If $y = 3x$, is y greater than x? Teachers Evolving Understanding of Operations on Quantities[1]

Analúcia D. Schliemann, Chunhua Liu, David W. Carraher, & Montserrat Teixidor-i-Bigas

Paper presented at the NCTM 2017 Research Conference, San Antonio, TX, April 3-5.

Learning about mathematics entails the study of numbers. Does it really entail the study of quantities? Or should considerations about quantities belong exclusively to the realm of science education? If not, how are students to make sense of word problems, where the objects of interest are not just pure numbers? Where, if at all, is it important for teachers and students alike to interpret a symbolic expression differently, according to whether the components are taken to represent pure numbers or quantities? What sort of understanding regarding quantities would be helpful, or even essential, for K-8 mathematics teachers to learn and to include in their teaching? In this study, we analyze how teachers consider the relationships between the elements in the algebraic expression of a function as quantities, before and after online discussions in an online teacher development course. We also analyze how their answers evolved, over the course of one week, as they discussed answers with their group instructor and their peers.

What are quantities? What is quantitative reasoning?

There is a general consensus in the physical and medical sciences that a quantity refers to a "property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference (BIPM et al, 2012)," the reference typically being either a unit of measure or, in the case of discrete quantities, an enumerable kind of entity (atom, molecule, cell, planet, chromosome, etc.). This definition will may serve us in mathematics education contexts, provided we include quantities outside the physical sciences (e.g. price, number of children, average test score, number of solutions).

In English, the term *quantity* has two distinct meanings (Bièvre, 2009): it may refer to the property itself (length, weight, volume, time, density) as well as the value of the quantity associated with a particular object or collection of entities (3.1 m, 7 g, 14.3 in^3, 3.5 h 11 pencils). In some languages, these meanings are expressly distinguished (e.g. *grandeur* and *quantité*, in French). In English, one might employ *quantity value* for the latter case, but, by itself, the term *quantity* will often be ambiguous without additional qualification. Furthermore, although we will carefully distinguish between quantities and pure numbers (e.g. 44, Pi, -3.1, 7 + 4i), we cannot assume that this convention is adhered to by either teachers or students.

For the present purposes, quantitative reasoning refers to reasoning about relations among quantities (as described above), whether they are generic (grandeurs) or associated with particular values (quantités).

Schwartz (1996) contrasts the mathematics of quantities to the mathematics of number devoid of referents. Lobato and Siebert (2002) consider quantitative reasoning as reasoning with measurable properties of objects. And the publication, Common Core State Standards for Mathematics (CCSSM), contrasts quantitative reasoning with abstract reasoning, describing the

[1] This study is part of a National Science Foundation (NSF) Math Science Partnership project (grant #0962863), The Poincaré Institute for Mathematics Education (https://sites.tufts.edu/poincare/). Opinions, conclusions, and recommendations are those of the authors and do not necessarily reflect NSF's views.

former as imputing a context to a problem formulated in symbolic terms and the latter as a way of "decontexualizing" the circumstances of a particular context to a wider one by means of symbolic formulations. In its Mathematics Practices, the CCSSM (2010) further describe quantitative reasoning as follows:

> Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (p.6).

Despite subtle differences in descriptions and definitions by different authors, quantitative reasoning may be instantiated in terms of reasoning about relationships among quantities as opposed to reasoning about pure numbers. Moreover, as highlighted by Thompson et al. (2014), Lobato and Ellis (2002), and Ellis (2007), quantitative reasoning is fundamental for mathematics education and to grades K-12 teacher preparation.

Quantitative reasoning in mathematics education

There are two simple views regarding the relationship between quantitative reasoning and reasoning about pure numbers. The first view is that one learns about numbers, their operations, and their properties and then learns to apply that knowledge to extra-mathematical contexts involving quantities. An opposing view is that a child's mathematical knowledge about number is constructed upon their reasoning about quantities.

The idea that mathematical knowledge is constructed upon reasoning about quantities is not a new one and a wealth of previous research points to the role of contexts, situations, and quantitative reasoning in learning and development. Studies of how children develop logical reasoning and come to understand mathematical relations point to the importance of children's reflections upon actions on objects as the source of logico-mathematical understanding (Piaget, 19xx). Studies of everyday reasoning show that even unschooled individuals can learn about mathematical relations and properties as they participate in socio-cultural activities involving physical quantities and measurement (Carraher, Carraher, & Schliemann, 1985; Nunes, Schliemann, & Carraher, 1983; Lehrer, 2003; Lehrer et al., 1999; Saxe, 1991). Vergnaud (1994) proposes that understanding mathematical concepts involves considering invariants, symbols, and situations. Mathematics education researchers have emphasized the importance of worldly phenomena and situations and agree that mathematical understanding builds upon and emerges as a result of student thinking about relations among quantities (e.g., Freudenthal, 1983; Lehrer, 2003; Liu et al., 2017; Thompson et al. 2016) such as length, weight, time, area, volume, speed, unit-price, density, or number of objects. Whitney (1968a, b) has proposed that quantities should be explicitly integrated into the mathematical models used to solve applied problems. Ellis (2007) found that, teaching grounded in quantitative reasoning produces more and stronger generalizations than teaching focused on numerical or calculation patterns and enable the learner to draw inferences about relationships that may not be present in the situation itself. Previous and current standards and frameworks for mathematics education in the United States highlight the importance of students' work with physical quantities and the preparation of students to use mathematics to solve every day or science problems, starting from the elementary school years.

