ALGEBRA NOTATION FOR FUNCTIONS IN GRADES 5 THROUGH 9

Sheree T. Sharpe
Tufts University
sheree.sharpe@tufts.edu

Analúcia D. Schliemann
Tufts University
aschliemann@mac.com

We examine how 1343 students in grades 5 to 9 solved an arithmetic problem and attempted to provide a general representation for a function described in the problem. We found that use of letters appears from grade 5 and continuously increases through grade 9 and that students who solved the arithmetic part of the problem were more likely to use letters as variables.

Keywords: Algebra and Algebraic Thinking; Assessment and Evaluation

Research shows that middle and high school students often fail to solve algebra problems (Knuth et al, 2011), have trouble generating equations from word problems (Kieran, 2007), and do not use letters to solve them (Booth, 1984; Küchemann, 1981). Difficulties with algebra have been attributed to the inherent abstractness of algebra (Collis, 1975, Kuchemann, 1981) or to teaching that emphasizes the "meaningless manipulation" of variables (Chazan, 2000).

More recently, a different approach to high school algebra (see Schwartz and Yerushalmy, 1992), which emphasizes the relationship between quantities and focuses on functions and their multiple representations (e.g., tables, symbolic expressions, and Cartesian graphs) have shown to promote a deeper understanding of algebra (Chazan, 2000). In fact, even in elementary school, a functions approach to algebra can promote students' understanding of algebraic principles and representations and use of variables to represent verbal statements (Carraher & Schliemann, 2007; Kaput, Carraher, & Blanton, 2008). These results support the proposal for a different curriculum, aimed at integrating algebraic reasoning across all grades (Kaput, 1998). To design such curriculum, we need to better understand students' intuitive and conventional ways of attempting to use letters as variables across grades and to identify their typical mistakes. This study contributes to this goal by examining the evolving use of letters as a variable by students in grades 5 to 9, as they attempted to represent a function described in a verbal problem, after they had solved a specific instance of the problem using arithmetic.

Method

A total of 1343 students from New England (U.S.A.) completed a mathematics written assessment at the start of the 2012-2013 school year; 233 students were in grade 5 (17.3%), 465 in grade 6 (34.6%), 378 in grade 7 (28.1%), 131 in grade 8 (9.8%), and 136 in grade 9 (10.1%). We analyzed students' answers to parts (a) and (b) of a five-part problem taken from a grade 10 State Assessment test. Part (a) relates to grade 6 Common Core State Standards (on expressions and equations) and part (b) relates to grade 8 standards (on functions). The problem situation and the two questions examined were:

> Liam and Tobet are going to walk in a fund-raising event to raise money for their school. Liam's mother promised to donate to the school $4 per mile that Liam walks, plus an additional $30. Tobet's father promised to donate to the school $6 per mile that Tobet walks, plus an additional $20. (a) If Liam walks 15 miles during the event, what is the total amount of money his mother will donate? Show or explain how you got your answer. (b) Write an equation that represents y, the total amount of money Liam's mother will donate if Liam walks x miles during the event.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

Results

Missing or incorrect answers were scored as 0 and correct answers as 1. Then, part (a) answers were coded into no response (blank answer or I don't know), incorrect answer, and correct answer; and part (b) into no response, do not use letters (e.g., 90, \$60+\$30=\$90; \$50=5 mi; 60, 15*4 = 60 or 20*4 = 80, 1120; 30 by 7.50 miles; if he walks 40 miles then his mom will donate \$110), and use letters. Answers to part (b) that used letters were also coded into the six categories in Figure 1.

