Teachers' Interpretations of a Linear Equation Regarding Physical Quantities[1]

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Over the years, quantitative reasoning—in particular, reasoning about relations among physical quantities--has increasingly been viewed as having a major role to play in elementary and middle school mathematics learning and teaching (e.g. Freudenthal, 1986, Lehrer, 2003; Lesh & Doerr, 2004; Lesh & Lehrer, 2003; Schwartz, 1988, 1996; Smith & Thompson, 2007; Thompson et al. 2016).

If an equation such as y = 3x + 5 is to model a relation between two kinds of physical quantities, say, weight and volume, certain assumptions and constraints come into play that do not arise when the relation entails two sets of pure numbers. Importantly, dimensional homogeneity must be respected: only quantities of the same kind can be compared, added, or subtracted (Bridgman, 1922). So if y is taken to represent weight, then the sum, $3x + 5$, as well as each of the addends, must represent weights.

Interpreting the $3x$ in y = $3x$ +5—or, more generally, interpreting the mx in the model, y = mx+b—demands close attention. For the case at hand, because x and y are quantities of different kinds, then the 3 is not a dimensionless factor. Consequently, $3x$ does not correspond to $x + x + x$. (Otherwise, $3x$ would be the same kind of quantity as x, thereby violating the property of dimensional homogeneity.) However, $3x$ may be interpreted as a "referent-transforming" operation (Schwartz, 1996). For instance, the volume (x) of an object multiplied by its density (3) yields a weight (3x). Because, in pre-secondary mathematics, the units and quantities are tacit, students will have to establish the contextual meaning of such symbols and operations largely through dialogue with teachers.

We wondered how mathematics teachers would interpret a pure linear equation of the form y = mx + b as a model representing a relation between different kinds of quantities. What quantities would they associate with the y-intercept, the slope, and the independent and dependent variables? Would they offer a coherent account of how the quantities are interrelated or composed, keeping in mind the issue of dimensional homogeneity and the subtlety of $3x$ in a modeling context?

The task was also being used to assess, along with a series of other items, progress made by in-service teachers during their 18-month participation in a mathematics education program. The teacher development program offers in-service teachers

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three graduate level courses focusing on algebra and the mathematics of functions and emphasizes the importance of probing and understanding students' thinking and of how mathematical understanding can emerge from modeling science and worldly phenomena.

Method

In this study, we investigated how in-service mathematics teachers interpreted a simple linear equation in terms of a relation between two variable quantities. The task was worded as follows:

> *The formula y = 3x + 5 represents a relationship between weight and volume.*
>
> *(a) Make up a brief story that could describe a relationship represented by the formula.*
>
> *(b) In the context of your story, what does the 3 in the formula represent?*
>
> *(c) In the context of your story, what does the 5 in the formula represent?*

The goal was to examine how teachers relate the symbols in the equation to the quantities in their stories. We also wanted to assess whether teachers were able to provide a coherent story modeled by the equation $y = 3x + 5$. Although some teachers provided stories involving a context other than the relationship between weight and volume, as long as the operations on quantities made sense, we considered a story to be coherent.

Coherence was determined in two steps. First, two judges rated the kinds of entity each teacher associated with the individual symbols, y, x, 3, and 5. Each entity, y, 3, x, and 5, was first judged as (1) ratio or per quantity, (2) weight or mass (3) volume (4) another kind of extensive quantity, or (5) a dimensionless number. Additional codes were reserved for (6) unclassifiable responses (7) "I don't know" responses, and (8) blank or missing data. When a code of (1) or (4) was applicable, judges made note of any units or dimensions associated with the data.

Ratios could be expressed in various ways. Sometimes the ratio was expressed as a quotient of two different units of measure, as in "The weight of a pail of water."

Judges then evaluated whether the response exhibited dimensional homogeneity. C*oherent* responses were those judged that clearly and consistently identify dimensions associated with y, 3, x, and 5, and for which the dimensions of y, 3x, and 5 agreed. Otherwise, a response was coded as *incoherent*. The overall inter-judge reliability was 89.8%. Disagreements were resolved through discussions with a third judge[2].

