Integrating Disciplinary Perspectives: The Poincaré Institute for Mathematics Education[1]

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Abstract: We describe the development of the Poincaré Institute, an NSF-MSP supported program developed through Tufts University Departments of Mathematics, Education, and Physics and by TERC, in partnership with nine school districts in Massachusetts, New Hampshire, and Maine. We focus on the challenges of developing an inter-disciplinary program aimed at improving the teaching and learning of mathematics from grades 5 to 9, the choice of mathematical and educational content of the program, the course structure, and the progress of the first cohort of participant teachers. We also outline the changes we are implementing for future cohorts.

Keywords: middle school mathematics, algebra, functions, collaboration between mathematicians and educators

Overview of the Institute & Aims of the Article

In 2010 Tufts University, TERC, and several school districts from Massachusetts, New Hampshire, and Maine created the Poincaré Institute for Mathematics Education, a graduate program of studies providing professional development for in-service teachers. The Institute was named in honor of Henri Poincaré, a distinguished mathematician and physicist from the turn of the 20th century who recognized the importance of mathematics

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education. Naming the Institute after Poincaré reflects our view that teachers need to broaden and deepen their grasp of mathematics, how children think and learn, how teachers teach, and how mathematics can be used to understand scientific and worldly phenomena.

The Institute seeks to transform and improve the teaching and learning of mathematics in middle school and the connections between the elementary, middle, and high school curricula. It highlights the connections by showing how functions implicitly permeate and potentially unify content throughout the K-12 curriculum. In particular, it uses the language of algebra as well as the geometry of functions to bring together otherwise disparate mathematical topics.

The Institute leverages expertise from mathematicians, educational researchers, physicists, teachers, and teacher leaders in school districts to: (a) offer graduate-level online courses on mathematical content, research in mathematics education, and knowledge relevant for teaching mathematics to three cohorts of 60 in-service teachers each (grades 5 to 9) from participant districts and a small group of pre-service teachers; (b) support long-term discussion forums in schools, where teachers plan, review, and improve their lessons; and (c) conduct research on teacher development and student learning.

The idea is to help teachers develop expertise suitable for whatever curriculum their school has adopted rather than provide them with ready-made lessons. Along with course activities aimed at deepening mathematical content, the teachers regularly examine video clips from classroom research on teaching and learning. They interview students on mathematics problems related to the curricula, and they plan, implement, and document their own learning activities in the classroom.

The attainment of substantial improvements in middle-school mathematics education requires special kinds of interdisciplinary and cross-institutional collaborations that must be carefully nurtured and sustained. In this article we describe the behind-the-scenes evolution of structures, working relations, and decisions that took place in the first two years of the Institute's existence, as we collectively sought to negotiate an interdisciplinary yet reasonably coherent and collaborative approach to a diversity of topics and issues.

The focus of this article will be on how we are merging the different perspectives brought to the project by mathematicians, mathematics education researchers, scientists, and the administrators and teachers in partner districts. In our analysis, we highlight key decisions we faced while attempting to set the scope and sequence of topics, as well as the roles of various contributors to the Institute. As the Institute moves forward and on the basis of what we are learning, we are revising the courses and improving the way we are working and collaborating. We hope the following discussion, although based on our limited experience with an ongoing program of professional development, may prove useful for other groups who are attempting to develop interdisciplinary approaches to middle-school teacher education.

We begin by describing and examining previous interdisciplinary collaboration by the Institute partners at Tufts and TERC that contributed to its content and pedagogical approach, taking into account perspectives from mathematics, mathematics education, and science. Next we provide an outline of the courses offered to teachers. We then introduce some issues that called for special adjustments in the roles, expectations, and interactions of the partners. At the end of the article, we outline how we plan to evaluate the impact of the project on the teaching and learning of teachers and their students, as well as some preliminary findings on changes we have observed among teachers in the first cohort.

Groundwork

Creating a truly interdisciplinary professional development program requires special sorts of collaboration. The Poincaré Institute needed mathematicians to do justice to the mathematics content, specialists in mathematics education to give proper due to issues of student learning and teacher development, and scientists to contribute expertise related to mathematical reasoning about physical quantities and modeling phenomena in the world beyond mathematics. We were fortunate to be able to draw on a decade-long program of early algebra research conducted by members from Tufts and TERC who would participate in the Institute. The algebra research furnished vivid video classroom examples related to the mathematics content of the courses. These video clips complemented future videotaped presentations by the mathematicians and software applets later designed by Poincaré teams. In-house teams carried out the Institute's own research and provided support for teachers as they designed and implemented their course-related projects for their students in the districts.

Any hopes that the Institute might exert a lasting contribution to classrooms require the input of teachers and other professionals from the participating schools and districts. However, teachers and district leaders' work primarily for schools and districts. They understand that their role as graduate students at a university is a temporary one, and the success of their graduate studies is valued according to its perceived benefits for their work in schools. This simple fact underlies many decisions we undertook during the first two years of the Institute, including:

1. The creation of, or support of existing, long-term teacher discussion groups in the schools;
2. The inclusion, in the online courses, of weekly challenge questions in which teachers were encourage to explicitly respond by taking into account their work in classrooms.
3. The designation of every third week of each unit as revolving around the theme, "Engaging Students". During this week participants partner with colleagues from their schools in planning lessons or interviewing students about the topics of the prior two weeks.

As we will describe next, the Institute, in its current form, has its roots in years of previous work and discussions among the partners in the project. By working closely with the districts from early on, we realized that it would be better to offer courses throughout the school term instead of during the summer or over a few weekends. The teacher leaders helped us identify and handle issues such as defining clear expectations for participants, compensating cohort and non-cohort teachers for attending after-hours meetings, and managing the technical resources provided to each participant.

Despite excellent reviews, the first proposal we submitted was not funded. We were instead encouraged to expand the work beyond Greater Boston and beyond Massachusetts. This delay in initiating the work ultimately proved beneficial. It allowed us to expand the program to target districts in rural Massachusetts, as well as districts in New Hampshire and Maine. It also gave us an additional year to establish the identity of the Institute and the roles of the various contributors. Buoyed by the enthusiastic commitment of the nine school districts, we submitted an improved proposal for the "Poincaré Institute, An MSP

Partnership for Mathematics Education", in August of 2009. The Institute officially began to function in June of 2010.

Initial Interdisciplinary Collaboration

The Poincaré Institute's interdisciplinary partnership was built on over a decade of prior collaboration rooted in research on algebra in the early grades, in the education of teachers and researchers, and on the efforts of Education, Mathematics, and Science faculty at Tufts University to improve mathematics teaching and learning at all levels.

