Teachers' Quantitative Reasoning When Using Graphs

Chunhua Liu, Montserrat Teixidor-i-Bigas, and Analúcia D. Schliemann
Tufts University

Abstract

We examine mathematics teachers' evolving quantitative reasoning, as they participated in a teacher development program focused on algebra, functions, multiple representations, student reasoning, and quantitative reasoning. A total of 47 teachers of grades 5 to 9 who completed in the program's three semester-long courses are included in the study. At the start of the program most of teachers considered variable quantities without thinking about their referents but just as pure numbers. By the end of the program, after they had explored mathematics representations of worldly and scientific situations, more of them, but not all, displayed awareness of quantitative reasoning. These results call for further studies on how to promote teachers' quantitative reasoning.

Keywords: Quantitative Reasoning, Graphs, Teacher Development

Thompson (2011) distinguished and defined two kinds of operations in mathematics education: quantitative operations (operations of thought by which one constitutes situations quantitatively) and numerical operations (those by which one establishes numerical relationships among measures of quantities). Research in mathematics education shows that a focus on quantitative reasoning increases the likelihood of success with algebra and makes arithmetic and algebraic knowledge more meaningful and productive (Ellis, 2007; Smith & Thompson, 2007). The Common Core State Standards for Mathematics (CCSS) emphasize the importance of quantitative reasoning, describing it as "habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them" (p. 6). However, traditional approaches to teaching mathematics tend to promote operations on numbers and numerical procedures, with no linkage between mathematical representations to the situations, problems, and ideas they refer to (Smith & Thompson, 2007).

One of the main features of quantitative reasoning is thinking with referents. Schwartz (1996) illustrated that manipulating functions depicted symbolically or graphically could lead to new insights about the referent situations the functions describe. Previous studies mainly focused on whether students or teachers could think with referents when quantifying relationships between quantities (Thompson, 2011) and when using number tables (Ellis, 2007, van Reeuwijk & Wijer, 1997) and symbolic representations (Thompson, 2011, Thompson et al., 2014). To the best of our knowledge, only Thompson et al. (2014) examined high school mathematics teachers' quantitative reasoning when using graphs, by asking middle school teachers to evaluate the numerical value of m if the y-axis were re-scaled. They found that most teachers in the study were not prepared to support students' learning and thinking with the magnitudes of physical referents. Further studies are still needed to clarify how to help teachers consider referents when analyzing graphs.

This study aimed at examining (a) in-service teachers' quantitative reasoning and (b) the potential effect of a teacher development program on their quantitative reasoning. The teacher development program builds upon the need for promoting teacher content and pedagogical knowledge (Ball & Bass, 2000; Shulman, 1986), focus on algebra and functions and their representations as ways to unify topics in the middle school curriculum, engage teachers in cooperation and analyses of student reasoning in the implementation of planned lessons, and emphasize the emergence of mathematical understanding from the analysis of relations among physical quantities (Authors, 2013).

In this report we evaluated teachers' responses to one of the questions in an individual written assessment given at the beginning and the end of the program. Our analysis aimed at determining whether or not the teachers considered the referents for different variables, as they considered graphs representing two different functional relationships in a real-world situation.

Method

The teacher development program, mostly offered online, consisted of three graduate-level semester-long courses. As part of the courses, weekly assignments offered the teachers opportunities to explore mathematics, the applications of mathematics to worldly and scientific situations, and the connections to topics they were teaching. Emphasis was given to the relationships among different representations, namely, verbal description of everyday and science phenomena, data tables, number lines, graphs, and algebraic notation, and to how students in all grades tend to approach and understand topics discussed in the courses.

A total of 47 teachers of fifth to ninth grades took the written assessment at the start and at the end of the program. We focus on the following question regarding the graphs in Figure 1:

> James walks at a constant speed of 4 meters per second. Which of the above figures would you choose to show both the distance he walks and the speed he walks over time: (A) Figure A is better; (B) Figure B is better; (C) Figure A and Figure B are equally good. Explain your choice.

Figure 1

The two functions of distance and speed depend on the same variable --- time, so the x-axis represents time in both Figures A and B. In Figure B, the y-axis of the upper graph represents distance and the y-axis of the lower graph represents speed. In Figure A though the y-axis represent both distance and speed. Because distance and speed have different units, which are meters and meters per second, respectively, it is impossible to label the y-axis in option A (4 cannot represent both 4 meters and 4 meters per second). In addition, there is no significance of the time for which the distance and the speed functions intersect. Saying that before the first second the distance is smaller than the speed would be equivalent to saying that 3 meters is less than 4 meters per second. So, if

the teachers thought about referents when they interpreted the graphs, they would choose option B (Figure B is better.)

Results

Table 1 shows that, in the pre-assessment, only 19 (40%) teachers chose option B, with four of these teachers failing to provide a good justification for their choices. In the pos-assessment 25 (53%) teachers thought that option B was better but, still, five of them did not provide a good justification.