Teaching and the mathematics of quantities

It is commonly expected that mathematics teachers are to promote quantitative reasoning among their students, as fundamental steps towards understanding arithmetic as well as algebraic representations and syntactic rules. To this goal, teachers need to be aware of the mathematics of physical quantities. However, in a study of teachers' quantitative reasoning, Thompson et al. (2014) has found that quantitative reasoning may present challenges to teachers. Some of teachers' difficulties with quantities have also arisen in the context of our work in a teacher development program (Teixidor, Carraher, & Schliemann, 2013; Schliemann, Carraher, & Teixidor-i-Bigas, submitted). For example, as part of an online discussion held in the early stages of a teacher development program, teachers were asked whether one could represent the relationship between distance and time with an arrow diagram, on a single line (Liu et al, 2017, in preparation). We were surprised that, at the start of the discussions, only 12% of the 58 teachers in the program stated that the two variables could not be represented with arrows in a single number line and justified their answers by saying that these were two different quantities or that distance and time had different units. This suggests that they were not aware that units of distance and time were fundamentally different and hence could not be added and subtracted nor that their magnitudes could be compared.

Even though mathematics as a discipline has started as part of everyday activities and may emerge, especially among young students, from considering physical quantities and situations, traditional teaching of mathematics has often been dissociated from considerations about the possible correspondence between the representations and structures of mathematics to worldly phenomena. As a result, despite the core role of physical quantities in mathematical reasoning, even when they discuss the multiple representations of, for instance, a linear function (table, graph, algebraic expressions of the function), teachers and students may make multiple connections among these representational formats without ever considering how they may correspond to a given situation or event.

The absence of a focus on the type of quantities mathematical notation refers to prevents students from using their own intuitions and reasoning as a basis for developing mathematical understanding and for appropriating conventional mathematics representations and procedures. It also leads students to attempts to solve problems by focusing on a string of pure number computations and failure to interpret results in terms of the problem situation or question. As middle and high school students proceed to the study of algebra and functions, the dissociation between quantities and mathematics representations and conventions becomes even more problematic. Then, the preparation of teachers to promote awareness of how phenomena in the world, their geometrical and spatial representations, and the numerical and algebraic structures of mathematics relate to each other should become a priority.

Multiplication and division are a good place to examine the distinction between number and quantity. The relationship between operations on numbers and operations on quantities is particularly difficult in problems in the field of multiplicative structures, as opposed to those on additive structures (Vergnaud; 1988; Schwartz, 1988, 1996). For example, when one deals with quantities, there are two kinds of operations, referent-preserving and referent-transforming operations. Referent preserving operations combine two quantities with the same (or equivalent) referents and produce a new quantity with the same referent. This is the case for addition and subtraction. In contrast, multiplication and division, as referent transforming operations, allow to combine two quantities, with the same or differing referents, to produce a new quantity whose referent differs from either or both referents of the original quantities (e.g. a speed multiplied by time resulting in a distance, a measure in feet times a measure of inches per foot, yielding a measure in inches; or a length times a length yielding an area).

Here we analyze how teachers consider or do not consider quantities when they answer questions about a function expressed in algebra notation, before and after online discussions taking place during an online teacher development course. We also analyze how their ways of answering the question evolved, over the course of one week, as they participated in online discussions with their group instructor and their peers.

Method

The data we analyze are part of teachers' online work during the fifth week of the first of three graduate level courses offered to a third cohort of teachers by the Poincaré Institute for Mathematics Education program (see Teixidor-i-Bigas, Carraher, & Schliemann, 2013 and Schliemann, Carraher, & Teixidor-i-Bigas, 2016). The three courses were offered online to teachers in grades 5-9. They aimed at meaningful and deep learning of key topics in the middle school curriculum viewed through the lens of algebra and functions and emphasized the importance of multiple representations (verbal statements, number lines, function tables, Cartesian graphs, and arithmetical-algebraic notation) for expressing relations among mathematical objects and quantities in science and everyday situations.

The online activities involved a wide range of discussions, among teachers and course instructors, about course topics (expressed in written notes, video lectures, apps, and video clips from K-8 classroom activities), the solution of mathematical problems (presented as challenge questions), and classroom activities planned and implemented by the teachers. Solving and discussing the challenge questions constituted a substantial part of the courses, taking place online over two weeks of each of the four units in a course. The Notes for the week the discussions took place included material about the treatment of numbers versus quantities, multiplication and division as referent transforming operations, and results of multiplication operations when quantities were involved (see examples in Figures 1 and 2).

