Poincaré Institute: Course Descriptions and Impact on Teachers and Students

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Abstract

We report on a teacher development program aimed at improving mathematics teaching and learning from grades 5-9. The 18-month online program arose through a partnership of mathematics education researchers, mathematicians, physicists, and nine school districts. The program employs functions as a lens to reinterpret, study in depth, and interconnect topics in the curriculum and to promote mathematical understanding by drawing upon students' reasoning about relations among quantities and numbers. The rationale, content, and activities for teachers drew inspiration from prior studies on early algebraic reasoning, teacher effectiveness, and student learning. We report on changes in teaching and learning after teachers participated in the program and consider whether these changes may be attributed to specific characteristics of the program.

Keywords: teacher development, arithmetic, algebra, functions, variables, representations, teacher interaction, classroom teaching, student learning, student reasoning, grades 5 to 9, online discussions.

Poincaré Institute: Course Descriptions and Impact on Teachers and Students

We report on a development program for teachers in grades 5 to 9 built upon the view that functions can take on a productive role throughout the K-12 mathematics curriculum, serving as a lens through which to view not only the teaching of algebra, but also key curriculum topics regarding numbers, arithmetical operations, transformations, and change. The program builds upon previous research on functional reasoning among young students and on studies of teacher effectiveness and student learning. We describe the program's theoretical foundations, its underpinnings in research about teaching and learning, and its implementation. We also consider evidence regarding the possible impact of the program on teachers and students as well as its shortcomings and highlight some of the challenges that lay ahead as the field moves forward.

The program:

(a) Employs functions as a means of interpreting, studying in depth, and highlighting interconnections across a wide range of topics in the curriculum, such as fractions, divisibility, ratio and proportion, graphs, equations, growth, and change.

(b) Aims to integrate content and pedagogical knowledge by requiring participants to solve mathematical problems and to design and implement learning activities on the same topics.

(c) Encourages teachers to identify and take into account students' reasoning and ideas in planning and implementing learning activities.

(d) Includes a wide variety of open-ended problems about everyday and scientific situations.

(e) Promotes teachers' discussions and collaboration about mathematics and pedagogical activities.

The program requires three semesters of intensive work (10-12 hours/week). So far, it has been offered (mostly online) to nearly 250 teachers who constitute a major part of the middle and beginning high school teachers in 10 New England school districts, with combined population of more than 40,000 students a year.

Functions in Mathematics and in the School Curriculum

The concept of function evolved over many centuries and now plays "a central and unifying role" in mathematics (Selden and Selden, 1992). A **function,** $f: S \to T$ (also called a **map**), consists of three pieces of data: two sets S and T and a rule that associates to each element of S (the **domain**) a unique element of T (the **codomain)**. This modern view has been built with the rigor of set theory. It differs markedly from earlier, "classical" approaches. Galileo employed the concept of function when trying to express fundamental quantitative principles or laws by applying mathematical reasoning to quantitative statements to deduce new physical laws (Kline, 1962). The concept of function was implicitly used well before 1694, when Leibnitz first employed the Latin term, *functio* (Piaget et al., 1977). As early as 1361, Oresme envisaged constant acceleration in map-like format as a linear relation between time (displayed as latitude) and velocity (displayed as longitude), thereby foreshadowing modern graphical representations of functions (Boyer, 1968).

The view that reasoning about functions has a major role to play in *secondary* mathematics education has been increasingly embraced over the years by mathematics educators (e.g. Harel & Dubinsky, 1992; Oehrtman, Carlson, & Thompson, (2008); Schwartz & Yerushalmy, 1992). Following the National Council of Teachers of Mathematics Standards (1989, 1991), functions and their representations became a core feature in the teaching of algebra. Chazan (1999, 2000) reports that making functions central to an algebra course helped his students solve problems before learning standard methods to solve equations and changed his experience as a teacher. Schwarz & Dreyfus (1995) and Schwarz, Dreyfus, & Bruckenhart (1990) examined the effectiveness of a 9th grade curriculum built around a software environment that integrates graphical, tabular, and algebraic representations. Students in their curriculum reportedly better recognized and coordinated properties of functions across different representations while solving verbal problems. Huntley et al.

(2000) evaluated the impact of a high school algebra curriculum built upon investigations of real life contexts, interweaving strands of algebra and functions with other math content, and making frequent use of graphing calculators. Students who had received that curriculum performed better at solving contextualized problems, while those in courses prioritizing manipulation of equations performed better in manipulating decontextualized symbolic expressions.

Much as functions appeared historically well before a theory of functions arose, we believe that functions have an important place in mathematics teaching well before they are formally introduced (usually in grades 8-10). We suggest that much is to be gained by having pre-secondary students engage in expressing associations between physical variables before they are formally introduced to functions or encounter functions in equations, where variables are "unknowns" constrained to one or two values, namely, solutions.

In fact, functions are already part of the school curriculum before the teaching of high-school algebra, even if not named or rigorously defined. The following informal definition of the term, *function*, offered by Gowers, Barrow-Green, and Leader (2010), illustrates how functions enter mathematics before being formally introduced:

> One of the most basic activities of mathematics is to take a mathematical object and transform it into another one, sometimes of the same kind, and sometimes not. "The square root of" transforms numbers into numbers, as do "four plus," "two times", "the cosine of," and the logarithm of." (p. 10).

Just as "four plus", and "two times" (commonly represented as $x + 4$ and $2x$), can be regarded as functions, arithmetical and algebraic operations and combinations of operations may be treated as functions.

Functions can be usefully exploited in the early mathematics curriculum by employing problems where generalized variables appear as variable quantities. Consider the following problem:

Mike has $8 in his hand. The rest of his money is in his wallet.

Robin has exactly three times as much money as Mike has in his wallet.

Describe in your own words or show in a drawing how much money they have.