Table 1

Now let us examine how teachers justified their answers. Table 2 shows that most of the teachers who chose Answer A justified their answers by saying that the two graphs are on the same plane, you can see the relation between them, or you can see that the two graphs intersect at (1,4). Here are examples from their written justifications:

> "Being able to look in one location to gather information from a graph is easier to do than to have to look up and down in multiple locations."

> "I like that figure A has all of the information in one graph. You can look at the graph, and see the constant rate of speed as well as the distance traveled. It is easier to see the relationship between the two when they are on one graph."

> "I like figure A more because it reinforces that James' speed is 4 meters per second because the audience can see that after 1 second the two lines meet at 4 meters."

Table 2

From teachers' responses we can infer that those who chose option A did not pay attention to the fact that the meanings of the numbers on the y-axis were different for the different quantities (distance and speed). Some teachers said: "It is easier to see the relationship", but they did not specify what relationship they wanted to see and how they would infer it from Figure A. Those teachers that paid attention to the intersection of the two graphs did not seem to be aware that a different choice of units (say hours instead of seconds and feet instead of meters) would have changed the point of intersection of the graphs.

Table 3 shows that most teachers who chose answer, B, thought that it is better to put distance and speed on different axes because distance and speed are different things, have different units, the y-axis needs to be labeled differently, or the intersection is meaningless. The following are examples of their written justifications:

> "The y axis must represent either distance or speed. It cannot represent both on A 4 represents 4 miles and 4 miles per hour, these are not the same thing."

"These graphs can't share a single coordinate plane because their respective axes need to be labeled differently. James' distance graph needs to have a y-axis of distance. James' speed graph has to have a y-axis of speed. If we mix the graphs the line labeled "speed" takes the meaning of someone perched 4 meters (distance) away and as time progresses they just stay at a distance of 4 away from us."

"Speed and distance have different units and should be shown on different axes. By showing them on the same axes you are saying that the point at which the two lines intersect is where they are equal however speed and distance cannot be equal to each other."

Table 3

These teachers considered the referents represented on the graphs and noticed that you could not compare distance with respect to time with speed with respect to time. Some of them even realized that the intersection between the two lines in option A does not mean anything.

Most teachers who thought Figure A and Figure B are equally good justified their answers by saying the two Figures A and B include the same information, they are both accurate, or Figure A is a combination of Figure B (see Table 4). Here are some examples of their written responses:

"They are the same thing, just two different ways to look at the situation."

"The figures are both accurate. Your choice would be dependent upon what you are trying to show in terms of relationships."

"They are the same but figure B is split into two coordinate planes."

"I think both graphs are equally as good. The first graph represents a system and the second represents two separate graphs. I think it would depend on what you are teaching your class at the time to decide which is better. I like being able to see the distance and speed all on one graph personally because it is less to look at, but the second graph is nice since you are looking at less variables in one graph."

"I would normally pick figure B and say that it's better to keep the two graphs separate from each other, but the first figure does show the relationship between speed and distance in showing where the two graphs meet at (1, 4) so, depending on what you are trying to show, either could be used."

Table 4

The teachers who chose option C work with the same framework as those who chose option A: they fail to take into account the units corresponding to the output of the two functions and the meaning (or lack thereof) of the point of intersection of the graphs.

Discussion

Teachers' responses at the start of the program shows that most of them did not consider that the two quantities in the problem, distance and speed, are different in nature, have different units, and cannot be compared with each other. This makes them ill-equipped to teach quantitative reasoning and suggests that the mathematics they teach deals only with pure numbers. Such an approach is likely to hinder their students' effective understanding and use of mathematics in science or in everyday life.

Teachers' responses at the end of the program suggest that, inviting teachers to explore the applications of mathematics to worldly and scientific situations led some of them to display awareness of quantitative reasoning. However, this was not enough for about half of the teachers. We need to draw teachers' attention to quantitative reasoning more explicitly and from different aspects, as they use different representations. Future interview studies would also further help understand teachers' reasoning. It will be relevant to understand what teachers had in mind when they talked for example about the relationship of the two graphs and their point of intersection.

Another aspect to consider is the fact that, even though some teachers seemed to suppress or ignore quantitative reasoning or not to consider the different nature of the different types of quantities, it was clear that most of them knew that distance and speed were different types of quantities, with different units. If this is the case, an additional question to be analyzed in the future concerns what inhibits them from thinking quantitatively when they represent the relationship between different quantities using graphs and other representations.

References

Authors (2013) Reference deleted for blind review.

Ball, D. L. & Bass, H. (2000). Ball, D.L., & Bass, H. (2000a). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.

Common Core State Standards for Mathematics. Downloaded on January 11, 2015, from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. Cognition & Instruction, 25(4), 439-478.

Smith, J., & Thompson, P. W. (2007). *Quantitative reasoning and the development of algebraic reasoning*. In J. Kaput, D. Carraher, & M. Blanton (Eds.), Algebra in the early grades (pp. 95-132). New York: Erlbaum.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4–14.

Schwartz, J. L. (1996). Semantic aspects of quantity. Unpublished manuscript, Massachusetts Institute of Technology, Division for Study and Research in Education.