- **The Arithmetic of Quantities is More Complicated.** When dealing with pure numbers, we can always add or multiply any two real numbers together and get back the same kind of object, namely a real number. This is different with quantities. First, we can only add or subtract together measures of the same type of quantity. For instance, 3 feet + 5 feet = 8 feet is fine, as is even 12 inches + 2 feet = 1 yard, since these are all measures of the quantity *distance* (even if they are expressed in different units). But we will run into trouble with the sum "2 meters + 15 kilograms", since these are measures of different quantities (namely *distance* and *mass*). Similarly, we cannot compare the sizes of two different types of quantities, e.g., it makes no sense to ask whether 2 meters is bigger or smaller than 15 kilograms.

  Second, we *can* multiply or divide measures of any two quantities, but the output is always a different type of quantity. For instance, 6 miles $\div$ 3 hours = 2 $\frac{\text{miles}}{\text{hour}}$. Even when the two input measures are of the same type of quantity, their product or quotient will *still* be a *different* type of quantity! For instance, 2 feet $\times$ 3 feet = 6 ft^2, or 6 square feet, which is a measure of area and not distance! We will return to the division of quantities in Unit 4.

Figure 1: Arithmetic Operations on Quantities, from Course I notes, the Poincaré Institute for Mathematics Education, Tufts University, 2016.

A total of 49 teachers (14 teaching the elementary grade levels, 17 in middle school, and 18 in high-school) and 20 coaches, special educators, or interventionists participated in the cohort we examine here. They were from nine low-income school districts in three New England states.

Multiplication of Quantities: Are They All Dilations?

One could ask if the multiplications we use in real life can be thought of as dilations. The answer is "it depends":

- **Stretching or Shrinking**: When a builder builds a house by scaling the dimensions based on the architect's model, when we adapt a recipe to our number of guests, or compute a 10% tip in a restaurant, we are not changing the nature of the amount. These can be considered as scaling a physical quantity, and as such are dilations.
- **Conversions**: A conversion changes the units of measure for expressing quantities without changing the nature (or amount) of the quantity. Mathematicians sometimes call this a change of coordinates: you are keeping the 0 where it was but choosing a new 1. There are some conversions (like changing temperatures from Celsius to Fahrenheit) that do not keep 0 fixed and so can't be represented by a simple multiplication. In converting an interval of 90 minutes to 1.5 hours one does not alter the amount of time, but simply adopts a different unit for expressing time. We will not call this a dilation, although in a certain sense, it is a dilation with scaling factor 1.
- **Change of Dimension:** Sometimes multiplying quantities of one sort yields quantities of an entirely different sort. For instance, a time of 3 hours times a speed of 2 miles per hour, yields a distance of 6 miles. Since the quantities of the domain and codomain are different, we will not call this a dilation.

Figure 2: Multiplication of quantities, from Course I notes, the Poincaré Institute for Mathematics Education, Tufts University, 2016.

One of the six challenge problems teachers were asked to answer, justify, and discuss during the week we analyze here consisted of the following set of seven questions:

(1) Can you add a distance to a distance? Either give an example and explain which kind of quantity you would get as a result or justify why this is not possible.
(2) Can you add a distance to a different type of quantity (such as, a time, volume, speed, force or weight)? Either give an example and explain which kind of quantity you would get as a result or justify why this is not possible.
(3) Can you multiply a distance by a distance? Either give an example and explain which kind of quantity you would get as a result or justify why this is not possible.
(4) Can you multiply a distance by a different type of quantity (such as, a time, volume, speed, force or weight)? Either give an example and explain which kind of quantity you would get as a result or justify why this is not possible.
(5) Can you divide a distance by a distance? Either give an example and explain which kind of quantity you would get as a result or justify why this is not possible.
(6) Can you divide a distance by a different type of quantity (such as, a time, volume, speed, force or weight)? Either give an example and explain which kind of quantity you would get as a result or justify why this is not possible.
(7) If y = 3x, and x and y represent different quantities, can you say y is bigger than x?

We posed questions 1 to 6 to invite the teachers to consider the meanings of different operations (addition, subtraction, multiplication and division) on the same or different types of quantities. Question 7, the focus of the current analysis, aimed at examining whether they would interpret the question on the algebraic expression y=3x in terms of pure numbers or in terms of quantities and how, before, during, and after they participated in online discussion with one another and with each group's instructors. Because multiplication of unlike quantities is a referent transforming operation, the coefficient 3 transforms x into a different type of quantity, that is, y. In such a case, where x and y represent different types of quantities, one cannot say *y* (e.g. a distance) is greater than, smaller than, or equal to *x* (e.g. a time).

Results

Teachers submitted drafts of their responses to the challenge questions early on in the week and their final answers by the end of the week, after feedback by instructors and by peers. Differences in early and later answers would conceivably reflect how the views of the teachers evolved over the week. It is not possible to determine the extent to which changes in answers reflected autentic progress in understanding as opposed to mere adoption of conventions being endorsed by peers and instructor. We examine the results with this caveat in mind.

