Education, which is associated with the categories Ordered Pairs of Numbers and Pictorial Representation. Finally, responses from the teachers in the modality *Others* were associated with the category Pictorial Representations.

Discussion

Little is known about how mathematics teachers conceive of the role and function of using multiple representations in the teaching and learning of mathematics, more specifically in the context of functional relationships. In this study, we have analyzed two questions posed at the beginning of an online course focused on the mathematics of functions. In a nutshell, Question #1 was designed to evaluate the representations teachers present and/or refer when thinking about the "*Candy Box Problem*" for themselves, whereas Question #3 focuses on the representations they expect from students.

First, the results indicate that the average number of representations for functional relations provided in both questions is considerably low (around 3 representations, in a scale from 0 to 9). These findings, especially those for Question #3, seem to indicate that teachers are not fully aware of the potential of representations to foster students' mathematical learning (Cuoco, 2001). Certainly, if teachers had answered according to the proposals of both curriculum designers and researchers (e.g., Brizuela & Earnest, 2008; Goldin, 1998; NCTM, 2000), they should have mentioned and/or presented a much higher number of representations, as well as more inter-relationships among them.

Second, our findings indicate that the number of representations identified in Question #1 is slightly higher than those identified in Question #3. In order to test these tendencies more rigorously, we are aware of the fact that we should apply parametric statistical analysis (something not possible in our case due to different factors, such as the small sizes of our sub-samples, their differences in N, etc.). However, assuming that these differences could actually be statistically significant, we should ask: Why do teachers expect *fewer representations* from their students than those they consider useful for themselves? As suggested in Stylianou's (2010) study, it might be the case that these mathematics teachers conceive of representations as useful tools for them as specialists in the field, but not really as learning tools for students.

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Third, the types of representations teachers used to show what they knew about the "*Candy Box Problem*" are considerably different than those they expected their students to use. In the former case, they state that the most useful representational means are Natural Language, Algebraic Expression (no form of function), Tables, and Algebraic Expression in the Form of an Equation. In contrast, the representations most expected from students are Tables and Pictorial Representations. Notice that tabular representations are used in both scenarios. Besides, it is interesting to notice that Natural Language was not among the most commonly mentioned categories in Question #3. As a hypothesis, we speculate that teachers might have assumed and/or interpreted that students would be presented the problem in written or verbal form and would therefore be less likely to respond to the problem using the same representational medium.

Fourth, in relation to Question #3, it is very interesting to notice the growth in the average number of representations referred to from Grade 5 to Grade 8, and then the decrease in Grade 9. This finding might be reflecting an inaccurate estimation of students' educational needs for learning mathematics. The fact that many Grade 9 students are generally able to master Graphs, Algebraic Expressions in both functional and non-functional forms, and Algebraic Expressions in the Form of an Equation does not mean that other representational systems should be radically eliminated from the classroom (e.g., Tables, Arrow Diagrams, Pictorial Representations). As mentioned in the Introduction, the use of multiple representations allows students to grasp different and complementary dimensions of the same mathematical ideas (Cuoco, 2001). Thus, we think that mastering certain representational systems should not necessarily result in the suppression of others.

Finally, our study suggests that the educational background received by the teachers might influence their implicit conceptions about representations for functions. Considering the results from both SCFA's together, it seems that teachers with degrees in *Mathematics* and *Science-Related Disciplines* find it more useful/desirable the use of Algebraic Expressions (Functions), Equations, Tables, Graphs, and to a lesser extent Pictorial Representations. In contrast, teachers from the groups *Mathematics Education* and *Others* seem to focus on fewer representations, most commonly Natural Language, Pictorial Representations, and Ordered Pairs of Numbers. In our

viewpoint, teacher preparation programs should stress the importance all types of representations, according to the recommendations of both researchers and curriculum designers.

The present study is limited for numerous reasons: a) because it does not inform us about what these middle school mathematics teachers actually do in the classroom, and about the way in which they teach and use multiple representations for functions; b) because our analyses are exclusively based on the data gathered from two specific questions (among many others); c) because all the statistical analyses carried out are descriptive in nature, and consequently do not allow us to describe more generalizable conclusions; d) because this study does not allow us to know if the conceptions of these teachers are changing as a result of their participation in the online courses; e) because we have not yet explored the relationship between the teachers' conceptions and their classroom students' learning. As mentioned above, the results presented in this article constitute part of the "base-line" in the study of these teachers' conceptions. Further studies should be conducted to analyze the impact of our project on these middle school teachers' professional development.

We would like to close with a reference to the educational implications of this study. Our educational/research project rests on the premise that to improve students' mathematical learning, teachers need to broaden and deepen their understanding of mathematics and mathematical knowledge for teaching. This is precisely the goal of our project, which is serving a relatively small sample of in-service teachers. In order to successfully meet the educational demands of current mathematics curricula, we think that both in-service and (especially) pre-service middle school mathematics teachers should be provided with further educational opportunities, designed to develop more sophisticated knowledge to make algebra more accessible to students.

Acknowledgements

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