Third and fourth grade students will typically first interpret this problem as describing a story about two children having fixed amounts of money. Many will claim that one of the children has more money than the other by arguing, for instance, that Mike has more because "he started with $8". By having the students carefully reconsider the wording of the problem and prodding them to systematically walk through various scenarios they come to realize that the story could correspond to any of several possibilities: if there were $1 in the wallet, Mike would have a total of $9 ($8 + $1) and Robin would have $3 (3 times $1); if there were $2 in the wallet, Mike would have $10 and Robin $6, and so forth. The students gradually come to treat the amount of money in the wallet as a generalized variable, thereby treating Mike's amount and Robin's amounts essentially as functions of the amount in the wallet. Young students, with careful support from the teacher, can come to express these functions through algebraic notation, such as w+8 and $3 \times w$, and interpret the problem through graphs of linear functions (Authors, 2007).

Recasting mathematics topics with the aid of functions allows for classroom activities based on representing, discussing, and solving open-ended problems. Different representational forms for functions, notably, verbal statements, number line diagrams, data tables, graphs, algebraic expressions, and equations serve as models of relations among physical quantities. Ratio and proportion problems can be explored as problems about linear functions and equations and inequalities may be interpreted as comparisons between the outputs of two functions. In such ways, functions hold the promise of offering a means for unifying a wide range of topics commonly introduced in a piecemeal fashion across the curriculum.

Teaching mathematics from the perspective of functions and variables aligns with Piaget's theory of cognitive development and Vygotsky's ideas on the role of cultural tools and social interaction in learning and development. Whereas Piaget's theory focuses on the development of logical and mathematical reasoning by the child, in interaction with the physical and social world, the Vygotskian view highlights the degree to which mathematical knowledge entails an appropriation of cultural practices and conventions. Mathematics teaching requires that teachers constantly shift from one perspective to another. For instance, classroom activities often begin by evoking students' views about a problem and the interrelations among quantities in the problem and gradually introduce conventional mathematical representations and tools. In this way, mathematical representations and models of informal mathematical activity may pave the way for conventional mathematical representations (Freudenthal, 1973, 1991; Gravemeijer, 1999).

There is a growing body of studies proposing and providing evidence that, from an early age, under conducive circumstances, children can use algebraic thinking and, specifically, functions in the broad sense of associations among two sets of values (Blanton et al., 2017; Blanton & Kaput, 2000; Brizuela, 2016; Brizuela & Schliemann, 2004; Cai & Knuth, 2011; Carpenter & Franke; 2001; Kaput, Carraher, & Blanton, 2008; Moss & Beatty, 2006; Carraher, Schliemann, & Brizuela, 2000; Carraher, Schliemann, & Schwartz, 2008; Carraher, Schliemann, Brizuela, & Earnest, 2006, 2016; Carraher & Schliemann, 2007, 2016, 2018; Kaput, 1995; Kieran, 2018; Schliemann, Carraher, & Brizuela, 2007, 2012). As a whole, these studies have shown that students in elementary school, including those from disadvantaged backgrounds, can come to display a firm grasp of variables, not merely as "mystery numbers," but, more generally, as place-holders for arbitrary members of large, possibly infinite, sets of values). They can learn to express functional relations through verbal statements, number lines, tables, graphs, or algebraic notation and interpret graphs as representations of (usually, linear) functions, noting how features of the graph convey significant information about word problems they were crafted to model. Such results far exceed the expectations of conventional wisdom and of current standards in mathematics.

The contribution of early algebra and experience with functions to the future learning of algebra and mathematics in general is still a work in progress. So far, results of an exploratory follow-up study by Carraher, Schliemann, & Brizuela (2012) have been encouraging: In a week-long algebra summer camp offered three-years after students had participated in an early algebra intervention from grades 3 to 5, the intervention students performed better than a control group on written assessments given before and after the camp. Moreover, the difference between the two groups increased after participation in camp lessons. These long-term results, although based on relatively few children, suggest that participation in early algebra activities based on the multiple representations of functions in elementary school may help prepare students to learn topics of algebra in middle school. We would argue that teaching key topics in the mathematics curriculum with a focus on algebraic reasoning, variables, and functions, adapted to the needs of school grades, may contribute to algebra learning in middle and high school.

In what follows, after briefly reviewing studies of teacher development and teacher effectiveness, we describe the teacher development program and provide evidence suggestive of its possible contribution to changes in teachers' ways of teaching and to their students' learning.

Research on Teacher Development and Effective Teachers

Shulman's (1986) reflections on "knowledge growth in teaching" have been a point of reference for many contemporary studies in teacher development. Shulman proposed that teachers need to develop *pedagogical content knowledge*, or "the ways of representing and formulating the subject that make it comprehensible to others (p. 9)." A teacher's mastery of fractions in no way ensures that she will be successful when introducing the topic to students whose experience has been restricted to counting numbers. She needs to have a variety of well-mastered examples suited to helping students understand why and how operations with fractions build upon, as well as depart from, operations with natural numbers.

Teachers' knowledge for teaching mathematics or teachers' pedagogical content knowledge (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill & Ball, 2008), as revealed through teachers' surveys, classroom practice, and/or lesson logs, has been found to be related to student performance and to value-added scores on students' achievement tests (Hill, Rowan, and Ball, 2005; Hill, Kapitula, & Umland, 2011).

In recent decades, as mathematics educators and researchers have made headway in identifying and clarifying characteristics of successful teachers of mathematics, researchers have sought to identify critical features of successful teacher development programs. Hill (2007) highlighted the need for programs of long duration that focus on both subject matter and teaching methods and rely on multiple formats such as lesson study and collaboration among teachers. Bautista and Ortega-Ruiz (2015) also emphasized the need for long programs to provide teachers with a deeper understanding of the subject matter, pedagogical strategies to teach specific content, and understanding of how students think of and learn about the content. Desimone (2009) remarked that there is a consensus about critical characteristics of professional development that promote teacher knowledge, improve their practice, and, potentially, student achievement. These are content focus, active learning, coherence, program duration, and collective participation. She highlights further the importance of student work, administrative support, curriculum materials and implementation, and high expectations.