Teachers Initial and Final Responses to Question 7

We first determined whether or not the teachers considered that *x* and *y* could represent different types of quantities and, therefore, could not be compared. As shown in Table 1, in their initial answers only 19% of the teachers considered physical quantities in their answers stating, for instance, that "*If y = 3x, and y and x represent different types of quantities we cannot generalize that y is bigger than x because, depending upon the type of quantity, we could be comparing unlike units which would have no meaning.*"

Table 1: Teachers Answers at the Start and at the End of the Week

Teachers' Interpretations of *x*, *y*	Initial Answer	Final Answer
As different kinds of quantities	13 (19%)	45 (65%)
As quantities of the same kind but different units.	4 (6%)	3 (4%)
As pure numbers	22 (32%)	12 (17%)
Not classified	4 (6%)	5 (7%)
No answer	26 (38%)	4 (6%)
Total	69 (100%)	69 (100%)

Among the remaining teachers, 6% considered unit transformations, that is unit conversions, in their answers ("When I first did this I had inches and feet and same distance but with different units. My conclusions were if my y was the larger unit, then yes, it will work--but if my y was my smaller unit ---no." Still, 32% interpreted the problem as entailing a comparison between numbers, arguing, for example that, if x is a negative number, x is greater than y. The other 30 teachers, either did not answer the question (38%) or gave an unclear answer (6%).

By the end of the week, after reflecting upon questions and feedback by instructors and by peers, the percentage of teachers who correctly treated x and y as different kinds of quantities in the final versions of their written answers, noting that it made no sense to compare the magnitudes of the different quantities, increased to 65%. Only 17% persisted in considering x and y as standing for pure numbers. The percentage of teachers giving no answers and of those giving unclear answers dropped to 7% in each case.

The association between time of answer (initial versus final) and type of response (different quantities versus pure numbers) was highly significant (Fisher exact probability < .0001).

Examples of Online Discussions and Changes on Teachers' Responses

The following excerpts of teachers and instructors' online postings in two of the groups document how some teachers engaged in considering the quantities in the algebraic expression. In examining these, we also try to identify strategies the instructors use to challenge and engage teachers in considering variables and numbers in the algebraic expression as quantities and in determining relationships among the different types of quantities represented as a function.

Let's look at the discussions that took place in two of the groups where the instructors and teachers engaged in the discussion of the answers to the question "If y = 3x, and x and y represent different quantities, can you say y is bigger than x? Please justify your answer."

Example 1

This group was composed by the instructor, seven classroom teachers (one in grade 6, three in grade 7, one in grade 8, and two in grade 9), and one math school coach.

Initial answers by five teachers in this online group treated x and y as numbers. In such cases, they noted that, if x is a negative number, x is greater than y, if x is zero, x and y are equal, and, if x is positive, then x is less than zero (three teachers didn't post an answer to this question early in the week). XX of them presented the diagrams similar to those in Figure 1 to support their answers[2].

Such answers suggest that the teachers were treating pure numbers as valid examples of quantities, and perhaps even different kinds of quantities. This assumption was contrary to the notion of quantity we were trying to promote. The group's instructor then posted the following comment:

> *Instructor:* In the last part of question 4 you look at the product purely as a product of numbers and ignore the fact that we are talking about quantities. Would your answer change if you think in terms of quantities?

One of the teachers answered the instructor as follows:

> *Teacher 1:* I actually had thought of it for a very brief moment, initially thinking that we would not be able to tell that necessarily if y is bigger given that x and y represented different quantities. However, I dismissed that thought because we were not multiplying the 2 different quantities by each other. Then I got focused on the actual number scenario. However, looking at it again, I am not sure. The reason I am confused is that you are multiplying x just by a number, not by another quantity.[3] I am not sure how we would get y of a different quantity. For example, if x is distance (miles) and it is multiplied by 3, we would always get more miles so y would be the same type of quantity and the resulting amount of miles (y) would always be equal to or greater than x.

---

[2]Typos in the transcriptions have been corrected.

[3] It is true that the equation, y=3x +y, does not expressly associate the numeral, 3, with a quantity. However, assuming that x and y are different kinds of quantities, the equation can only be valid if 3 itself represents a quantity.

> Basically, when you talk about quantities of something not plain numbers it seems that the answer would depend on the type of quantity (if it could be negative). However, when I think about my product being a different quantity I am unsure because I cannot think about an example of multiplying a quantity by just a number and getting a different type of quantity. I hope that my answer/thinking is not too confusing... I'll wait to hear responses and others thinking. I might be missing something completely here!

The instructor then focused on the possible meaning of 3 in the expression y=3x and provided two examples for the quantities 3 could represent:

> *Instructor:* If you multiply a type of quantity by 3 and get a different type of quantity, then the 3 is not a plain number. For instance, if every child in your class needs to have 3 notebooks, you can think of the 3 as notebooks per child. Multiplying the 3 with the number of children will give you the number of notebooks. Or if you walk at 3 miles per hour, multiplying 3 by the hours will give you the miles. Try to come up with a few more examples and think about the question again.