The collaboration began through NSF-funded research projects such as the TERC-Tufts Early Algebra, Early Arithmetic project (http://earlyalgebra.org). This series of classroom investigations led to key publications about young students' learning of algebra. The research contributed in a fundamental way to the directions of the Poincaré Institute.

While Tufts University's Education Department became increasingly engaged in mathematics education research, it also created structures that fostered interaction with faculty from the Mathematics, Physics and Engineering Departments of the same university. For example, candidates in Tufts (Masters of Arts in Teaching) program for the preparation of middle or high school teachers take a minimum of two courses in the discipline they would specialize, in consultation with faculty from the corresponding departments. Each math teacher has two advisors, one from the Department of Education, another from the Department of Mathematics. This led to initial collaborations among the mathematicians and mathematics educators at Tufts.

In 2003, Tufts University created a masters and doctoral program in Mathematics, Science, Technology, and Engineering Education (MSTE). The program prepares researchers and future leaders in Math, Science or Technology Education and demands a greater knowledge of math, science, and technology. This led to increased collaboration among Mathematics, Science, and Education faculty. For example, faculty members from the different departments commonly serve together on doctoral dissertation committees. The graduate students often take part in Math Club activities and interact regularly with their peers from graduate programs in Mathematics.

In 2005 Tufts University created the Fulcrum Institute for Leadership in Science Education, an NSF funded MSP project with contributions from faculty from Tufts University's Departments of Education and Physics and from TERC. This program has prepared science educators to implement and lead research-centered science learning and teaching in their schools and districts. Participants advance their professional knowledge and status through the Institute's three online graduate course sequence. These courses, created during the NSF support period, are now part of Tufts' regular course offers and form the basis for a new program, the Tufts University Certificate Program for Science Education teachers. At the end of 2007, we began planning the Poincaré Institute for Mathematics Education, an interdisciplinary project focusing on the needs of school districts in the Greater Boston area. Our first challenge was to find a unifying topic for the math curriculum in grades 5-9 and engage mathematicians, scientists and education specialists around the topic.

Function as an Unifying Concept in K-12 Mathematics Education

We soon realized that functions could provide such a common ground. The concept of function is exceedingly important in modern mathematics. It traditionally enters the curriculum only in high school and beyond. Yet there were compelling arguments, which the mathematics educators themselves had championed (Carraher, Schliemann & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007), that functions underlie much of early mathematics, including the operations of arithmetic. The scientists also viewed functions as critical tools for fitting data to models. In short, there was a strong consensus that functions would offer a basis for substantial contributions from all three fields, even though each field had slightly different takes on what functions were about, how they were used, and why they were important. Provided we defined functions in a coherent way, we decided it would be useful to allow approaches from mathematics, education, and science to highlight different facets of functions. In a sense, this reflected our view that the teaching of mathematics requires respect for mathematical concepts and definitions while considering its applications, as well as sensitivity about how students and teachers make sense of it. Maintaining an eclectic perspective has been a constant concern throughout the development of the Institute.

The school districts were deeply concerned about the discontinuities in mathematics education across the K-12 curriculum, especially concerned about the transition from Elementary to Middle School and Middle School to High School. They also identified algebra as the topic that created or brought down barriers in these transition processes. They favorably viewed the prospect of teachers from early grades working alongside colleagues from later grades. One district suggested that the Institute range from grades 5 through 9, rather than 4 through 8 (as we had originally proposed), in order to address the transitions between elementary, middle, and high school mathematics.

Most districts were already committed to the idea that algebra needed to be made accessible to all of their students. Although most districts had not focused on the concept of functions as one means of helping them achieve this, they were invariably receptive to the idea.

Multiple Representations

During the proposal development phase that led to a second proposal, and after the project was approved, the core members of the Institute met regularly to map out the content and rationale of the three graduate courses to be offered. This allowed the members from different disciplines to identify key topics and ideas for framing the course content.

Early on we recognized that the notion of "multiple representations" would be very useful to the teachers, allowing them to recognize the connections among a number of topics that they normally teach in isolation. It was also of great importance to the mathematicians and the specialists in mathematics education. To illustrate what is meant by "multiple representations" it is useful to recall that functions are conventionally represented mathematically through tables of values, algebraic expressions, arrow diagrams, displacements on number lines, graphs in a coordinate space, input-output "machines," and various kinds of descriptions in natural language. In the field of mathematical learning, one also includes personal representations of functions that may or may not be consistent with standard mathematical conventions. The team scientists commonly referred to representations as *models* of extra-mathematical phenomena (data, processes, mechanisms). Meanwhile, teachers normally consider the teaching of algebra as manipulation of symbols and the geometric representation of graphs of functions as separate lessons. We decided to leave the definition of representations somewhat open to interpretation so that it could serve well in mathematical, learning, and scientific contexts and to present multiple representations to the teachers as often as possible throughout the courses.

Interdisciplinary Perspectives

The individual members of the Poincaré Institute often have experience in more than one of the Institute's three foundational disciplines (Mathematics, Mathematics Education, and Science). For example, all of the research mathematicians serve as mathematics educators at Tufts University, and at least some of the Institute's researchers in mathematics education and science have familiarity with mathematics beyond the high school level.

Different disciplines tend to emphasize different aspects regarding *what* teachers should learn to become better teachers of mathematics, *why* they should learn it, and *how* they might best engage students in learning. Such assumptions are not set in stone nor necessarily fully consistent within any discipline. Nonetheless they are important to mention, insofar as they underlie recurring discussions about how the graduate courses should be structured and how the work in the school districts should proceed.

Here we will outline some of the thinking behind various perspectives in the Institute.

Perspectives from Mathematics Education

Our pedagogical approach has its roots in Piaget's constructivist theory of cognitive development and in socio-cultural approaches to learning and development inspired by Vygotsky's work. Their insights into the long-term development of children's understanding of basic logical and mathematical principles provide a rich starting point for mathematics education work. However, their contribution does not directly consider how learning and understanding is reorganized through appropriation of specific mathematical symbol systems and tools such as the conventions of the decimal system, fractional and graphical notation, transformations across conventional measuring units, etc. (Carraher & Schliemann, 2002; Schliemann & Carraher, 2002). While teaching and learning of mathematics as a discipline should unfold from children's basic logical and mathematical understandings, they must lead to more general, complex, and explicit knowledge. To acknowledge this, however, is not enough. We need to analyze how children's logical and mathematical intuitive understandings can be further expanded as children learn mathematics (Vergnaud, 1996). Ultimately, as Piaget stressed, we need to find "the most adequate methods for bridging the transition between (…) natural but nonreflective structures to conscious reflection upon such structures and to a theoretical formulation of them" (Piaget, 1970, p. 47).