In a study of nearly 3000 randomly-assigned volunteer teachers and students, the Measures of Effective Teaching project (Kane & Staiger, 2012) found significant associations between ratings of videotaped classroom lessons based on measures of teacher effectiveness and student performance. Lessons were evaluated in terms of students' intellectual engagement, interactions between students and teachers, accuracy with regard to content, emphasis on meaning rather than memorization, considering student perspectives, probing techniques for questioning and discussion, and promoting student participation. On state-mandated tests of mathematics, students whose teachers were at the 75th percentile of effectiveness performed fully eight months ahead of peers whose teachers were at the 25th percentile.

Some teacher development programs have been found to contribute to teachers' performance on written tests of mathematical knowledge for teaching (e.g., Hill and Ball, 2004; Hill, Ball, and Schilling, 2008). However, only a handful of programs have been the focus of systematic research and even fewer succeeded in showing impact on student achievement (Sztajn et al., 2017). Examples of these are studies by Franke, Carpenter, & Levi (2001), McMeeking, Orsi, and Cobb (2012), Santagata et al. (2011), and Saxe, Gearhardt, & Nasir (2001). In general, as Bautista (2015) and Bautista and Ortega-Ruiz (2015), Hill (2007), and Sztajn et al. (2017) stress, most teacher development programs have not shown significant changes on student learning. By our count[1], of the 25 National Science Foundation Math and Science Partnership program projects sponsored between 2002 and 2015 (http://hub.mspnet.org), only six (including the present one) report progress among both teachers and students.

The Teacher Development Program

Since 2011 the program has been engaging successive cohorts of teachers in grades 5 to 9 in in-service professional development. The program, consists of three-semesters of work, with a mix of online activities, associated classroom-based assignments, and regular face-to-face meetings. It was developed through a partnership of mathematics education researchers, research mathematicians, and physicists drawing on contributions from the three disciplines (mathematics, mathematics education, and physics). Key stakeholders included nine participating school districts in New England (in MA, NH, and ME). From the start, the intended beneficiaries were both the

[1] We employed two independent raters to arrive at this figure.

teachers and their students, even though the program mentors did not work directly with the students.

The mathematical content of the program covers not only algebra but many other topics in the math curriculum. We designed activities and produced materials to help teachers develop their own insights about mathematical topics and, in particular, apply the lens of functions to the topics they teach. Our experience leads us to include not only mathematics content and teaching strategies, but also examples of lesson implementation and discussions in successful classrooms. Furthermore, teachers needed to be convinced of the relevance of such examples to their own teaching. We will shortly describe some of the ways in which such issues were addressed.

We worked under the assumption that successful instruction in the middle school classroom requires that teachers be scrupulously attentive to both mathematical content and to the mathematical reasoning of their students. Teachers must understand the content they teach in depth and should understand how different topics in the curriculum can become interrelated. This requires long-term programs that focus on understanding mathematical content (rather than mere mastery of algorithms) where teachers interact with each other as they discuss mathematics and pedagogical activities. For teaching to be effective, teachers must engage students in the analysis of relationships between quantities or magnitudes in everyday and science situations, considering students' intuitive ideas and reasoning as starting points for classroom discussions and for the introduction of new concepts and strategies. Viewing mathematics through the lens of functions allows such an approach. As the devil is in the details, we will illuminate certain characteristics that might otherwise remain hidden.

In what follows, we will describe the program's structure and pedagogical foundations and exemplify its content, requirements, and activities. Then we discuss data we gathered on (a) changes in classroom teaching along with changes in student engagement and (b) improvements in performance on state mandated tests for students in each target district, in comparison to those in districts matched along dimensions known to be associated with educational outcomes. We consider the extent to which the results may be due to the program content and activities as opposed to other factors. In doing so, we attempt to further understand how our specific approach to mathematics can make a difference in teaching and learning and, hopefully, contribute to theoretical ideas about mathematics teaching and learning which, as discussed by Herbst and Chazan (2018), could enrich research on mathematics teaching.

General Features of the Program

The program is structured around three semester-long courses for in-service teachers of mathematics in grades 5 to 10. The courses are taken in succession over an 18-month period. Teachers participate in online discussions several times a week and receive feedback from their peers and from instructors. At the beginning of each semester they attend a two-day meeting at Tufts University campus and meet weekly at their schools, with program mentors joining these meetings once a month. The instruction offered online includes video lectures, written notes, videos of classroom activities, and education research papers and summaries. Office hours are offered online via regularly scheduled meetings as well as individual appointments.

Asynchronous online delivery was chosen to allow teachers flexible scheduling of supporting extensive interaction with other teachers. When suitably employed, online course delivery can actually hold advantages in terms of participants' interactions (see Hawkes and Good, 2000). We designed and improved our online activities to facilitate interactions among teachers and instructions and to comply with Tufts University's requirements in terms of quality of content and quality and quantity of course instructor-participant interaction.

The program's main features are:

1. *The interconnection and depth of analysis of mathematical topics*: While other programs may highlight functions as a core concept, mainly in the teaching of algebra, we use functions and their representations as a lens to explore and unite various topics of the curriculum. This offers, we

believe, a foundation for deeper understanding and facilitates the use of multiple mathematical representations to explore verbal and contextualized problems, even when a problem could be solved using a single representation or strategy.

2. The integration of mathematics and pedagogical content. There was a deliberate effort to integrate the content and pedagogical approaches throughout the curriculum, in terms of instructional materials and of activities required from the teachers. Course materials, jointly developed by the interdisciplinary team, were presented in an integrated format. Teachers discussed videotaped classroom activities related to the content of the courses in terms of the mathematical content, teaching strategies, and students' ideas and achievements. They were made aware of the need to be precise and clear in their assumptions, clarify their ideas, and justify their assertions in a way that works in any situation rather than for specific numbers.

