How Teachers Interpret Equations and Graphs in terms of Situations [1]

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Objectives and Theoretical Framework

It is generally acknowledged that algebraic word problems require that one think about the relations between mathematical representations and extra-mathematical quantities both when one sets up an initial mathematical representation of the problem and when one attempts to interpret a notational or graphical solution in terms of the specific question(s) one had set out to answer. Less often noted, and less often exploited in instruction, is the fact that intermediate representations in algebraic-arithmetic notation—notably, intermediate equations--can be interpreted and give rise to additional, intermediate graphs that can likewise be interpreted in terms of the problem context.

We examine how middle and high school teachers of mathematics enrolled in a professional development program produce and interpret equations and graphical representations when they are asked to do so at various moments in solving an algebraic word problem. The problem was devised to be well-suited to the notion of an equation as a comparison of two functions—an idea heavily emphasized in the program's courses.

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We will discuss with the audience how to help teachers and students consider how quantities relate to equations and graphs.

Viewing the function as the fundamental object of mathematics offers a novel interpretation of an equation as a statement about a relationship between two functions, one on each side of the equals sign (Schwartz, 1996). Specifically, the unsolved equation can be interpreted as posing, and holding the answer to, the question “For what values of some variable(s), do the two functions have the same value?” This supports the idea that solving an equation entails repeatedly applying equivalent transformations to the right and left sides of an equation until one reaches, if possible, one or more statements of the form, x = <some value>. Because functions can be displayed as graphs, any equation can be represented through two graphs, the solution(s) of which correspond to the x-value(s) of points of intersection. As Schwartz highlights, graphical representations allow for visualizing how transformations, beginning with an initial equation, generate successive equations, each with the same solution set. Additionally, it becomes relatively easy to spot illegal transformations (they alter the x-value of the intersection points).

Some have suggested that treating equations as entailing comparisons between two functions help promote the learning of algebra (Carraher and Schliemann, 2016; Chazan, 2000; Kaput, 1988; Schwartz, 1996), but further systematic studies are needed in this area. This study contributes to this effort by examining how teachers (a) make sense of the practice of comparing functions in solving equations (b) relate their operations on equations to the transformations of the graphs of the functions being compared; and (c) interpret their algebraic and graphical representations in terms of relations between quantities in the word problem under consideration.

Method

Participants were 58 fifth through ninth grade teachers in a teacher development program in which they took three graduate-level courses, offered mostly online. The courses were designed to encourage teachers to view major topics in the middle school curriculum through the lens of algebra and functions. Functions tacitly permeate K-12 mathematics (the operations of arithmetic are themselves defined as functions). The fact that they are given expression in a multitude of forms (verbal structures, arithmetic and algebraic notation, tables, Cartesian graphs, diagrams on number lines, etc.) makes them well suited for integrating topics and techniques that might be otherwise taught and learned in isolation. And they facilitate, through the introduction of variables, the process of making generalizations.

The teachers read notes, worked on apps, watched videos of lectures and classroom lessons, and worked in small online and face-to-face groups, solving problems and planning, implementing, analyzing, and discussing student learning in interviews and classroom teaching activities related to the topics in the courses. In the third unit of the second course teachers answered the following question:

> *A delivery truck from the Leakit ice-cream factory leaves Boston at midnight driving towards Maine at a constant speed of 40 miles per hour. At the same time, a manager from Leakit leaves from the headquarters in Providence, 50 miles south of Boston, at a constant speed of 60 miles per hour, driving also towards Maine via Boston. Write an equation whose solution gives you the number of hours from midnight till the manager catches up with the truck.*

Teachers were to explain how they solved the equation, identify the functions being compared at each step, draw graphs of the functions, and interpret each step of the solution in terms of the quantitative referents in the verbal description. In the notes for the week, they had read about how an equation can be considered as a comparison between two functions and in a video lecture they saw an example of how to interpret each step of the solution in terms of pairs of graphs and of the problem situation.

Results

In this report, we will focus on whether the teachers were able to identify the two functions to be compared in the original and intermediate equations and on how they interpreted each solution step in terms of the given problem context and quantities. In the presentation we will discuss examples of teachers' work with the audience.

The following represents one *valid sequence of* equations that a teacher produced for solving the problem. The functions next to each solution step are the ones being compared.

$$60x-50=40x \quad (y=60x-50 \text{ and } y=40x) \qquad (1)$$

$$60x=40x+50 \quad (y=60x \text{ and } y=40x+50) \qquad (2)$$

$$20x=50 \quad (y=20x \text{ and } y=50) \qquad (3)$$

$$x=2.5 \quad (y=x \text{ and } y=2.5) \qquad (4)$$

Figure 1 shows one teacher's correct graphs corresponding to each step of the equation solution and how the solution was preserved across transformations.

Most teachers correctly set up the equation, solved it and identified the two functions. Solving the equation and identifying the two functions (y=60x-50 and y=40x) in the original equation, the two functions (y=60x and y=40x +50) in the second equation, and the two functions (y=20x and y=50) in the third equation was easy for most of the them: of the 58 teachers, 54, 51, and 51 teachers, respectively, correctly identified the functions and drew their graphs. However, only 31 of them identified the functions y=x and y=2.5 in the last equation. Among the other 27 teachers, three did not answer the problem, eight did not draw the final graphs even though they had solved the equation, and 16 drew incorrect or unclear graphs. Figure 2 shows examples of incorrect graphs.