Thompson, P.W. (1994b, April). Bridges between mathematics and science education. Paper presented at the research blueprint for science education conference, New Orleans, LA.

Thompson, P. W. (2011). *Quantitative reasoning and mathematical modeling.* In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), New perspectives and directions for collaborative research in mathematics education WISDOMe Monographs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming Press.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), Epistemic algebra students: Emerging models of students' algebraic knowing. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming. http://bit.ly/1aNquwz.

van Reeuwijk, M., & Wijers, M. (1997). Students' construction of formulas in context. Mathematics Teaching in the Middle School, 2(4), 230–236.

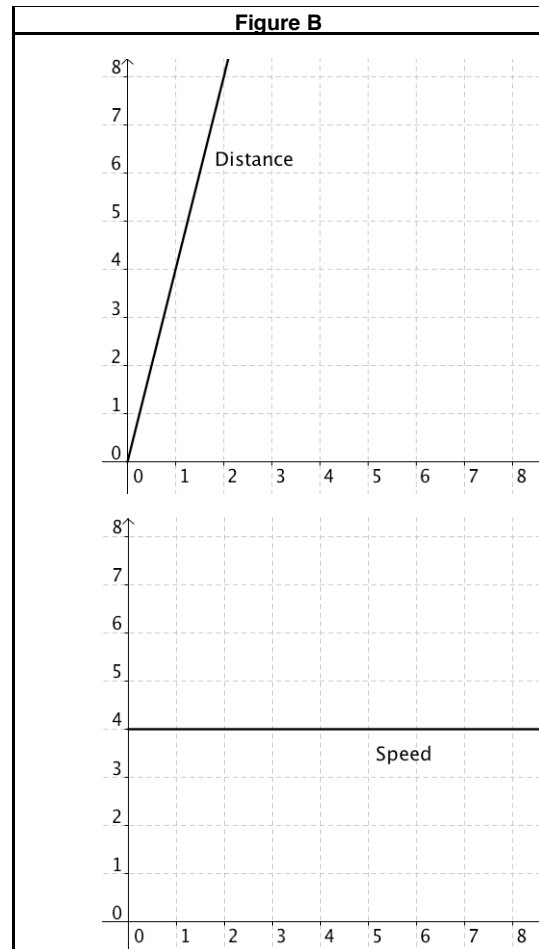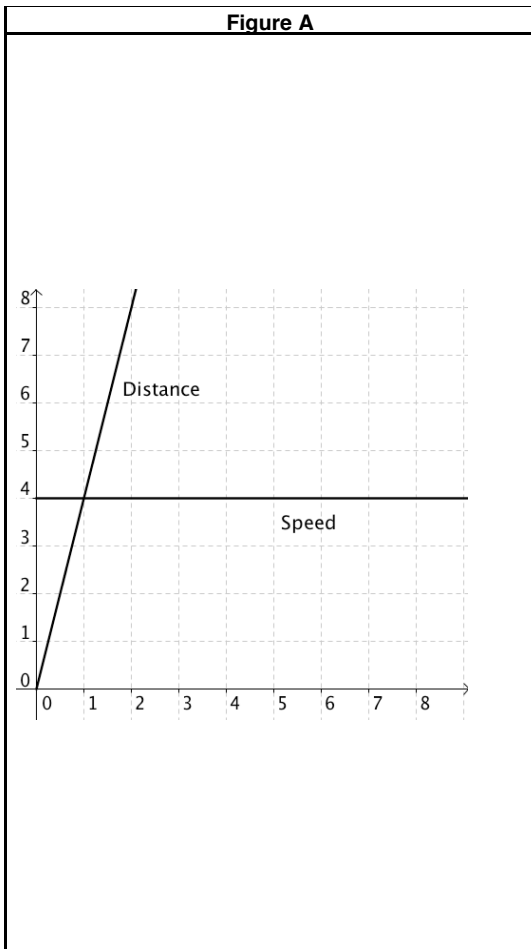Figure A

Figure B

Figure 1: Which figure is better?

Table 1: Number of Teachers who Gave Different Answers in the Pre- and Post- tests

Test periods	Answer A	Answer B	Answer C	No answer	Total
Pre	12 (26%)	19 (40%)	15 (32%)	1 (2%)	47
Post	9 (19%)	25 (53%)	13 (28%)	0 (0%)	47

Table 2: Justifications for Choosing Answer A (Figure A is Better)

	Support B	For same event	On the same plane (Concise)	See the relation of them	Intersect at (1, 4)
Pre	0	1	5	3	3
Post	1	0	3	4	1

Table 3: Justifications for Choosing Answer B (Figure B is Better)

	Different units and/or intersection	Different things	Label differently	Other reasons	Easier to read	No explanation
Pre	12	4	0	2	1	0
Post	10	4	6	2	2	1

Table 4: Justifications for Choosing Answer C (Figure A and Figure B are Equally Good)

	Support B	Same information/ Both accurate	A is combination of B	Intersection	Other reasons	No explanation
Pre	1	4	2	2	5	3
Post	0	8	1	0	2	2