Use of Letters

a. **Conventional Correct Notation**: (e.g., y=4x +30)
b. **Different Correct Notation**: uses different letters to represent x (number of miles walked) and y (total donated) (e.g., d=4m+30 or 4y+30=x)
c. **Reverse Notation**: reverses order of number and letter
   i. **Correct Answers** (e.g., x4+30 = y)
   ii. **Incorrect Answers** (e.g., y = x6+20)
d. **Incomplete Notation**: provides an incomplete equation
   i. **Correct Answers** (e.g., 4x+30)
   ii. **Incorrect Answers** (e.g., y = x + 30 or y=4x)
e. **Wrong Information** (uses wrong number information in a complete, incomplete, or incorrect set-up; e.g., 30 + (x*5)=y; y = 4x + 40; y = 15x + 20; y = 6x, y = x + 20; y *x = 450; y=90)
f. **Unidentified Use of Letters** (Randomly combines letters given/not given in problem; e.g., x + y = m; (x*y)+20=90; y/x+30=60; 16xy; 15xy=y; 4x=y+30=y; y=x)

Figure 1: Categories for Use of Letters as a Variable in Part (b) of the Problem

Representing and solving the arithmetic part of the problem seems to have been easier than providing a general algebraic representation of the situation. Here, the percentage of correct arithmetic answers (part a) ranged from 38.2% in grade 5 to 77.9% in grade 9, with no significant differences (analyzed with an ANOVA) between grades 7 to 9, but significant differences between the other grades. Correct algebra representations (part b) ranged from 4.7% in grade 5 to 70.6% in grade 9; differences across all grades were significant.

The arithmetic part of the problem was significantly more difficult for 5th graders, with 38.2% correctly solving the problem, than for 7th through 9th graders, where more than 70% did so.

The algebraic representation of the problem was difficult for 5th, 6th, and 7th graders, with from 4.7% to 36.5% of the students correctly using letters. These percentages increased to 57.3 in grade 8 (the grade where variables are introduced) and to 70.6 in grade 9.

Table 1 shows, for students in each grade, how the percentage of no response, incorrect, or correct responses to part (a) relates to no response, did not use letters, and used letters in part (b). Percentages in the cells for each grade level add up to 100%.

The last column in the table shows the total number of responses to part (a), for each category and grade level. Here, 5th graders show almost equal percentages for each of the three categories. Among 6th graders, there was a slight improvement in results. For the three higher grades over 70% of students provided a correct response to part (a).

Overall, in response to part (b), 33.6% of students in all grades gave no response, 5.3% did not use letters, and 61.1% used letters as a variable.

Even though a high percentage (33.9%) of 5th graders provided no response for both parts (a) and (b), as many as 21.9% correctly solved part (a) and used variable(s) in their solution to part (b). From grades 6th through 9th, the percentage of students who correctly solved part (a) and used variables in part (b) increased from 42.4% to 75%.

As expected, across all grades, a high percentage of students who provided no response to part (a) also provided no response to part (b), while those who correctly solved part (a) had a higher

percentage of use of variables in part (b). This suggests that students who understand the arithmetic relations in the problem are more likely to use variables in part (b), at any grade level. Also, the

Table 1: Percentage of Each response by Grade

Grade	Part (a) Response	Part (b) Response			Total
		No Response	Didn't use letters	Used Letters	
5 (N=233)	No Response	33.9%	0.9%	0	34.8%
	Incorrect	10.7%	9.0%	7.3%	27.0%
	Correct	12.0%	4.3%	21.9%	38.2%
6 (N=465)	No Response	21.7%	0	0.9%	22.6%
	Incorrect	9.5%	2.4%	9.5%	21.4%
	Correct	10.8%	3.0%	42.4%	56.2%
7 (N=378)	No Response	12.2%	0	0	12.2%
	Incorrect	3.7%	0.8%	13.0%	17.5%
	Correct	7.9%	1.6%	60.8%	70.3%
8 (N=131)	No Response	10.7%	0	0.8%	11.5%
	Incorrect	0.8%	0	14.5%	15.3%
	Correct	0.8%	2.3%	70.2%	73.3%
9 (N=136)	No Response	10.3%	0	0.7%	11.0%
	Incorrect	0	0.7%	10.3%	11.0%
	Correct	2.9%	0	75.0%	77.9%
All Grades (N=1343)	No Response	18.9%	0.1%	0.4%	19.4%
	Incorrect	6.3%	2.7%	10.6%	19.6%
	Correct	8.4%	2.5%	50.0%	60.9%

percentage of students who correctly answered part (a) and used variables in part (b) positively correlates with grade level.