3. *Eliciting and building upon students' ideas.* Teachers interviewed students to understand how they initially approach particular mathematics topics and discussed videotaped lessons and interviews with colleagues and mentors. The design and implementation of classroom activities was expected to take into account the information gathered. The activities demonstrated and promoted in the courses focused on open-ended questions about relationships between sets of quantities or values and were conducive to students' engagement and discussions.

4. *Conceptual understanding and representations emerging from analyses of situations and relationships between sets of quantities*: Our approach to functions allowed for building upon the role of quantification in learning and development (Abrahamson, 2012a, b; Abrahamson et al., 2014; Lehrer et al., 2001; Liu et al., 2017; Piaget, 1964) and on the importance of quantitative reasoning in the preparation of teachers (Lobato and Ellis, 2002; Thompson, 2015). Teachers were encouraged to introduce problems involving physical quantities and sets of values, starting with natural language and progressing towards conventional representations such as tables, number lines, graphs, and algebra notation.

5. Interaction among teachers and of teachers with instructors: In keeping with lesson study approaches to teaching (Lewis, 2002; Perry & Lewis, 2008; Stiegler & Hiebert, 1999; Yoshida, 1999), cooperation among teachers is a key component of the program. Throughout the courses, teachers held online and face-to-face discussions with their peers as they solved mathematical challenge problems, analyzed videotaped demonstrations from previous research or from prior cohorts, and planned, implemented, discussed, and evaluated their lessons, with feedback from mentors at each step of the process. We encouraged teachers to frame and interpret ideas and activities for themselves, rather than rely solely on ready-made models.

6. Scope and length of the program: The program extends over three successive 15-week semesters, with an expected workload of roughly 10-12 hours per week. The required work includes understanding written texts and video lectures, some about the mathematics alone, most relating this mathematics to the way they can be presented and used in the classroom. It includes weekly responses to and discussions of challenge questions on mathematical concepts and examples of student work and reasoning.

7. Financial support: The courses were offered tuition-free. The teachers also received a $1,000 stipend for each completed course and were given a laptop computer and a camera at the start of the program to easily access course materials, interact online with instructors and colleagues, and collect and analyze classroom data. Upon program completion, teachers kept the equipment. Since salaries, particularly in urban areas, are often insufficient to cover living expenses, these resources allowed teachers to dedicate to the long-term program time they might otherwise use working part-time jobs.

8. Institutional support: Teachers' institutional support differed widely from district to district. While in some districts, administrators (usually a math coordinator) encouraged teacher participation and facilitated practical issues arising from it, such as meeting times, in other districts, the teachers were left to solve such issues on their own.

Program Implementation

The program's three semester-long credit bearing graduate level courses were offered by Tufts University Departments of Mathematics and of Education, as part of the Poincaré Institute for Mathematics Education. Credits awarded by the courses have been accepted as mathematics content requirements for master degrees in teacher preparation programs at other universities.

The *first course* builds on the idea of numbers, from representations of quantities to relatively abstract conceptions of numbers as mathematical objects; it shows how functions are intimately associated with diverse topics in the early mathematics curriculum. Teachers work on fractions and decimals, rational and irrational numbers, and the many ways numbers can be represented, for example, as points on a line or an oriented segment. Course activities focus on relations between sets of values culminating in the idea of arithmetic operations as functions of a single or of two variables. The *second course* treats equations and inequalities as entailing the comparison of two functions and uses transformations of the line and of the plane to explain solving processes. Questions regarding divisibility appear in the study of numbers (course 1) and in the solution of Diophantine equations and factorization of polynomials for solving polynomial equations (course 2). The *third course* compares linear to non-linear functions with the aim of examining constant and variable rates of change.

From January 2011 to May 2014, three cohorts of approximately 60 teachers each, from grades 5 to 9 from nine school districts in the northeastern USA (five in Massachusetts, three in New Hampshire, and one in Maine), took the courses. A fourth cohort of 70 teachers from a single Massachusetts school district took the courses from 2014 to 2015.

A total of 12 districts with a high percentage of low-income families were initially contacted via e-mail messages addressed to their superintendents and mathematics coordinators to discuss the possibility of becoming partners in a teacher development program proposal to be submitted to the National Science Foundation. Three districts declined our invitation because they were already committed to teacher development programs offered by other institutions. Each of the nine districts accepting the invitation were then visited by two or three senior members of the proposal team. In these visits we met with mathematics coordinators, coaches, and teachers, discussed details of our initial plans, and gathered suggestions on how to structure the program. As the proposal developed, we invited mathematics coordinators and teacher representatives to meetings, to further get their contribution to the proposal.

The courses are now part of Tufts University regular offerings and are open to pre- and in-service teachers. During the courses, online discussions are held in groups of about eight teachers and one online instructor. Instructors are faculty from the Departments of Mathematics and the Department of Physics or advanced PhD students in mathematics. Work by small groups of three to five teachers in schools, discussing, planning, evaluating, and writing reports on classroom activities, is monitored by and receive feedback from senior researchers in mathematics education, postdocs, and PhD students in mathematics education or science education. Most instructors had participated in the planning of the program.

During the first two weeks of each of the four units of a given course, teachers from the three cohorts in this study worked on course materials, and answered and discussed, online, four to six challenge questions, one on pedagogical aspects, the others on mathematical content. Each week, individual teachers posted successive versions of their answers, commented on postings by others, asked questions, responded to instructor questions and comments, and posted a final improved version of the work. The amount of feedback by each group’s mentor was fairly high compared to what happens in mathematics college level courses.

In the third week of each unit, in interaction with their small group of teachers from the same school, they interviewed students about particular topics to understand their ways of thinking (about, for instance, fractions on the number line), analyzed videotaped lessons, and planed, implemented, evaluated, and improved classroom activities.