[2] Sara Bray served as the third judge.

Results

In this section, we qualitatively explore some examples of teachers' responses (see Table 1) and present quantitative results.

Example 1 shows a response rated as coherent whereby y represents total weight of sand and the bucket, x represents the volume of the sand, 3 represents the ratio of weight to volume of sand, and 5 represents the initial weight of the bucket.

Example 2 is a story about the relationship between cost of entering a park and the number of persons and was also coded as coherent.

Table 1: Sample responses[3] of teachers

Ex.	*brief story*	3 represents…	5 represents…
1	"You are making a sand castle at the beach. Your buckets weights 5 pounds and each time you add a shovel of sand to the bucket, the weight increases by 3 pounds." (T450)	"The 3 is the unit rate in weight/volume (pounds/shovel). Shovel is not a typical volume measure however it could be viewed as a measure of volume as the volume of the shovel itself does not change."	"The 5 is the initial weight of the bucket in pounds."
2	"I'm not sure how to use weight and volume for this formula. I could use other things - for example the cost of entering a park is 5 dollars plus three dollars per person." (T250)	"cost per person"	"flat entry fee"
3	"y=density, 3x=weight 5=volume. An object weighs 3 times the weight of a penny. The object has a volume of 4ml. [5ml.] What is the density of the object?" (T323)	"3 times a penny"	"5 represents the volume of the object"
4	"The volume of an object is equal to five more than three times the weight of object." (T588)	"3 times the weight"	"Add 5."
5	"The weight of the bottle increased for every 3 ml added to the bottle with the dropper. The starting weight of the bottle was 5." (T245)	"In my story the 3 represents every pour of liquid into the bottle. The variable"	"The 5 in the formula represents the starting weight of the bottle the constant."
6	"You're filling a container up with gas at the gas station. Three gallons of gas weights 1 pound. The container is 5 pounds when it's empty." (T321)	"3 gallos = 1 pound"	"Weight of the empty container"
7	"A bucket weighs 5 pounds and for every additional pail of water you add to the bucket, the weight go up by 3 pounds." (T234)	"The weight of a pail of water."	"The weight of an empty bucket."

3 Responses were written; quotation marks convey that entries are literal transcriptions.

Not all the teachers explicitly mentioned how the coefficient 3 transformed the quantity volume x into weight, 3x that, summed with an additional weight, 5, was equal to the weight, y. But in all the coherent stories, the teachers implied that 3 represents a ratio of two types of quantities or an intensive quantity such that 3x was the same kind of quantity as 5 and y.

Examples 1, 2, and 7 in Table 1 were judged coherent; examples 3-6 were judged incoherent.

Example 3 fails to meet coherence in several ways. It does not make sense to add a weight to a volume, nor to claim that their sum represents a measure of density. This is similar to what we found when we worked with the teachers in the courses. They tended to solve an equation, even though that was neither called for nor possible. It seems like that they were not used to representing a general relationship between two quantities with a function.

In example 4, the teacher treats y as a volume. However, the teacher appears to have treated the 3 as a pure number that did not transform a weight into a volume; 3 was not conceived as an intensive quantity. This occurred often in teachers' responses. In Example 3, the teacher also said: "An object weighs 3 times the weight of a penny. (The teacher also wrote "3 times a penny", without referring to the property of weight.) So, 3 was treated as a pure factor to make "3x" 3 times as large as x.

In Example 5, the teacher did not clearly describe what 3 represents, saying "The weight of the bottle increased for every 3 ml". Note that the 3 is taken to express the magnitude of the volume. In addition, it is unclear how much the weight will increase for every 3 ml of volume. The teacher also said, "the 3 represents every pour of liquid into the bottle". It is not clear what property of every pour of liquid the 3 is taken to represent. For many teachers, it was particularly challenging to provide an account using an intensive quantity or a ratio of two quantities.

The equation, "3 gallons = 1 pound", in Example 6, violates dimensional homogeneity and does not match the assertion that "3 gallons weights [sic] one pound". Additionally, it is unclear from the story what x stands for. Even if one assumes it was tacitly taken to stand for volume (in gallons), additional work would be required to derive an intensive quantity that, multiplied by x, would yield a 3x in pounds.