Mathematics educators have been arguing for many years that algebra should pervade the curriculum instead of appearing in isolated courses in middle or high school (Schoenfeld, 1995). The weaving of algebra throughout the K-12 curriculum could lend coherence, depth, and power to school mathematics, and replace late, abrupt, isolated, and superficial high school algebra courses (Kaput, 1998). To this goal, in our approach (Brizuela & Earnest, 2007; Carraher, Schliemann, & Brizuela, 2000; Carraher, Schliemann, & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007), functions and their multiple representations (e.g., natural language, line segments, function tables, Cartesian graphs, and algebra notation) play a critical role as an integrative concept, as proposed by Seldon and Seldon, (1992), Dubinsky and Harel (1992), and Schwartz and Yerushalmy (1992, 1995).

Our approach rests on the premise that a deep understanding of arithmetic requires mathematical generalizations and understanding of basic algebraic principles. We view algebra in elementary and middle school as a generalized arithmetic of numbers and quantities and the introduction of algebraic activities as a move from computations on particular numbers and measures toward thinking about relations among sets of numbers and variables. A key idea behind this view is that an algebraic, functional approach to arithmetic topics will lead to better teaching and learning of arithmetic operations, fractions, ratios, proportion, and geometry, main topics in the middle school curriculum. It also leads to considering isolated examples and topics as instances of more abstract ideas and concepts. Multiplication by two, for example, is a table of number facts (1 x 2 = 2; 2 x 2 = 4; 3 x 2 = 6; 4 x 2 = 8) but it also can be understood as a subset of a function over the integers, f(n)=2n, that maps each element from the domain to the co-domain. As such it lays the groundwork for the real-valued, continuous function, f(x)= 2x, which can be represented as a line in the Cartesian plane. In this approach, topics of ordinary arithmetic foreshadow increasingly abstract and symbolic topics.

In addition, in elementary and middle school, the contexts and situations in which mathematics problems are embedded play important roles in learning. Research from diverse perspectives (e.g., Moschkovich & Brenner, 2002; T. N. Carraher, Carraher, & Schliemann, 1985, 1987; Nunes, Schliemann, & Carraher, 1993; Schwartz 1996; Smith & Thompson, 2007; Verschaffel, Greer, & De Corte 2002) has shown that the young learner uses a mix of intuition, beliefs and presumed facts coupled with principled reasoning and argument, instead of relying solely on logic and syntax. However, although rich problem situations provide important points of departure for identifying and working with more abstract structures and syntax, students will eventually need to derive conclusions directly from written system of equations or x-y graphs drawn in the plane.

Likewise, we have often found it useful to begin focusing on students' current ideas, including those that may have arisen outside the classroom. The challenge for teachers in their classrooms, as well as for us in the planning of Poincaré courses, has been to design problems and situations that would trigger the learners' motivation for understanding, their own representations, and their initial intuitive approaches towards solutions. The role of the teacher should then be to further promote reasoning about specific situations, to provide access to new concepts and conventional representation tools, and to allow for abstract knowledge about mathematical objects and structures to emerge. Thus, when working on a given problem, we hope to provide conditions that engage learners in using their own perspectives, ideas, and ways of representing the problem as they come into contact with more advanced mathematical content. Consequently, teachers need be aware of students' typical ways of approaching specific mathematical content, as documented by mathematics education research or by his or her own explorations about actual students in the classroom, together with a view of how students' ideas may relate to the mathematical content to be learned.

Our three longitudinal classroom research investigations revealed the positive impact of this approach (Schliemann et al., 2003; Schliemann, Carraher, & Brizuela, 2012). For example, in a classroom intervention study we implemented from third to fifth grades, teaching weekly early algebra lessons based on the above described views, we found that, at the end of fifth grade treatment students fared better than controls on algebra problems included in the project's written assessments, as well as in problems included in State mandated tests. And the benefits of the intervention appear to have persisted two to three years later, when the treatment students were more successful than their peers in learning to solve more advanced algebra problems (see Schliemann, Carraher, & Brizuela, 2012).

The following is an example of classroom activities we developed in the early algebra project that proved relevant to the work of Poincaré teachers. We presented the following problem to fourth grade students (see Carraher, Schliemann, & Schwartz, 2007):

> Mike and Robin each have some money. Mike has $8 in his hand and the rest of his money is in his wallet. Robin has altogether exactly three times as much money as Mike has in his wallet. How much money could there be in Mike's wallet? Who has more money?

Fourth graders in our intervention study easily accepted the suggestion that *w* can stand for "whatever money there is in Mike's wallet." The instructor then listed, in a table drawn on the blackboard, the various amounts in the wallet in the first column, followed by Mike's total amounts in the middle column, and Robin's amount in the third column. For the first several rows in the table, students determine Mike's and Robin's amounts by recalling the story. For each possible amount in the wallet, they compute the values in each column. They discuss whether Robin has three times as much money as Mike, or three times as much money the amount in Mike's wallet. At a certain point a student notes that Mike's amount is always 8 greater than *w*. Someone suggests writing *w* and *w*+8 as headers for the left and middle columns. Later someone suggests that, because Robin's amount is three times the amount in the wallet, Robin's column be labeled $w \times 3$. From this moment on, students are able to immediately determine the values of columns two ($w+8$) and three ($w \times 3$) from those in column one (w). Inferences can be made solely on the basis of the written forms without having to refer back to the story that generated the forms.

Eventually the students conceptualize $w + 8$ and $3 \times w$ as functions free to vary across all values of w. When they plot these functions in the Cartesian space with w along the x axis they recognize that at one and only one value of w do the graphs intersect, namely, when $w = 4$. They come to realize that this is the only value of w for which the equation, $w + 8 = 3 \times w$ happens to be true. When Mike has less than $4 in his wallet, then Robin will have more than Mike. The situation is reversed when Mike has more than $4 in his wallet. The only time they have the same amount is when $w = 4$.

In the activities of the first cohort Poincaré Institute teachers, we have seen children taking this big step towards more abstract thinking and the use of variables. In particular, in a fourth grade classroom, while a teacher was introducing the idea of displacement of a graph in the plane using both tables and graphs, children spontaneously started to use letters instead of numbers and wrote relationships among these symbolic representations in the form of equations (with two variables).

Perspectives from Mathematics and Science

Building upon the pedagogical and research expertise described above, the interdisciplinary work undertaken since the first planning steps of the Institute has greatly expanded, transformed, and deepened by the joint contribution of mathematicians, mathematics education researchers, and physicists. The following ideas are perhaps the most salient, for they constituted some of the original key topics on which the mathematicians, educators, and scientists first focused their attention upon. And quite a few of the ideas ultimately assumed prominent roles in the courses for teachers. They are:

1. Elementary and middles school children are far more capable of algebraic reasoning than they were thought capable of just a couple of decades earlier.
2. The mathematical concept of function, normally introduced at the onset of high school, has considerable potential in uniting diverse topics in early mathematics and bringing out the algebraic character of arithmetic.
3. Mathematical concepts are intricately associated with representations that are used for making sense of diverse situations, inside and outside of mathematics.
4. Much of young students' burgeoning knowledge about algebra and functions is bound up in trying to explain extra-mathematical situations, hence modeling.