At the end of each course teachers individually implemented classroom lessons jointly planned with their peers. These lessons were videotaped and analyzed by the group.

Examples of coursework

The following examples illustrate how course material is presented in notes, video lectures, and software apps and how teachers, working in small online groups, discuss and solve mathematics questions.

First example: Fractions

Course 1 materials cover fractions and their traditional representations (verbal descriptions, pie charts, area and bar models, and number lines), as well as the assignment of a point on the real line for every class of equivalent fractions and the representation of fractions as slopes in the Cartesian space (see Figure 1). This aimed at helping teachers to relate fractions to linear functions, ratio, proportionality, multiplication, and division and to the algebraic treatment of these concepts, replacing specific numbers with variables. The figure shows points (*b, a*) representing fractions $\frac{a}{b}$, each equivalent to either $\frac{3}{2}$ or to $\frac{-1}{2}$. In the case of $\frac{3}{2}$, the fraction corresponds to points, (b, a) lying on the line, $y = \frac{3}{2}x$, and on the unit grid. For example, the highlighted points along that line refer to the equivalent fractions $\frac{3}{2}, \frac{6}{4}, \frac{-3}{-2}$ *and* $\frac{-6}{-4}$.

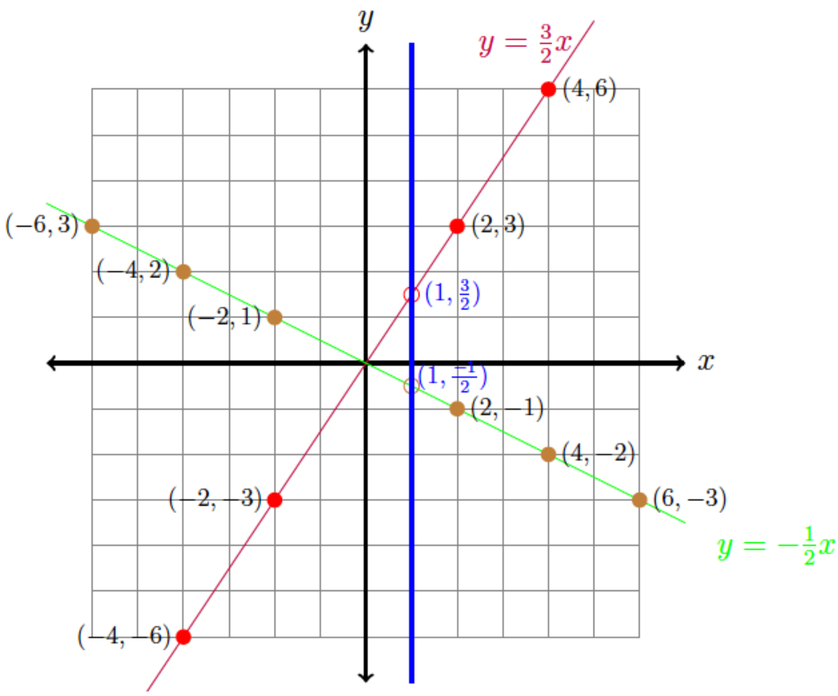

Figure 1. Slope representation of fractions in the Cartesian space.

Identifying the (vertical) line, $x = 1$ as analogous to the real line, the point of intersection of the line $y = \frac{3}{2}x$ with the line $x = 1$ gives the point corresponding to $\frac{3}{2}$ on this real line. This highlights the direct correspondence between this slope representation of fractions and the more traditional representation of a fraction as a real number and its corresponding point on the real line. It also supports visually interpreting equivalent fractions from the correspondence of the intersections of the line $y = \frac{3}{2}x$ with the unit grid.

This model helps the discussion of some normally difficult topics such as fractions with negative numerators and/or denominators or the ordering of fractions in increasing or decreasing magnitude. It was drawn upon later when looking at increasing and decreasing functions. It was also used to explain the rules of signs for multiplication of numbers (such as “the product of two negative numbers is positive”), when comparing the signs of the numerator and denominator of a fraction with the sign of the slope of the line in which it lies.

If two fractions are equivalent, then the fraction obtained by adding the numerators and the denominators is a third fraction equivalent to the other two. In the unit about fractions, teachers discuss the following situation:

> Children in an art class are divided into two groups. The yellow group receives a bag of beads that the children completely share equally. The red group gets a different bag and, again, when they share them equally, none are left over. It so happens that each child whether in the yellow or red group has the same number of beads. It is intuitively clear that if the two groups of children had been joined into one single group and the two bags of beads had been dumped in a single bag, then the number of beads a child would have received would still be the same. Show algebraically that this is indeed the case. You can develop some intuitions by picking specific values for the number of beads and the number of children in each group and producing tables exemplifying the situation. But, in the end, we would like to see you being able to explain this in general, perhaps using variables in place of numbers.

While most teachers analyzed this question using specific numbers, many teachers were able to introduce variables and use algebra in their explanations.

Second example: Equations.

The second course presents equations as the comparison of two functions. For example, the equation $60x + 50 = 40x$ can be considered as a comparison of the two functions $f(x) = 60x + 50$ and $g(x) = 40x$. Building upon the study of transformations of functions and equations, teachers read and discussed examples of how to interpret each step of the solution to an equation in terms of transformations of pairs of graphs and of the quantities in the problem situation. As one of the online challenge questions, teachers considered the following description of a situation:

> A delivery truck from the Leakit ice-cream factory leaves Boston at midnight driving towards Maine at a constant speed of 40 miles per hour. At the same time, a manager from

Leakit leaves from the headquarters in Providence, 50 miles south of Boston, at a constant speed of 60 miles per hour, driving also towards Maine via Boston.