This led to the following response by a teacher, who elaborated on her changing perspective:

> *Teacher 1:* Thanks, that now makes much more sense; I was stuck looking at the 3 as a plain number! I am going to think of this some more and come up with other examples like you suggested.
>
> However, based on the examples that you gave me I would probably go back to when I first considered that x and y were different types of quantities; at that point I was thinking that we really could not say that y was bigger than x because you are talking about different things. In one of your examples, we would be comparing notebooks and children; we couldn't possibly order them because they are different. Even though we would want to just look at the magnitude of the quantity they are just different things altogether! The same would be true with your other example in which y is a quantity of miles and x is a quantity of hours. You can't really say which is bigger because they are different. I will go back and think of more examples now....

Another teacher then also expressed:

> *Teacher 2:* I just read through what the Instructor and Teacher 1 were saying about [sub-question] 7 of question 4. I did not think of it in terms of quantities at all and am having some difficulty fully understanding the concept. The example of crayons and children does help. I will be interested to read what others do with this problem.

Other teachers joined the discussion. By the end, all teachers in this group had expressed, with examples, that, if x and y represented different quantities, they couldn't be compared. Figures 3 and 4 show examples of their final answers.

> "After considering quantities such as the number of notebooks per student or the number of eggs you would need to make a batch of cookies (3 eggs per batch; where x is the number of batches and y is the number of eggs needed), I can look at the quantities as numbers and say that I need more notebooks than I have students and I need more eggs than the number of batches of cookies I am making. I can't compare the two though. I can't say that 6 notebooks are greater than 3 students or that 9 eggs are greater than 3 batches of cookies."

Figure 3. Final answer by a teacher.

(7) If $y = 3x$, and x and y represent different quantities, can you say y is bigger than x? Please justify your answer.

If we just think of $3x$ as purely a product of numbers and ignore the quantities, then no, y will not always be bigger and the explanation of this is shown in the bottom part of this question.
However, the question states that "x and y represent different quantities" so we need to think about these in terms of quantities. Thinking about x and y as quantities, it is not possible to compare or order x and y because they are different types of quantities (different attributes).

For example, if I give 3 tests in a term and I have x number of students on my team, I will have graded y tests at the end of the term. I cannot create an inequality of the tests I graded and the students on my team because they are completely different quantities, it just doesn't make sense to compare them.
y = 3x x = number of students; 3 tests per term; y = number of tests graded

y = 3(100) = 300 → I graded 300 tests; there are 100 students on my team.
I cannot compare these two quantities 300 test vs. 100 students

Another example:
y = 3x x = number of kids at the party; 3 slices of pizza per kid;
y = number of slices of pizza served at the party

y = 3(15) = 45 → 45 slices of pizza were served; there were 15 kids at the party.
I cannot compare these two quantities 45 pizza slices vs. 15 kids.

Explanation when thinking of just numbers not quantities:

3*x*	***y***
3 (-2)	-6
3(-1)	-3
3(0)	0
3(0.5)	1.5
3(1)	3
3(2)	6

Is y bigger?
- If x is a negative number, then x is greater than y.
- If x is zero, then x and y are equal.
- If x is a positive number, then x is less than y (y is bigger).

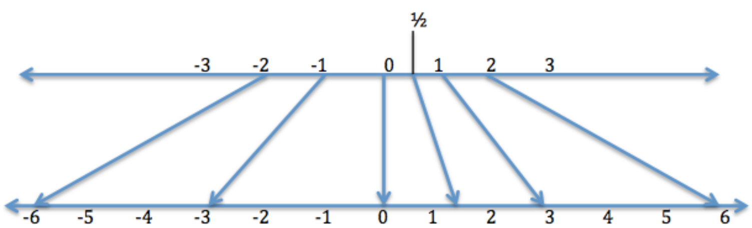

Figure 4. Final answer by a teacher contrasting operations on quantities to operations on numbers.

Example 2

This group was composed by three teachers of grade 5, one of grades 6 to 8, one high school teacher, and four special education or teachers of English learners. As we will see, these teachers' ideas were more varied. The instructor tried to trigger the discussion among the teachers and let them figure things out by themselves, by asking questions to let them reflect on their own answers and letting them compare each other's ideas. Some teachers only wrote their ideas in the pdf documents they added to the online forum space. The instructor cited answers in the pdf documents, so that other teachers knew what these teachers said, even if they did not open the documents.

Before any answer to the questions the following discussion on quantities took place in this group:

> *Teacher 1*: For Question #4 Quantities is being used to mean different values in parts 1-6. My question is for part (7): are we using quantities as values or units?

Teacher 2: I don't know. When I read the word quantities I automatically thought any number, but now I am wondering if a quantity is always going to be a positive number.

Teacher 2 (again): In the Notes it says that a "quantity is some measurable property, like height, weight, distance, cost, or speed." I don't know if this totally solves our questions.

Instructor: Hi Teacher 2 and Teacher 1, Yes, a physical quantity is a physical property of a phenomenon, body, or substance, that can be quantified by measurement. A value for a quantity is a numerical value in terms of a unit of measurement, such as 3 meters, 5 seconds. The values for some quantities can be negative. For example, if we consider the ground level is the position of zero, then the position above the ground level would be positive and the position bellow the ground level would be negative.