The focus on functions was one of the critical decisions we faced early in finding a common ground on which the three basic disciplines could work together with the middle school teachers from the partner districts. This meant having a clear sense of the objects of study as well as some sense as to how these objects could contribute to teaching and learning in the districts. "Algebraic reasoning" and "early algebra," although generally consistent with our planned focus, are not well defined mathematically and thus do not offer the needed traction for an interdisciplinary partnership. Algebra itself is a vast domain of mathematics as well as a language for expressing mathematical ideas in many sub-domains of mathematics.

It should be recognized, however, that functions are rarely prominent in middle-school curricula. On the contrary, they are mainly associated with high school grade levels in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010). Although NCTM's (2000) standards are generally compatible with function-based approaches to middle school mathematics, implementation of the NCTM's standards are often framed in terms of pattern extension, a relatively ill-defined notion, as opposed to assignment rules of functions.

In mathematics, functions have proven to be a high-level construct of special importance in the history of modern mathematics. Functions are well defined and susceptible to rigorous mathematical examination. For scientists, functions are perhaps *the* key mathematical tool for modeling properties and processes of the physical world through relations among measured variables. Scientists regard functions as lying at the heart of modeling. Their focus on physical quantities and on functions to describe and explain physical and real world phenomena is at the core of our pedagogical approach. Furthermore, the educational research team had gathered compelling evidence that functions could be introduced early on in the context of the four basic arithmetical operations (Schliemann, Carraher, & Brizuela, 2007).

By focusing on functions as the core concept in the development of middle-school teachers, it may have seemed that we were introducing new, more advanced topics into already-overcrowded middle-school curricula. In fact, we were proposing functions not as additional content but rather as organizers of existing content. To this end, we needed to first challenge the widely held premise that there is no wiggle room in the middle school math curriculum. We had to convince teachers that many topics taught in isolation are in fact different aspects of the same idea. Teaching them together not only leads to a better understanding but also economizes instruction so it can be devoted to a deeper exploration of topics. For example, rational numbers, ratio and proportion, and linear equations and slope can be combined in a lesson that would help students notice the true meaning of all these notions and their use. Similarly, in any given class, teachers are encouraged to explore problems though multiple representations, especially diagrams, graphs, tables of values, written numeric and algebraic notation, and linguistic constructions.

Reaching Students Through their Teachers

A substantial amount of our work at the onset of the Poincaré project dealt with teaching students rather than their teachers. The "Early Algebra" project carried out research in which the investigators went into the classroom at regular intervals for an extended time and implemented their lessons as a supplement to what was regularly taught in a math class by the classroom teacher. Mathematicians had advised MAT and doctoral students in Math Education but their own teaching was only to undergraduates. While some members of the group participated in the Fulcrum Institute, this was a very different type of experience: Fulcrum was addressed to teachers at all K-12 levels, dealt with science, and teachers came on their own, while this project is targeted to 5-9 Math teachers that work together with their colleagues in their districts.

How could we expect that the Institute may impact student learning if our contacts are solely with the students' instructors? We address this challenge in several ways.

For one thing, we have chosen topics directly relevant to the middle school curriculum. In our case, these topics where numbers (fractions, rational numbers, integers and divisibility), arithmetic (the basic operations of addition subtraction and multiplication), functions and their representations through graphing and tables, slopes, solution of equations both linear and polynomial, modeling and applications. As we mentioned earlier, these can be unified under the umbrella of the study of functions. Then numbers become domains for these functions, arithmetic operations become examples of such functions. Slope is an important characteristic of a (nice) function and linear equations can be solved by applying suitable functions to the plane. Modeling and applications are in many ways a scientist's take on functions.

Our challenge then was to first provide the teachers with the background in mathematics they needed to understand these concepts, their interconnections, and their position in the big picture. Then we had to show them specifically how the topics they teach in the classroom relate to this big picture framework. And finally we had to get them ready to develop activities for their students that build on this approach.

The first two goals have been tackled with a series of lessons in written and video format. These lessons increasingly considered together the mathematics and pedagogical aspects of a topic, in an integrated way, rather than separately. Because both mathematical knowledge and its teaching need to be constructed by the learner, special attention was given to the choice of "homework" questions that go beyond confirming that information in the text has been rote learned. The homework questions are designed to trigger discussions and understanding at a deep level and allow multiple approaches. They are based on the lessons and relate the mathematical framework of the courses to the specific topics that are part of the middle school curriculum. Some of these assignments include analyzing a situation that appears in a classroom, presented either through a videotape of such a class or through written work of the students. Exploration, discussion and appropriate use of technology have been encouraged throughout.

The above last step aims at making sure that the teachers feel confident with the material to the point that they can bring it themselves to their students and that their teaching methods are conducive to learning mathematics with understanding. To encourage these attitudes, right after they have learned about selected topics, teachers either interview their students on the topic or develop a learning activity related to the topic and analyze its implementation. They present their work as written reports often accompanied by video clips. They discuss each other's reports and provide feedback for improving the activities. In the final two weeks of the course, they implement activities in the classroom based on their lesson design or interviews conducted during the term.

Integrating Perspectives

Our initial ideas had to be assessed against the needs of the teachers in the districts. In our preliminary visits to schools, during the grant-writing period, our suggestions to focus the courses around algebra met with considerable enthusiasm. Teachers and administrators alike recognized the need to improve the teaching and learning of algebra. Algebra I and II were regarded as major obstacles to success in high school and preparing students for these courses was seen as a goal for middle school. Generally speaking, strong students take algebra I in middle school, whereas those who lag behind take pre-algebra and leave algebra for high school.

In our discussions with teachers, we tried to determine some specifics topics for the courses but they were not clear on what would make a difference in their classrooms. Somehow, they were open to the topics we would choose. Although we had a clear idea of what type of mathematics is important and what type of understandings students should have by the time they leave the educational system, we were less confident about how to prepare the current teachers to teach in an effective way. Most of the previous work and expertise from educational researchers in the early algebra studies dealt directly with the students rather than their teachers.

Many of the fifth and sixth grade teachers had been trained and licensed to teach elementary school and most of them never expected, when they were in college and during their professional preparation, that they would be mostly math teachers. At the other end of the spectrum, those teaching ninth grade were licensed to teach high school and could find themselves in any given year teaching anything from algebra I, or even pre-algebra, to AP Calculus. Needless to say, the educational background of the teachers was also very diverse. Many teachers had only a bachelor's degree and provisional licensure; some had a master's degree. Majors ranged from mathematics and the sciences to the humanities.