Teachers were asked to write an equation for the number of hours from midnight till the manager catches up with the truck, explain what the numbers in the equation mean, solve the equation algebraically and graphically, show how each step in solving the equation corresponds to changes in the graphs, identify the functions at each step, and consider how the solution steps relate to the situation.

Figure 2 shows a graphical representation, similar to those produced by some teachers, of the equation $60\text{x} = 40\text{x} + 50$ and of the intermediary steps towards the solution, $\text{x} = 2.5$.

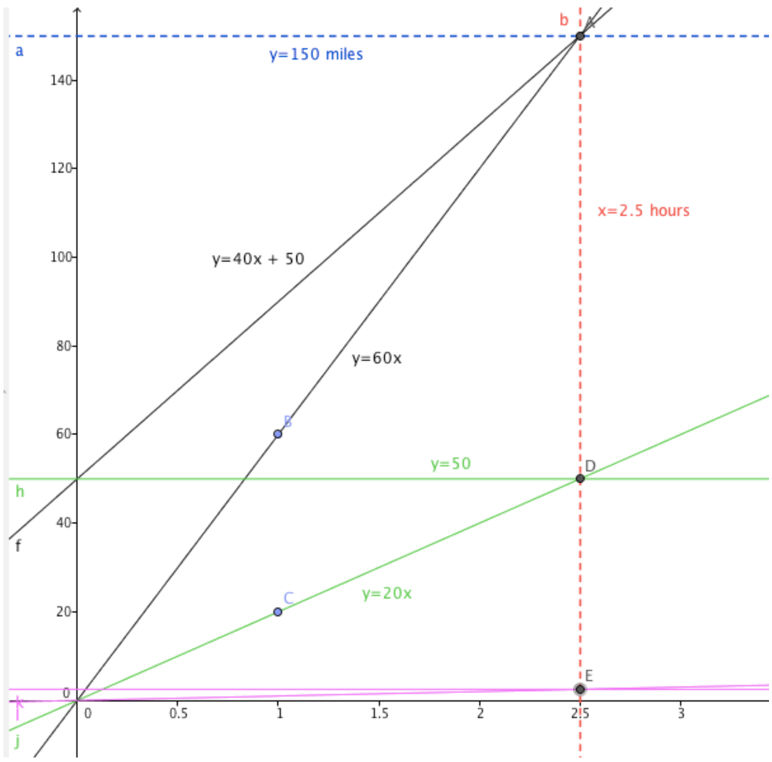

Figure 2. A graphical representation of the equation $60x = 40x + 50$ and of the steps towards the equation, $20x = 50$ and the solution, $x = 2.5$.

The following answer by a teacher illustrates how some teachers described the relationship between the situation in the problem and the transformations of the two functions by subtracting 40x, a transformation that lead to a visual representation of the solution to the equation:

"Essentially, the first transformation takes away the distance that they both covered at the same rate: 40x. What we did was to divide up the distance that the manager drove at 60 mph into two sections: one that he drove at 40 mph (which we then "matched up" or cancelled with the truck's side), and then determine how long he would have to drive at 20 mph in order to cover the 50 extra miles that he would have to cover from Providence to Boston".

During the third week of the same course unit, teachers analyzed classroom video footage of fourth grade students as they grappled with the Wallet problem discussed above. They then adapted and implemented it in their classrooms.

Planning for classroom activities.

The following excerpt from course materials exemplifies instructions teachers were given before planning and implementing a classroom lesson for the Final Project, a major assignment at the end of each course:

> When choosing the specific problem, questions, or representations you will include in the activity for your Final Project, consider what you have learned from students' previous responses during the Unit 4 Week 3 activity, so that they will further explore, rather than repeat, what they had done before. While computations and learning how to use algorithms has its place in the curriculum, try to choose, instead, an activity that will engage your students in exploring new concepts and new mathematical ideas and representations.
>
> Avoid starting with a request for a definition of a term or a straightforward computation. Choose instead an open-ended problem, that is, a problem in which students are likely to come up with different approaches to advance different claims and put forth various, possibly conflicting, ideas and representations that will lead to discussions and, hopefully, to reconsidering their initial views. After presenting the problem, make sure you elicit your students' ideas and representation. Then encourage discussions on what they may have proposed and raise new questions. We would like to see your students taking an active role in the discussion. Encourage them to make comparisons between their possibly different approaches and to support their arguments with evidence. Provide help and counter-arguments if needed. Bring them back to the topic if they seem to stray into an unproductive dead alley or into non-mathematical details.
>
> You may choose to implement the same plan as your peers or to develop your own plan. Remember, however, that the final implementation and report are individual. In any case,

continue discussing your ideas with your colleagues and instructor to get their input about your plan.

Changes in Teaching and Learning

External evaluators analyzed *changes in classroom teaching* using a pre-post intervention design. *Student achievement* in mathematics in the five districts in Massachusetts participating in the program was examined through state-mandated tests of the period (from the program's inception in the Spring of 2011 until early 2014) and compared to those of students from districts with similar student initial test results, size of student population, ethnic composition, and demographic variables.

We aimed at determining changes in teaching practices and in student achievement that may be due to participation in the program and consider aspects of the program that may have played a role in these changes.

Establishing evidence of impact is challenging goal. Establishing how and why the program leads to better teaching and learning is even more challenging, for one needs to make a credible case that certain characteristics of a successful program, and not others, are responsible for the results obtained. In addition, one hopes to acquire insight into the mechanisms underlying observed changes. Although, in principle, these challenges might seem likely to benefit from experimental design, long-term, broad-scope teacher development programs such as the present one entail a large number of critical features that may not work in isolation, rendering them unfit for experimental methods. We can however explore the possible contribution of different aspects of the program (or lack thereof) using quasi-experimental methods (Campbell & Stanley, 1963).