The first posting (by teacher 2) about the main question (by teacher 2) came after the discussion above and stated:

Teacher 2 (draft): This table (see below) proves that y is not always bigger than x. To say a number is larger than another it has to fall to the right of the first number on a real number line. Since -3 does not fall to the right of -1 it is not larger than -1 and therefore is smaller. Also, 0 is not larger than 0.

x	y
3	9
1	3
0	0
-1	-3
-3	-9

The instructor responded to this by clarifying to all group members that the question referred to different types of quantities:

Instructor: Hi everyone, In part 7 of Question 4, it says "If y = 3x, and x and y represent different quantities, can you say y is bigger than x? Please justify your answer." In fact, what we were trying to say is "x and y represent different types of quantities".

Another teacher then responded, still just considering the values of pure numbers:

Teacher 3: Here's what I'm thinking for this part... please let me know if my thinking makes sense. No, we cannot say that y is bigger than x because we don't know the value of it. It could be a positive or negative integer for that matter, and there are endless possibilities. (I'm thinking back to U1W1, knowing that there is an infinity amount of numbers between 3 and 4.) There is nothing that tells me what y equals, therefore, I have no basis to know what the value of it is and there is an infinite amount of possibilities.

The instructor considers the last answer and takes the opportunity to raise a new question aimed at better understanding the teacher's reasoning:

Instructor: Hi Teacher 3, If we know that both x and y are positive, can you say that y is bigger than x?

Following the instructor's question, the teacher responds considering only plain numbers and the instructor then reminds her that the question refers to quantities:

> *Teacher 3:* I just inputted numbers into a chart (not shown in the forum) for x and y...and besides when x and y are 0,0, the answer is YES, y will always be bigger than x when you use positive numbers. For example, when y=3x; 12 =3times 4 and y=3x when 6=3times 2 and y=3x; But, when you use (0,0), x and y are equal. Am I on the right track?
>
> *Instructor*: Hi Teacher 3, How about if the x and y represent different types of quantities? Can you say y is bigger than x, if they are positive?

The teacher tentatively switches her focus to quantities and gives an example, but she still thought that one type of quantity could be bigger than another type of quantity.

> *Teacher 3*: Rather than thinking mathematically, I thought of "x and y" as different types of quantities and used the formula for work that we are using in science. work (y) = force (3) times distance(x). Therefore, the work is going to be bigger than the distance, so, y will be bigger than x. Am I on the right track?

Another teacher answers by considering unit conversion situations (although he made a mistake in considering the conversion factor):

> *Teacher 4*: Do you necessarily have to convert whatever one unit is to the other? For instance, what if you have y = ounces and x = pounds. If x=5: so y=3(5), y = 15. Would you necessarily have to then convert the 15 to the proportional ounces to pounds? If not, then no, y is not always bigger than x. In this example, we would be saying that for every 5 pounds, you would have 15 ounces. Does this make sense?
>
> *Teacher 4* (again): Or maybe a better example would be if you were using gallons and ounces with: x = gallons; y = ounces, millimeters or centimeters. Once again, as long as you don't have to convert one unit to the other and are looking at it as a ratio, y would not be bigger than x.

The instructor reminds Teacher 4 that the question refers to different types of quantities:

> *Instructor*: Hi Teacher 4, In your example, both x and y represent the same type of quantity, volume. How about if x and y represent different types of quantity? Can you say that y is bigger than x?
>
> *Teacher 4*: I think so! What if I substituted ounces for inches. Once again as a ratio only you would have the x being bigger than y. I'm not sure if that is something one could do.

Apparently, Teacher 4 was still thinking about the same type of quantity with different units. Although he said inches, he seemed to mean cubic inches, which was still the unit of volume. The instructor drew his attention to what another teacher had said about this question to help him understand what "different types of quantities" meant.

> *Instructor*: Hi Teacher 4, Do you mean that, if you substituted ounces for inches (y = inches and x = pounds), then you think that y is bigger than x? Here is what Teacher 5 said, in her work, "If y = 3x, and y and x represent different types of quantities we cannot generalize that y is bigger than x because, depending upon the type of quantity, we could be comparing unlike units which would have no meaning. For example, if x = 60 miles per hour, and 3 signifies 3 hours then y = 180 miles. While the number 180 is, indeed, greater than the number 60, we cannot quantify that 180 miles is greater than 60 miles per hour because we would be comparing unlike quantities (distance and distance/time), which would essentially have no meaning." What do you think?

Teacher 4: I would agree with her. The only way that I could find that with y=3x, for y to be greater than x, it would have to be a completely different unit altogether. It would therefore not have any meaning.

Teacher 4 said that he agreed with *Teacher 5*, but he also said: "for y to be greater than x, it would have to be a completely different unit altogether. It was not clear what this meant. Then Teacher 1 joined the discussion. She also considered this function as a unit conversion formula and viewed x and y as the same type of quantity (length). In respond to Teacher 1, Teacher 5 explained her ideas again.

Teacher 1: Thanks for clarifying. That was my original question. So for example if x= 1 foot then 3x would = 1 yard. This example it does work, but when you use negative numbers x= -1 foot then y= -3 feet, it does not work. Am I on the right track?