Course Development

Our initial proposal had only course titles and a paragraph description for each course: the first course was to deal with functions and their representations, the second course with transformations and their use in the solution of equations, and the third course with change as modeled by functions. These big ideas served as the basis for the three courses offered to the first cohort of teachers. As described later in this article, this initial proposal has been constantly expanded and adapted, as we implemented course units, examined teachers' work, and carefully considered their suggestions and feedback to course content, structure, activities, and materials. The content and structure of the courses as offered to the first cohort of teachers are described below.

Course 1: Representations

The main goal of Course 1 was to present the idea of function and its multiple representations and uses, especially in modeling arithmetic operations from the middle school curriculum. We wanted to make invertibility a major focus of the course, not only because it united the arithmetical operations, but also because it was fundamental to algebraic operations on equations. It is a crucial and unifying notion that allows one to deal with a multitude of topics, from the relation between addition and subtraction to the fact that one cannot divide by zero or that positive numbers have two square roots.

The course was divided into three units: functions and relations, functions on the real number line, and representation of functions on the plane. Units were divided in weeks, each with a main focus on mathematics, education, or science. Eight of the fourteen weeks of Course 1 focused on the mathematics of functions and relations; two weeks were dedicated to mathematical modeling in science, and four of the fourteen weeks focused on teaching and learning.

Teachers were divided into online teams of six teachers per team, with two instructors (one educator and one mathematician or physicist) as tutors. For each week, teachers were first presented with an exploratory activity. In "math" weeks, the assignment came with a set of notes and videos containing mathematical background. In many cases not much formal knowledge was needed for this first exploration. When this preliminary assignment was completed, more materials and a second set of more complex questions would come up, along with short essays presenting a mathematician's, a scientist's, and an educator's perspective on the main topic. In this second phase, teachers were asked to comment on the work of their online team peers. They were also encouraged to make use of the general forums where they could post questions and ideas and discuss any topic related to mathematics or classroom practice.

The faculty had invested much time and effort in the course preparation and delivery. However, not everything ran smoothly. At the beginning, in the case of some units, we overestimated the level of mathematical knowledge of our participants and greatly underestimated the amount of time it would take them to complete an assignment. Coordination among the faculty designing different parts of the course was not optimal and integration among the disciplines while present, was not fully achieved.

Despite the above flaws, learning was taking place and enthusiasm towards the program pleasantly surprised us. Even in those units in which we had aimed too high, the teachers were heavily engaged and their effort and cooperation coupled with instructor support led to impressive outcomes and a great sense of achievement.

The teachers were particularly drawn to the “education weeks,” for which they analyzed video of classroom activities or samples of student work produced by the early algebra previous research. Teachers watched and listened carefully and marveled at how much algebra young children were able to grasp. Some teachers modified the activities and used them in their own classrooms.

By the end of the semester, we had learned as much from our teachers as they might have learned from the course. We had the luxury of the summer break between the two courses and we spent most of it preparing Course 2.

Course 2: Transformations

If the Poincare Institute was to have a real impact, teachers should be applying what they learned from the courses into their classroom. At the same time, in view of the needs of our participant teachers, mathematical content should not be shortchanged.

Taking into account what we witnessed during Course 1, we decided to revise the course structure, organizing Course 2 into five units, each integrating mathematics, science, and educational views. During the first two weeks of each of the first four units, mathematics, modeling applications, and educational insights were to appear together. As planned, in the first week of each unit in Course 2, the teachers explored the topic, discussed models of teaching the unit's specific subject, analyzed students' ideas and challenges in learning the subject, and solved problems relevant to their learning and teaching. In the second week, they were guided to develop a deeper understanding of the mathematical content of the unit, again through notes, videos, problem solving, and online discussions, working on assignments that would require them to think through the questions often from several of these points of view. Then, in the third week of each unit, groups of three to five teachers jointly designed a learning activity for possible future implementation, based on topics from the previous two weeks. For their final, individual project, each teacher implemented in their classroom one of the learning activities they had planned. They videotaped this activity and analyzed his/her teaching and their students' learning in a short individual report, which was posted online, along with selected classroom video clips, and discussed by other teachers.

At the request of teachers we opened Course 2 with a more in depth treatment of fractions and divisibility than what had been presented in Course 1. We then moved to transformations of the line, as a geometric model for arithmetic operations, followed by transformations of the plane. Transformations were then used to analyze graphs of functions and to present a geometric way of solving equations.

To exemplify our work, let us focus for a moment on the unit on transformations of functions (unit 4 of Course 2). In retrospect, this unit was overly ambitious, insofar as we asked each teacher to work through a number of new ideas as well as practical applications to their classroom. Nevertheless, it was well received.

In the previous unit, the teachers had been examining transformations of the line and of the plane, specifically, translations, dilations, and reflections. (We did not include rotations, which, although interesting, have a more complicated algebraic representation and are less useful for studying graphs of functions and for solving equations.) Through their familiarity with invertibility, the teachers had a rudimentary notion that one could move back and forth between functions. This would be greatly extended in this part of the course.

The transformation of functions unit opened with the story of a train first moving along a track at constant speed, then stopping for a brief period before restarting the journey. Teachers were asked to graph in the Cartesian plane the distance function in terms of time. They then considered variations of the initial trip, such as a train leaving later (but otherwise taking the same trip as the earlier train), or coming from the opposite direction, or moving faster or slower. They were asked to relate the story variant to the initial trip both geometrically and algebraically. They also applied the same type of analysis to other modeling options such as cost functions in terms of weight.

The following week, the teachers worked with the relation between algebraic and geometric presentations in the abstract. They were then presented with a linear equation interpreted in terms of the intersection of two lines and looked at the types of transformations that preserve solutions and their use in solving the equation. Finally, transformations were used to bring the equation of a parabola to the standard form and this was used to obtain the quadratic formula. Several of these topics were revisited in Course 3 and studied in more depth.

Course 3: Invariance and Change

The Course 2 structure, with three-week units and educational activities explored by the teachers in the third week, was very successful and was therefore utilized for Course 3. However, in week three of each unit the teachers could either develop plans for learning activities (as was the case in Course 2) or interview individual students on problems designed to explore student thinking, their spontaneous solution strategies, and difficulties they would face. Almost all teachers opted to interview students. This then became the basis for the development, implementation, and evaluation of a classroom activity they developed as a final project for Course 3.

The mathematical content of Course 3 began with an analysis of solutions of equations, starting with the meaning of the equal sign, moving from linear equations to quadratic and higher order, and understanding the relation between factoring and roots of an equation. We then explored change with the idea of slope and its meaning. The fourth unit looks at modeling and real life applications and how to teach children to make the connection between the math and word problems. As in Course 2, the final two weeks of Course 3 were dedicated to the development, implementation, and analysis of a learning activity.