In Examples 3-6, the teachers do not articulate the role of the coefficient, 3, in transforming one type of quantity into another.

It is noteworthy that the value 3 could be expressed as a ratio without employing a quotient. For example, in Example 7 (judged as coherent), the teacher described the "3" as "The weight of a pail of water," that is, as the weight of the contents of one pail of water. In this case, the referent-transformation is implied rather than explicit, as in "3 pounds per pail".

Interestingly, several teachers who provided coherent stories seem to have construed the meaning of 3x in a way that downplays or gets around referent transforming. When y is taken to be a weight, for example, they appeared to treat the 3 as the weight of some reference object and x as a count or tally of the number of such objects. (see Example 7). This sort of conceptualization emphasizes the accumulation of an extensive quantity (weight) rather than the transformation of one kind of quantity into another. This may be more consistent with teachers' view of multiplication as repeated addition than with multiplication as referent transformation.

Table 2 shows, that at the beginning of the program, only 30% of the teachers provided a coherent story. At the end, 47% of the teachers gave a coherent story in their answers. The number of correct answers was significantly higher in the posttest as compared to the pretest (Wilcoxon W = -85, z = -2.18, p=0.0146, one-tailed test).

Table 2: Coherence of Stories before and after Participation in the Program

Test periods	Coherent	Not Coherent	I don't know	Blank	Total
Before	14 (30%)	22 (47%)	6 (13%)	5 (11%)	47
After	23 (47%)	8 (17%)	8 (17%)	8(17%)	47
Total	37	30	14	13	94

Discussion

At the beginning of the teacher development program, more than two thirds of the teachers did not provide coherent accounts of how one quantity is composed from two kinds of quantities. Many teachers provided stories that violate dimensional homogeneity, for example, by suggesting that a weight and a volume might be summed, yielding either a weight or a volume. There is evidence that they find elusive the notion that an operation may transform a measure of one type of quantity into another by means of a rate or intensive quantity.

It is possible that the present results reflect the teachers' lack of familiarity with the particular problem context (the relationship among weight, volume, and density). One might expect them to fare better with highly familiar contexts, such as distance, time, and speed. Future research will hopefully determine the extent to which the problem difficulty is related to familiarity with the kinds of quantities under consideration.

After the three courses, almost half of the teachers implicitly considered the role mathematical operations plays on preserving or transforming types of quantities. This improvement is statistically significant.

We believe that, despite the significant improvement of teachers over the course of the program, there is more that we need to do to help mathematics teachers and researchers understand the mathematics of quantities.

It is worthy of note that the teachers were not explicitly exposed to reverse modeling problems (from a model to an extra-mathematical context) throughout their coursework in the teacher development program. Although the time lapse between examinations exceeded 18 months, it is plausible that a second exposure to the same reverse modeling problem played a part in the improved responses.

The significance of the work lies in helping to clarify the issues teachers (and students) face in employing symbols to represent relationships among different kinds of physical quantities. We proposed and employed a methodology for the evaluating the coherence of stories teachers produced to be represented by a pure linear equation of the form $y = \mathrm{m}x + \mathrm{b}$. By examining the stories and identifying which quantities corresponded to which symbols, it was possible to assess the coherence of stories from the perspective of dimensional analysis (was dimensional homogeneity met?), internal consistency (were the symbols clearly and consistently associated with identifiable quantities?) and conformity to the model (did the story have the underlying structure of the linear model?).

Perhaps because units of measure and quantities are largely tacit in pre-secondary mathematical notation, dimensional analysis, which focuses on the formalization of quantities and units in notation, may be considered by some to be unrelated to early mathematics teaching and learning. But the present results suggest that, in order for students to make sense of the correspondence between equations and extra-mathematical contexts, they need to develop a clear sense of how quantities are composed and interrelated homogeneously (in the case of addition, subtraction, and comparison) and also in a referent-transforming manner (in the case of the multiplication and division of unlike quantities).

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