Teacher 5: Hi Teacher 1, I'm not sure if this is correct or not, but I think that for this particular problem, the numbers we substitute for x and y don't matter as much as the types of quantities that we use. For example, if we assume that x = 60 miles per hour, and 3 = 3 hours then y would = 180 miles. While the number 180 is, indeed, greater than the number 60, we cannot quantify that 180 miles is greater than 60 miles per hour because we would be comparing unlike quantities (distance and distance/time), which would essentially have no meaning. I hope I am not confusing the issue!

Since most teachers have switched to consider different types of quantities, the instructor tried to let all the teachers compare different ideas from two teachers who had both considered different types of quantities.

Instructor: Hi everyone,

Teacher 3 said: "Rather than thinking mathematically, I thought of "x and y' as different types of quantities and used the formula for work that we are using in science. work (y) = force (3) times distance(x). Therefore, the work is going to be bigger than the distance, so, y will be bigger than x".

Teacher 5 said: "I'm not sure if this is correct or not, but I think that for this particular problem, the numbers we substitute for x and y don't matter as much as the types of quantities that we use. For example, if we assume that x = 60 miles per hour, and 3 = 3 hours then y would = 180 miles. While the number 180 is, indeed, greater than the number 60, we cannot quantify that 180 miles is greater than 60 miles per hour because we would be comparing unlike quantities (distance and distance/time), which would essentially have no meaning".

Who do you agree with?

Two teachers then responded in the following ways, which were different from their earlier postings:

Teacher 2: I think that if you strictly look at the numbers then yes, y will be bigger than x in both cases. If you look at the units they represent in each case you can't compare them because they are unlike quantities.

Teacher 1: So, the answer to question "can you say that y is bigger than x?" would be no as the types are different and we cannot compare two different values of different types of quantities. Would this be accurate?

Some general trends

For many teachers, the word *quantity* may be conflated with *number*. After reading the question, most teachers did not consider that x, y and 3 were quantities that described properties of objects and had different units. Then, many answered the question by examining cases in which x and y are either positive numbers, negative numbers, or zero. The trend of only considering pure number was very strong, given the question was posed right after the six sub-questions on operations on quantities.

Some teachers did consider x and y as quantities, but interpreted 3 as a pure number. That was the case of Teacher 1 in the first group above, when she said: "I cannot think about an example of multiplying a quantity by just a number and getting a different type of quantity."

Teacher 4 in the second group considered 3 to be a unit conversion factor, that is, x and y were viewed as the same type of quantity but with different units. His answer that x and y were the same made sense since he considered x and y as the same quantity expressed in different units.

Teacher 1 in the second group also considered x and y as quantities, but unlike Teacher 4, she focused only on the size of numbers when she said: "… if x= 1 foot then 3x would = 1 yard. This example it does work, but when you use negative numbers x= -1 foot then y= -3 feet, it does not work." Teacher 5's explanation seems to have helped Teacher 1 who, at the end of the week, realized that we could not compare the values of different types of quantities.

Teacher 5, also in the second group, gave a good answer at the beginning and her ideas and her further explanation in the discussion did help other teachers. Teacher 4 and Teacher 1's ideas on considering 3 as a unit conversion factor were also valuable, because these are also important situations for using functions.

Where and how did progress occur?

Table 2 shows that the percentage of teachers who changed from an exclusive focus on numbers in their first postings to considering quantities and expressing that one cannot compare x and y because they represent different kinds of quantities varied greatly across groups.

Table 2: Instructors' Postings and Teachers Answers at the Start and at the End of the Week

Groups	Instructors' Postings on Quantities	Teachers' Initial Answers on Quantities	Teachers' Final Answers on Quantities	Percent of Changes from Numbers to Quantities
A (N=9)	2	1	3	25% (2 out of 8)
B (N=9)	1	1	3	25% (2 out of 8)
C (N=8)	2	2	4	33% (2 out of 6)
D (N=9)	1	2	5	43% (3 out of 7)
E (N=9)	5	5	7	50% (2 out of 8)
F (N=9)	12	0	6	67% (6 out of 9)
G (N=8)	5	2	8	100% (8 out of 8)
H (N=8)	3	0	8	100% (8 out of 8)

The table also shows the number of instructor's postings on the y=3x question. In the four groups where fewer than half of the teachers changed from considering numbers or unit transformation to considering quantities (groups A to D), the discussions geared to other topics and only one or two questions were asked on the answers to the challenge questions analyzed here. In group H, where all teachers progressed from a focus on numbers in the initial responses to considering quantities in their answers, even though the instructor only posted three comments on the answers to the questions, these were detailed and geared to eliciting teachers' reflections from a very interactive group, where participants often reacted to postings by the instructor and by their peers.

Discussion

At the beginning of discussions, teachers were more likely to assess the question (if $y = 3x$, is $y > x$?) by assuming that x and y were pure numbers (32% vs. 19%). By the end of the week, the trend had reversed: 17% of the teachers were treating x and y as pure numbers. Fully 65% of the teachers were evaluating the question under the assumption that x and y were quantities of different kinds. Differences across groups may have been, at least in part, due to instructors' ways of addressing the issue as it emerged. In the groups where instructors raised questions aimed at eliciting reflection about relations between elements in the algebraic expression and about how these related to quantities, most or all teachers switched to considering quantities, instead of just pure numbers.