Weekly Meetings In Schools

As we mentioned above, the teachers meet after school in their districts once a week. They are free to choose what they want to discuss at their meetings so long as it is related to mathematics and its teaching in their classrooms. Once a month, the faculty pair assigned to that district attends the meeting.

The monthly meeting with Poincaré faculty has been a very useful forum for teachers to express their concerns and suggestions and a good way to further monitor their progress. Some of the teachers have built personal ties with their faculty mentors and are no longer hesitant to contact them when difficulties come up in course material or even in advanced mathematical topics they need to teach. Sometimes, however, especially during the first semester, the weekly meeting became a place to moan about what was wrong in the district. Technology glitches in Course 1 implementation also took a good amount of meeting time. The situation changed dramatically during the second semester, when Course 2 was offered. The main reason was that the new course structure, requiring a group project related to a teaching activity, became an important topic for discussion. All of our participants chose to form a group with other people in their district, most often with those in the same school as themselves. The weekly meetings became then the natural time to plan and discuss these projects. While this has not been the case at all the meetings, we found that, when it happened, it led to very fruitful discussions that helped the teachers develop substantially improved activities or to discuss in depth the thinking and learning of their students. For example, in three districts, after the teachers had submitted their analysis of interviews with individual students on the problem shown in Figure 1, the monthly meeting with Poincaré faculty was dedicated to the analysis of students' spontaneous ideas about how to represent the problem.

• Elizabeth Excited, Patty Planner, and Carly Catch-up are all cousins. Next year, they would like to send their grandmother on a big vacation for her birthday, but the trip will cost $3,000. Elizabeth, Patty, and Carly decide that they have one year to raise $1,000 each.

• Elizabeth starts saving a lot of money on the very first day and realizes that she would like to have some money for herself, too, so each day, she puts less money into her bank account than the day before.

• Patty figures out exactly how much money she will need to save each day to reach $1,000 in one year and she puts the same amount of money into her account each day.

• Carly begins by saving very little but she realizes that she will not save enough money in time, so each day she puts more money into her account than the day before.

• All three girls saved exactly $1,000 at the end of the year.

• Draw graphs showing how much money Elizabeth, Patty and Carly had during the year.

Figure 1: The problem students' were asked to represent during individual interviews (Adapted from Yerushalmy and Schwartz, 1995).

In three different districts, during the meetings with Poincaré faculty, the teachers discussed the graphs produced by the students in terms of:

- Use of bar graphs
- Attempts to transition from bar graphs to line graphs
- Representation of savings month by month versus representation of accumulated savings.
- Challenges of representing linear vs. no linear functions.
- Possible intuitive approaches to the representation of step functions.

Teachers discussed students' views as revealed in their interviews, explored the possible origin of students' difficulties, and considered ideas on how to develop learning activities taking into account what teachers found in the interviews. Teachers acknowledged that, even though the children did not know the formal conventions for graphs, many showed interesting and often coherent representations for savings by month or accumulated savings.

Difficulties identified and discussed were related to:

- What the axes represented.
- The tension between bar graphs and line graphs and syncretism.
- The arriving point for all lines (1-year, $1,000).
- The tension between the representations of linear vs. non-linear functions.
- The difficulty of representing Elizabeth's savings as starting from the origin (she saves more at the start).

Some teachers then decided to develop a learning activity based on this problem, considering how students' intuitive solutions can be a step towards learning about graphs on non-linear functions.

The participating teachers seemed to enjoy the weekly meetings for a variety of reasons. The most often cited reason for enjoying the meetings was that they allowed them to communicate with the other teachers in the district, understand the continuous progression of the syllabus, form personal bonds with their colleagues, and have a forum for discussion of teaching issues. For many, this was an opportunity they never had before and they seemed to be eager to keep these meetings once their participation in the Poincare Institute was over.

One goal we have, as the second cohort of teachers start taking the courses, is to make sure that teachers from the first cohort will join the new teachers in the weekly meetings, an important aspect to achieve permanent changes in teaching and learning at their districts.

Looking Ahead

Program Revisions

As the first cohort of teachers approached graduation, we started revising the courses for the next cohort, taking into account the written suggestions from our team members, our experience in the first round, some preliminary research results, the needs of participant teachers and their students, and the many suggestions provided by the teachers, online or during our face-to-face meetings in the districts. We began by asking all participant faculty, researchers, postdoctoral fellows, students, or staff members to give us a view of what they would like to do in the second round. Except for a couple of extreme opinions, we were surprised to see that most Poincaré team members recognized the importance of contributions from mathematics, mathematics education, and science. At least to some extent, these two years of working together made mathematicians, educators, and scientists more interested in the work of each other and more appreciative of the role of science and modeling in learning mathematics.

The collaboration process among mathematicians, educators, and physicists at first consisted in individual contributions that were made accessible in a given week. We then evolved into jointly producing course notes which, even though they emphasized one or another perspective, resulted from the collaboration and points of view from the different areas. Administratively, we also improved the process for developing course materials. In revising the courses to be offered to the second cohort of teachers, each unit is produced by a small interdisciplinary team of up to three people. Those in charge of each unit post the first draft of materials for feedback from all course team members, including a teacher from cohort 1. The feedback is compiled by an interdisciplinary editorial board who then asks the authors to implement the relevant changes. This process of feedback takes place twice, until the editorial board approves the final version of materials.

In terms of content, developing the courses goes beyond the list of topics that we want to cover. The three Poincaré courses are meant to develop habits of mind and foster appreciation for the subject, at least as much or even more than specific topics. We mostly agree on what these habits and ideas should be. We feel we have succeeded in passing some of these to some of our teachers, but we are far from our goal with others.

Among the mathematical abilities that we would like to promote are an awareness of the roles of conjecture and proof. On the one hand, while we do not expect or even desire that teachers be able to write detailed and polished proofs of the sort required of an advanced math major, we believe they should understand that checking a few examples of a result is not sufficient to confirm the truth of a statement that could be applied in much greater generality. On the other hand, playing with a few examples is the only way to get a feeling about the subject that would allow them to, then, formulate a conjecture. We would like teachers to feel sufficiently comfortable with these ideas so that they can model them in their lessons with their students.

We tried to incorporate some ideas about conjectures and proofs during face-to-face workshops offered in the kickoff meetings as well as in notes and assignments. While there seems to be a noticeable awareness of what conjecture and proof are, we are far from having reached our goal. With the second cohort, we will try to further incorporate proofs in the work of each unit of each course, using simple examples to draw the attention of the teachers to the method as much as to the final result. We will also ask the teachers to try their hands at it, providing help and frameworks as needed.