Changes in Teaching

External evaluators from the Intercultural Center for Research in Education (INCRE) examined changes in classroom teaching among teachers in the program's *first cohort* by conducting systematic observations of classroom lessons chosen by each teacher, one at the outset of the program and another at the end of the three-course series. They also observed a sub-sample of approximately half of these teachers in follow-up visits, six months and a year after program completion. External evaluators' findings were complemented by the project's research team ratings of the same first cohort teachers' written lesson plans, presented as part of the second course assignments, and video-taped lessons collected at the start of the first and at the end of the second course.

The External Evaluation Procedures

The external evaluators used two classroom observation tools in their program evaluation: the Reformed Teaching Observation Protocol (RTOP, AZ State University, Sawada et al., 2002) and the INCRE Analytical Observation Tool. Their overall goal was to evaluate teachers' support for student discussions and uses of multiple representations, in-class time spent addressing students' mathematical reasoning, the design of lessons and discussions of mathematics problems with students from various perspectives and representations, and the percentage of students actively engaged in the lesson. It also provided ratings of teachers' and students' mathematics mistakes and understanding of mathematical content. The two observation instruments complement each other and include similar items that help establish the consistency of the observation data.

The INCRE evaluators conducting the site visits were seasoned math teachers and administrators with years of experience in observing and evaluating mathematics lessons. To enhance the reliability of their observation ratings, teachers were generally observed by the same evaluator across the visits.

The RTOP serves to determine the degree to which teachers use student-centered and engaging learning practices that includes multiple opportunities for collaboration among students. The RTOP 25 items assess five areas of instruction: (1) lesson design and implementation; (2) propositional pedagogic knowledge; (3) procedural pedagogic knowledge; (4) classroom communication interactions; and (5) student/teacher relationship. The 25 RTOP items are rated on a scale of 0-4, yielding a 100-point rating scale for each lesson. The observers write detailed notes of teacher and student activity throughout the lesson being observed and, at the end of the lesson, complete the RTOP and INCRE rating scales.

The additional analytical observation tool developed by INCRE included systematic observations of the following aspects:

Teacher's practices:

(a) Structure of the lesson (who leads the lesson, how students are grouped, and instructional approaches).

(b) Teacher's responsiveness to students' interests, questions, and experiences.

(c) Use of problems related to real-life contexts.

(d) The degree to which the teacher gives evidence of having prepared the lesson and whether the teacher opens the lesson by connecting to prior lessons and to students' experiences.

(e) The teacher's understanding of the mathematical content (terminology, use of notation, modeling, use of appropriate problems, tools, and representations, correctness in presenting and discussing mathematics content).

(f) Use of questions to solicit students' ideas and conjectures, asking students to provide evidence for their ideas, probing for revisions of unclear ideas, promoting discussions among students to build on or refine responses and understanding.

(g) Providing students with clear directions and observing and assessing their understanding before proceeding further.

(h) Providing students with opportunity to review, summarize, reflect, and gain deeper understanding of the lesson.

Student participation:

(a) Engage in reflection, explain results, apply content of the lesson to solve problems, correctly apply and discuss mathematical concepts, representations, and operations.

(b) Work collaboratively with peers, respect each other's thinking, pay attention to other students' ideas and explanations.

(c) Assess their own work, demonstrate understanding, ask questions, support ideas with evidence.

(d) Use appropriate mathematics terminology.

(e) Apply procedures and concepts to new problems.

(f) Show interest, motivation, and on-task behavior.

Of the 56 participants in the first cohort (52 teachers and four coaches), 51 (91%) completed the program. Baseline data, using the RTOP and INCRE tool were collected in January 2011 (the first month of Course 1) for the classroom lessons of the 52 teachers. Post data, using the same tools, were collected in April 2012 (just before the end of Course 3) for 48 teachers (of the original 52; two of the teachers were on leave and two were not teaching in the post data collection year).

After the teachers completed the program, the evaluators interviewed them to gain insight into their views about the program and whether they thought it may have influenced their teaching. Six months after program completion (October 2012), classroom lessons of 28 of the teachers were once more observed. Finally, one year after completion of the program (April 2013), 33 teachers were again observed. The ratings for these two sub-groups of 28 and 33 teachers at the start of the program did not differ from those of the teachers who were not observed in the two follow-up dates. Overall, a total of 23 teachers were observed at all four points in time.

The External Evaluation Results

RTOP findings are presented as total score means. Changes in mean RTOP scores over time are expressed in terms of effect size (ES), calculated as the difference in means divided by the square root of the pooled variance for two-time points. An effect size of 0.5 of a standard deviation is seen as substantive.

The mean RTOP score for the 48 teachers observed at the start (January 2011) and at the end of the program (April 2012) increased from 43.1 to 49.1, a significant mean increase of 6.0 points and an effect size of .30 ($p<.05$) (see Table 1 and Figure 5). RTOP ratings were obtained for a subgroup of 23 teachers (44% of the cohort), six months (October 2012) and a year (April 2013) after the end of course 3. This subgroup of teachers, was similar in "pre-test" ratings to the remaining teachers. The subgroup's mean increased from 43.1 in January 2011 to 57.5 in April 2013, an increase of 14.4 points.

Table 1: RTOP mean ratings and gains over time from January 2011 (T0) to April 2012 (T1), October 2012 (T2), and April 2013 (T3)

	T0: Jan. 2011 (baseline)	T1: Apr. 2012 (courses' end)	T2: Oct. 2012 (follow-up)	T3: Apr. 2013 (follow-up)	T1-T0 Gain	T2-T1 Gain	T3-T2 Gain	T3-T0 Gain
Mean RTOP (N=48)	43.1	49.1			6.0* (ES=0.30)			
Mean RTOP (N=23)	43.1	48.9	56.9	57.5	5.8* (ES=0.25)	8.0* (ES=0.38)	0.7 (ES=0.03)	14.4** (ES=0.69)

(*) p<.05; (**) p<.01; ES: effect size, in standard deviations. Source: INCRE, 2013, 2014).