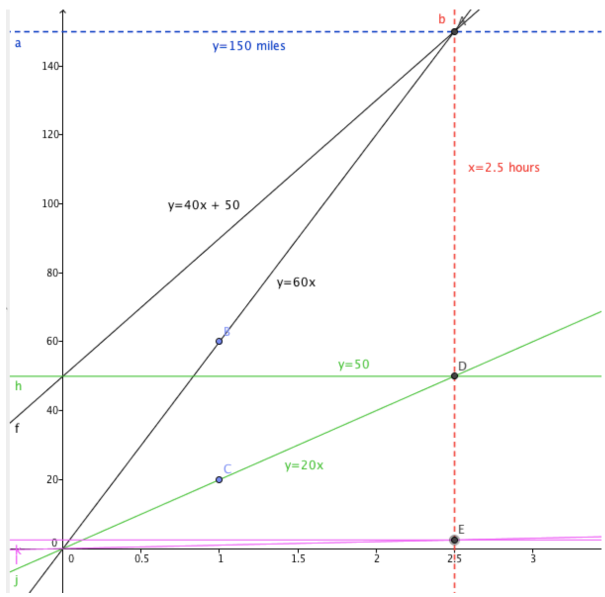


Figure 1: Graphical representation of the transformations in solving the problem

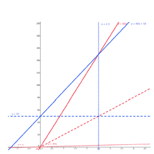
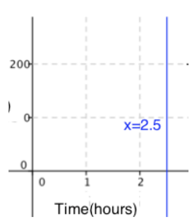
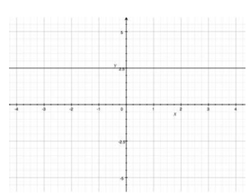
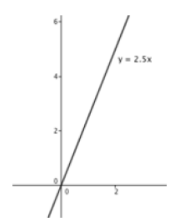
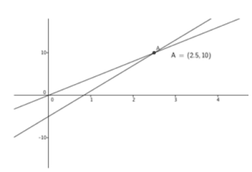
y=x; x=2.5	x=2.5	y=2.5	y=25x	y=4x; y=6x-5
				

Figure 2: Incorrect graphs of the two functions in the last step of the solution

When asked to *relate each step of the equation and graphic solution to the problem context*, only a few teachers did so.

For adding 50 to each side of the equation to get 60x=40x+50, only 14 teachers showed correct descriptions, as in the following examples:

> "The first transformation would add 50 to each side, thus it is measuring the vehicle's distances from Providence rather than Boston."
>
> "I used 40t + 50 = 60t by thinking in terms of how much further ahead the truck was by leaving from Boston, but I could have used 40t = 60t – 50 thinking in terms of how far behind the manager was by leaving from Boston."

For subtracting 40x from each side of the equation to get 20x=50, only four teachers correctly wrote, for instance:

> "Essentially, the first transformation takes away the distance that they both covered at the same rate: 40x. What we did was to divide up the distance that the manager drove at 60 mph into two sections: one that he drove at 40 mph (which we then "matched up" or cancelled with the truck's side), and then determine how long he would have to drive at 20 mph in order to cover the 50 extra miles that he would have to cover from Providence to Boston."

For dividing each side of the equation by 20 to get x=2.5, only three teachers gave correct answers, as in the following example:

> "… it is as if the truck started 2.5 miles away from the manager. If this was the case and the manager drove at 1 mph, it would still take 2.5 hours to get to the truck."

Discussion

Our results suggest that (a) middle and high school mathematics teachers can learn and understand a functional approach for solving equations but have difficulties in identifying the functions in the final equation (e.g. x=2.5) and (b) it is challenging for most teachers to interpret each step of the solution in terms of the quantities and problem context.

One source of difficulty in identifying the two functions in the final equation, x=2.5, may be their unfamiliarity with comparing the identity function to a constant function. Additionally, they may have confused x=2.5 with y=2.5. These difficulties, as significant and prevalent as they are, may well be easily addressed. The inability to interpret each step of the solution in terms of the situation, however, will likely demand much more effort given that: (a) reasoning about quantitative relationships in situations does not typically follow any standard pattern or routine like that of variable assignment and equation solving steps in traditional algebraic problem solving; (b) reasoning about quantities is grounded in how we conceive of situations by drawing upon experiences in everyday life or science classes; and (c) different people have different experiences with a wide range of situations. Thus, it may take a lot of practice to extract mathematics from the relationships among quantities in different situations and then interpret the mathematics models and operations back to the relationships among quantities.

The field of mathematics education will benefit from further studies aimed at clarifying how to build mathematical understanding, concepts, and representations from everyday or science experiences and how to employ mathematics to get insights into everyday or scientific situations.

References

Carraher, D.W. & Schliemann, A.D. (2016). Powerful Ideas in Elementary Mathematics Education. In L. English & D. Kirshner (Ed.). *Handbook of International Research in Mathematics Education* pp. 191-218. New York: Taylor & Francis.

Chazan, D. (2000). *Beyond Formulas in Mathematics and Teaching: Dynamics of the High School Algebra*. New York, Teachers College Press.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by 'algebrafying' the K-12 Curriculum. In *The Nature and Role of Algebra in the K-14 Curriculum* (pp. 25-26). Washington, DC: NAP.

Schwartz, J. L. (1996). Semantic aspects of quantity. Unpublished manuscript, Massachusetts Institute of Technology.