Something similar could be said for modeling and problem solving, in general. In the first round of courses, we might have been too explicit about modeling, trying to give the teachers words for a variety of phenomena instead of having them work more on developing mathematical models for particular situations. In addition, as assignments were normally related to a topic, those that were only loosely related to a particular mathematical content, or that used many aspects of the content at the same time, have failed to promote deep understanding of modeling and problem solving strategies. We attempted to address this limitation only towards the end of course 3. In planning the second round, we are making a point of offering the teachers a chance to work on these types of modeling and open-ended problems at regular intervals. The biggest obstacles to overcome arise from the fact that some teachers prefer to be sure that they will be able to give the right answer to all of the questions asked and feel uncomfortable when they have to deal with a problem that cannot be solved with the tools they have just learned.

Another aspect that we want to emphasize is "what lies beyond the horizon." Teachers should be aware that there is a lot more mathematics than what they teach and that, like a work of art, mathematics can sometimes be enjoyed just for the pleasure of it, even without understanding all the details.

Some of the structural aspects of the courses seem to have been working very well in Courses 2 and 3 and are being preserved in future cohort offers. For example, courses will continue to be divided into three-week units. The first two weeks of a unit will include mathematics, education, and science content in an integrated way, and the third week will be a teaching-related exploration of the content covered in the previous two weeks. The first course will include teaching and learning demonstrations that the teachers will analyze, as a training ground for the other courses. Teachers will interview some children about a topic related to what they learned in the unit and try to understand the students' ways of thinking, or they will design an activity related to the topic that could be used in their classrooms, as both types of activity proved to be useful for cohort 1 teachers.

Two of the issues we want to address are how to foster intense and focused online discussion and how to provide useful feedback to teachers. To be clear, there has been a substantial amount of discussion, often inspired by the lessons or, at other times, by teachers' experience in the classroom. Most of it takes place in a general online discussion forum that is part of the platform for course delivery. A lot of discussion happens also in face-to-face weekly meetings at the schools and during office hours regularly offered to help teachers as they work in the weekly assignments. Since the "third week" activities are teamwork, some discussion is happening as teachers work on the assignments. The regular work for Weeks 1 and 2, however, are posted on-line and can only be viewed by teachers that are members of that team (and by all faculty members). In some of the on-line teams, there is regular discussion of assignments with teachers posting drafts of their answers and helping each other gain a better understanding. Other teams, however, hardly ever discuss their peers' work. We are trying to develop a new model that will insure that discussion on their work happens for all teams and in all weeks of each unit.

In terms of feedback on teachers' responses to the course assignments, we spent a substantial amount of time on a task that teachers might not take so much advantage of because, by the time they receive it, they are already working on the next unit. For the second cohort, instead of giving feedback once a weekly assignment has been completed, we will provide on-line help to each group while the work is being done and will post some model answers at the end to help teachers decide for themselves if they were on the right track. As before, on-line office hours will still be available but individualized feedback on each participant's submission will be briefer.

Regarding mathematical content, it is not substantially different from the first round, with one exception. In round one, we introduced functions as sets of ordered pairs from the Cartesian product of elements from the domain and co-domain. Although this makes sense, mathematically, it was too abstract a starting point for middle school mathematics. We decided that in the second round we would emphasize, in the beginning, the notion of functional dependency; namely, that output values (the image) were "dependent" on input values (from the domain). This also allowed us to highlight, early on, mappings involving the real numbers.

Presently we start with a study of the real line and incorporate functions as a transition between arithmetic and algebra, skipping our previous attempt with relations. We also agree that an earlier introduction of a variety of functions and a focus on rate of growth would help teachers understand that not everything is linear. The content of the courses offered to teachers in the second cohort is described in the Appendix.

Evaluating the Impact of the Program

Given that our first cohort of teachers has just graduated, a large amount of data remain to be analyzed. The impact of the Poincaré Institute will be analyzed in terms of teachers' and students' evolving understanding of mathematical content and representations and in terms of teachers' implementation of effective teaching activities, as demonstrated in written assessments designed by the project, videotaped classroom discussions, and course assignments.

Teachers' written assessment data and videotaped lessons have been and will be collected among Poincaré teachers and their colleagues, at the start and end of the five-year project and, for teachers in each of three cohorts, at the start and end of each three-course sequence. Data on student learning are being collected through written assessments designed by the project, state-mandated assessments (MCAS, NECAP), and videotaped classroom discussions. Comparisons between pre-and post-written assessment measures and between participant and non-participant teachers and their students will allow for evaluation of the impact of teachers' progress and of their students' success.

Dependent measures cover the mastery of mathematical content (Numbers, Fractions, Ratios, Proportions, Relations, Linear and Non-Linear Functions, and Algebra Equations), algebra in modeling, and use and interpretation of mathematical representations. Our analysis will focus on willingness to explore problems in depth, considering all potentially relevant aspects before proposing solution methods and answers, use of multiple representations for functions (natural language, tables, number lines, graphs, written notation), and use of algebra as a modeling tool in extra-mathematical contexts. Detailed qualitative analysis of students' questions, answers, argumentation, justifications, solutions, and written work, as they participate in videotaped lessons before and after their teachers are taking courses, will allow further insights into the project's impact on student success.

The Poincaré Institute aims to substantially improve the teaching and learning of middle school mathematics and the project's research team is working at collecting data that will allow us to show that this is happening. While it is too early to present quantitative data on teachers' and students' progress, we do have some anecdotal evidence and preliminary analyses showing that change is actually happening, if not in how much children are learning, at least in how teachers are teaching.

As we mentioned in the course descriptions, during Course 2, each team of teachers was asked to design four activities related to the content of the course that could be implemented in their classroom. Then, at the end of the course, each individual teacher had to implement one of these activities in his or her classroom, videotape the implementation, and analyze its results.

In most groups, there was a notable progression in the quality of the activities designed over the semester. While the first activity was usually an immediate adaptation of something in a textbook, without much thinking about how it could help students' learn, the last few showed a much richer and careful design, with examples carefully adapted to the goal, and much better use of a variety of approaches and representations. For instance, teachers' learning activity plans show, from the start to the end of Course 2, a clear increase in the number of alternative representations for the math content they proposed to teach, with an average of 2.56 kinds of representations for Unit 1 (with half of the teachers only using one or two kinds of representations), to 4.88 kinds in Unit 4 (with only one plan using fewer than three kinds of representations). Most of all, teachers see a much clearer connection between the algebraic and geometric presentations of a given concept. The teachers, themselves, are very aware that this is something that has permanently changed in their understanding of mathematics and are very happy to discover for themselves and present to their students this new way of looking at algebra. Here is a teacher's comment in one of the discussion forums for Course 3:

> … my biggest walk-away will be the ability to show kids all the great connections between algebra and geometry. The connection between the two when we were working with transformations on the number line and the plane were very enlightening for me and gave me a deeper understanding, which will definitely benefit kids that I work with.