In part, the initial responses may have resulted from a different interpretation of the term "quantity", something that could be easily addressed. However, we need to keep in mind that considerable effort was put into explaining and exemplifying what was meant by quantity, both in the course notes and in the challenge question itself. This suggests that it's not enough to "clearly define the term, *quantity*, for the teachers". They need to discuss their views and the instructors' role in these discussions are an important factor in the teachers' development.

Whether one interprets the variables as quantities leads to very different answers to the question "if $y = 3x$, (and y and x are different kinds of quantities), is $y > x$?" More generally, whenever one is modeling, it is important to distinguish between numbers and quantities.

A focus on relationships between pure numbers is of utmost importance in mathematics. However, a focus on quantification is essential in promoting students' mathematical understanding. A balance between the two views is certainly the ideal approach.

Students need to be given the opportunity to consider the variables and numbers in the algebraic representation of functions as quantities much earlier and more often. This would allow for a deeper understanding of mathematics and its representations and would prevent the view of functions as pure numbers, a view that would be very to change after it has been settled in the early years.

Future analysis of instructors and teachers' online interactions will examine ways to promote teachers' consideration of how mathematical notation represents quantities in terms of what is represented, and on how to engage students in considering quantities.

References

Bièvre, P. B. (2009). The 2007 International Vocabulary of Metrology (VIM), JCGM 200:2008 [ISO/IEC Guide 99]: Meeting the need for intercontinentally understood concepts and their associated intercontinentally agreed terms. *Clinical Biochemistry* 42, (2009) 246–248.

BIPM, I., IFCC, I., IUPAC, I., & ISO, O. (2012). The international vocabulary of metrology—basic and general concepts and associated terms (VIM), 3rd edn. JCGM 200: 2012. *JCGM (Joint Committee for Guides in Metrology)*.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British journal of developmental psychology*, 3(1), 21-29.
Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194-229.
Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*, Riedel Publishing Company, Dordrecht: The Netherlands.
Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *Research companion to principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.
Lehrer, R., Jacobson, C., Kemney, V., & Strom, D. (1999). Building on children's intuitions to develop mathematical understanding of space. In E. Fennema & T. Carpenter (Eds.), *Mathematics Classrooms that Promote Understanding* (pp. 63–87). Mahwah, NJ: Erlbaum.
Lehrer, R., Schauble, L., Strom, D., & Pligge, M. (2001). Similarity of form and substance: Modeling material kind. In S. M. Carver & D. Klahr (Eds.), *Cognition and instruction: Twenty-five years of progress*. Mahwah, NJ: Erlbaum.
Liu, C, Carraher, D.W., Schliemann, A.D., & Wagoner, P. (2017). Learning to Quantify Relationships Among Weight, Size, and Kind of Material. *Cognition and Instruction*, 35(1), 20-54.
Lobato, J. & Ellis, A.B. (2002). The Teacher's Role in Supporting Students' Connections Between Realistic Situations and Conventional Symbol Systems. *Mathematics Education Research Journal*, 14(2), *99-120.*
Lobato, J. & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior, 21*, 87-116.
Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. New York, Cambridge University Press.
Saxe, G. B. (1991). *Culture and cognitive development: Studies in mathematical understanding*. Hillsdale, NJ: Erlbaum.
Schliemann, Carraher, & Teixidor-i-Bigas (submitted). Evaluating the Impact of a Teacher Development Program on Mathematics Teaching and Learning.
Schwartz, J. (1988) Intensive quantity and referent transforming arithmetic operations. In In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades* 2: 41-52.
Schwartz, J. L. (1996). Semantic aspects of quantity. Unpublished manuscript, Massachusetts Institute of Technology, Division for Study and Research in Education.
Teixidor-i-Bigas, M., Schliemann, A. D., & Carraher, D. W. (2013). Integrating disciplinary perspectives: The Poincaré Institute for Mathematics Education. *The Mathematics Enthusiast, 10*(3), 519-562.
Thompson, P. W. (2016). Researching teachers' mathematical meanings for teaching mathematics. In L. English & D. Kirshner (Eds.), *Third Handbook of International Research in Mathematics Education* (pp. 435-461). New York: Taylor and Francis.
Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.
Vergnaud, G. (1988). Multiplicative Structures. In H. Hiebert & M. Behr (Eds). *Research Agenda in Mathematics Education. Number Concepts and Operations in Middle Grades* (pp. 141-161). Hillsdale: Erlbaum.
Vergnaud, G. (1994). Multiplicative conceptual field: What and why. In G. Harel & J. Confrey (Eds.), *Multiplicative Reasoning in the Learning of Mathematics* (pp. 41–59). New York: SUNY Press.
Whitney, H. (1968a). The mathematics of physical quantities: Part I: mathematical models for measurement. *American Mathematical Monthly*, *75*(2), 115-138.
Whitney, H. (1968b). The mathematics of physical quantities, Part 2: Quantity structures and dimensional analysis. *American Mathematical Monthly*, *75*(3), 227-256.