A total of 672 (50%) of the 1343 students were able to correctly solve part (a) and then use letters as a variable in part (b). These were 51 students in grade 5 (21.9% of 5th graders), 197 in grade 6 (42.4% of 6th graders), 230 in grade 7 (60.8% of 7th graders), 92 in grade 8 (70.2% of 8th graders), and 102 in grade 9 (75.0% of 9th graders).

The percentage of students correctly using letters as a variable increases with grade level as does the percentage of those using the conventional notation. Accordingly, the percentages of most types of incomplete, wrong, and unidentified use of letters decreased with increasing grade levels but was still found among 27.3% and 12.8% of 8th and 9th graders, respectively. Only 9th graders used the correct conventional notation in more than 50% of responses, with 71.6% of them in this category. However, 5th to 8th graders would also use correct notation but reversing the order of variables and constants. Total percentages for correct use of variables, including cases of inversion and incomplete answers, show that 17.6% of the 5th graders, 49.2% of the 6th graders, and 56.5% of the 7th graders who used variables did so correctly, even though this is an 8th grade standard.

Discussion

In keeping with previous research on high school students' difficulties with algebra, our results show that generalized use of letters as a variable, although more frequent in later grades, is still problematic for more than half of 7th graders, more than 40% of 8th graders, and nearly 30% of 9th graders. However, among students who could solve the arithmetic part of the problem and used

variables in its algebraic part, use of letters in the correct conventional notation or reverse correct notation start to appear from grade 5 and continuously increase through grade 9.

In terms of solving the arithmetic part of the problem, even though 7th to 9th graders results were better than those for earlier grades, as many as 38.2% of 5th graders and 56.1% of 6th graders understood what the problem was asking them to do. This is noteworthy because this study's data were collected at the beginning of the school year, before 5th and 6th graders were taught the content of part (a).

Our results show that even 5th graders are capable of using letters to represent variables described in verbal statements and suggest that instruction about variables can start much earlier in the K-12 curriculum. The results also suggest that the ability to understand and solve arithmetic problems is a basis for considering variables and using letters for representing them.

Concerning students mistakes, reversing the conventional order for representing variables and constants was present at all grade levels. This could be easily addressed with practice. Other mistakes such as incomplete, wrong, and unidentified use of letters, present even in 8th and 9th grades, call for introducing algebra and functions through activities that promote a clear understanding of variables and of how algebra notation relates to verbal statements.

This study's results and previous findings on young students' use of letters as a variable, suggest that the integrated teaching and learning of arithmetic and algebra may successfully start in elementary school to promote a deep understanding of mathematics across the curriculum.

References

Booth, L. (1984). *Algebra: Children's strategies and errors. A report of the strategies and errors in secondary mathematics project*. Windsor, UK: NFER-Nelson.

Carraher, D.W. & Schliemann, A. D. (2007) Early Algebra and Algebraic Reasoning. In F. Lester (ed.) *Second Handbook of Research on Mathematics Teaching and Learning*. Vol II (pp. 669-705). Charlotte, NC: Information Age Publishing.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school Algebra classroom*. New York, NY: Teachers College Press.

Collis, K. (1975). *The development of formal reasoning*. Newcastle, Australia: University of Newcastle.

Kaput, J., Carraher, D. & Blanton, M. (2008) (Eds.), *Algebra in the Early Grades*. Mahwah, NJ: Erlbaum.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by 'algebrafying' the K-12 Curriculum. In NCTM and NRC (Sponsors), *The Nature and Role of Algebra in the K-14 Curriculum* (pp. 25-26). Washington, DC: National Acad*emies Press*.

Kieran, C. (2007). Learning and teaching Algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. Lester (ed.) *Second Handbook of Research on Mathematics Teaching and Learning*. Vol II (pp. 707-762). Charlotte, NC: Information Age Publishing.

Küchemann, D. E. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (pp. 102-119). London: Murray.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2011). Middle school students' understanding of core algebraic concepts: Equivalence & variable. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives* (pp. 259- 276). Heidelberg, Germany: Springer.

Schwartz, J. & Yerushalmy, M. (1992). Getting students to function on and with algebra. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 261-289). Washington, DC: MAA.