Or from another teacher at the end of Course 2:

> My textbook presents equations in chapter with solutions using transformations, no graphs. Graphs of linear equations come in chapter 4. When reading the notes for unit 4 week 2, I had an epiphany: I need not wait for the chapter on linear equations to ask the students to represent their solutions graphically.

Summary

The implementation of Poincaré courses has been generally successful for the first cohort of teachers. As we plan and approach the offer of courses to the second cohort, we hope to improve the collaboration between all Poincaré participants and to correct possible flaws in the design of the different components of the project.

References

Carraher, D. W. & Schliemann, A. D. (2002). Is everyday mathematics truly relevant to mathematics education? In J. Moshkovich & M. Brenner (Eds.) Everyday Mathematics. *Monographs of the Journal for Research in Mathematics Education, 11*, 131-153.

Carraher, D. W., & Schliemann, A. D. (2007) Early Algebra and Algebraic Reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669–705). Charlotte, NC: Information Age Pub.

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Harel, G. & Dubinsky, E., (Eds.), (1992). *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes, Vol. 25, pp. 261–289). Washington, DC: Mathematical Association of America.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by 'algebrafying' the K-12 Curriculum. In National Council of Teachers of Mathematics and Mathematical Sciences Education Board Center for Science, Mathematics and Engineering Education, National Research Council (Sponsors). *The nature and role of algebra in the K-14 curriculum* (pp. 25-26). Washington, DC: National Academies Press.

Kaput J., Carraher D. W., & Blanton M., (Eds.). (2008) *Algebra in the early grades*. Mahwah, NJ, Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Seldon, A., & Seldon, J. (1992). Research perspectives on conceptions of function: Summary and overview. In E. Dubinsky & G. Harel (Eds.), *Concept of function: Aspects of epistemology and pedagogy* (pp. 1–21). Washington, DC: Mathematical Association of America.

Schliemann, A. D. & Carraher, D. W. (2002). The evolution of mathematical understanding: Everyday versus idealized reasoning. *Developmental Review, 22*(2), 242-266.

Schliemann A. D., Carraher D. W., & Brizuela B. M. (2007) *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice*. Hillsdale, NJ: Erlbaum.

Schliemann A. D., Carraher D. W., & Brizuela B. M. (2012). Algebra in elementary school. In L. Coulange & J.-P. Drouhard (Eds.), Enseignement de l'algèbre élémentaire: Bilan et perspectives. Special Issue of *Recherches en Didactique des Mathématiques,* Hors Série, pp 109-124.

Schoenfeld, A. (1995). Report of Working Group 1. In C. B. Lacampagne, W. Blair & J. Kaput (Eds.), *The algebra initiative colloquium* (Vol. 2, pp. 11–18). Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement.

Schwartz, J. & Yerushalmy, M. (1992). Getting students to function in and with algebra. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes, Vol. 25, pp. 261–289). Washington, DC: Mathematical Association of America.

Schwartz, J. L., & Yerushalmy, M. (1995). On the need for a bridging language for mathematical modeling. *For the Learning of Mathematics, 15*(2), 29–35.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning* (pp. 219-239). Hillsdale, NJ: Lawrence Erlbaum.

APPENDIX

Content of the Courses offered to teachers in the Second Cohort

Course 1: From numbers to functions

UNIT 1: *Real numbers*. An introduction to the real line, fractions and their multiple representations, classroom applications and use of numbers in modeling.

UNIT 2: *From numbers to functions*. An introduction to functions: the intuitive idea of function, its use as assignments and as a constraint between two types of quantities, and the formal definition of function. Composition of functions. The vertical line criteria. Use of functions in modeling. Examples include simple arithmetic operations (addition, product) and also functions on objects other than numbers. Special attention to multiple representations of functions (verbal, arrows, tables, algebraic expressions and graphs).

UNIT 3: *Examples of functions*. An expansion of the previous unit focused mostly on examples of functions of one real variable, especially those examples that appear commonly in mathematics and science: linear functions, absolute value, monomials, exponentials and step functions. Some examples of "compound functions" like those obtained from the simpler pieces by composition, addition or product.

UNIT 4: *Division*. The various interpretations and applications of division. Functional approach to ratio and proportion. Division with remainder, decimals and decimal representation of rational numbers. A basic introduction to divisibility for integers and decomposition into product of powers of primes.

Course 2: Transformations and equations

UNIT: 1 *Transformations of the plane*. Functions of two variables, in general, building on the examples of addition, multiplication and division already introduced. Translations, dilations and reflections on the plane and comparison with similar functions on the line. Compositions and inverses of these functions.

UNIT 2: *Transformations on the graph of functions*. Translations, dilations and reflections acting on the graphs of functions. Interpretation of changes in the data modeled by a function in terms of transformations to the graph. Algebraic representation of transformations for the graph of a function. Solution of linear equations using transformations and the connections between algebraic manipulations and geometric representations.

UNIT 3: *Equations*. Geometric and algebraic representation of equations and their solutions. Parabolas and their equations under transformations. The quadratic formula.

UNIT 4: *Divisibility for integers and polynomials*. Recall of the concept of divisibility for integers. Unique factorization for integers as product of primes. The Euclidean algorithm for the greatest common divisor. Review of basic facts about polynomials. Divisibility for polynomials, unique factorization. The relations between roots and factoring for polynomials. The number of solutions of a polynomial equation.

Course 3: Change and invariance

UNIT 1: *Slope and rate of change*. Slopes as indicators of the rate of change of a function. Average rate of change of a function over an interval and its geometric representation as slope of a secant. Instantaneous rate of change as the limit of an average rate of change over small intervals and its geometric counterpart as slope of a tangent line. Comparison of the growth of linear functions to other types of functions

UNIT 2: *An example-based introduction to the idea of limit.* Decimals with an infinite number of digits as limits of sequences of some special functions. The idea of limit and of vertical and horizontal asymptotes (1/x, exponential). Comparison of the growth behavior of these functions to other types of functions. Applications to arithmetic operations and the middle school classroom (dividing by zero, dividing by large numbers). Approximating solutions to equations.

UNIT 3: *The slope function*. Introduction of the derivative as the function "slope at the point" or rate of change at the point. Comparison of derivatives for different types of functions (constants, linear quadratic, exponentials, 1/x). Reconstruction of a function given its derivative. Applications to issues relevant to middle school students, to modeling and science.

UNIT 4: *Change and invariance of shapes under transformations*. Transformations that preserve and do not preserve the shape of graphs. Lines through a point and solutions of linear equations.