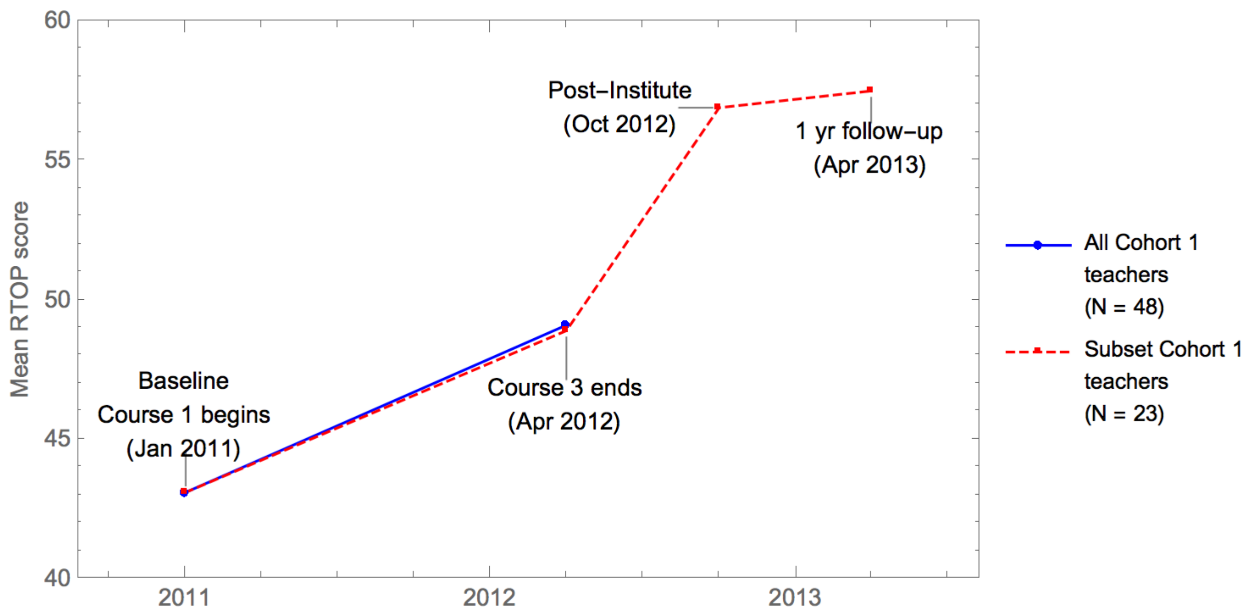

Figure 3: Mean RTOP scores for the first cohort at beginning and end of courses and for a sub-group of 23 teachers observed six months and a year later (source: INCRE, 2013, 2014).

RTOP mean rates increased after program completion and then remained stable until at least one year later. In interviews conducted by the external evaluators, several teachers explained that, while taking the courses, they did not have enough time to take into account what they were learning and to prepare new lessons inspired by the program. Over the summer, before the following semester had started, they were apparently finally able to further put what they had learned into practice.

Data collected using INCRE's own observation tool also showed statistically significant gains in student engagement and active participation in mathematical questioning, reasoning, and problem solving. As noted in the external evaluator report (INCRE, 2013), students were found to increasingly explain their thinking, engage in dialogue and questioning, and collaborate more with their peers. Teachers were found to be more inclined to use drawings, bar diagrams, number lines, data tables, algebra notation, and graphs. They also increasingly made connections among different math concepts and between algebraic and geometrical representations. Furthermore, the teachers more often used tools and technology, worked with contextualized problems, encouraged students to use variables and to think algebraically, listened to students and displayed longer wait times.

As the reader may have noted, the changes in classroom teaching described by the external evaluators are consistent with the aims of the three courses and also consistent with features associated with teacher effectiveness. Moreover, the external evaluators found that these results were consistent with teachers' own judgments, as revealed in individual interviews, about the influence of the program on their teaching and development, both in terms of content and approach and in the timing of the changes. We will not systematically analyze these self-reports, given our focus on direct, rather than on indirect, data about the possible contribution of the program.

Internal Analyses of Lesson Plans and of Videos

At the end of each of the four units of the second course, teachers in the *first cohort* were asked to design, in small groups of three to five teachers (18 groups in all), a learning activity related to the content of the unit. Our analysis of the *written plans* for all groups, conducted by two judges, showed that the learning activities teachers produced for each of the four units in Course 2 were increasingly designed so as to engage students with multiple representations, a core feature of the program (see Table 2).

Table 2 - Multiple Representations in Teachers' Lesson Plans

Content of Course 2 Units	Average Number of Representations	Percent of Plans with More Than Two Representations
Unit 1: Fractions	2.56	50%
Unit 2: Divisibility	3.00	56%
Unit 3: Transformations of Line and the Plane	4.67	83%
Unit 4: Transformations of Graphs of Functions	4.88	95%

For example, in Unit 1 of Course 2, on fractions, 44% of the plans used the length model, 11% included the representation of fractions on the number-line, and 17% used Cartesian graphs and slope to represent equivalent fractions. These models were proposed and illustrated by course materials. For the corresponding assignment in Unit 3 of Course 2, on Transformations on the Line and on the Plane, the plans included Translations (44%) and Dilations (44%) on the Cartesian plane, as means to clarify issues related to arithmetic operations (56%) or to fractions, ratios, and proportions (10%). In the plans, coherent with course materials and activities, Transformations provided opportunities for reflection upon scaling and similarity of geometric figures (56%), perimeter and area (33%), or everyday situations (28%), as well as opportunities to practice plotting points on the plane (94%), as the other goals were addressed. Among the 18 plans, 44% and 39% included questions aimed at eliciting students' predictions and generalizations, respectively. Representations promoted by the courses such as use of verbal descriptions, manipulatives, tables, number lines, coordinate points, graphs in the Cartesian plane, algebra notation, and software demonstrations, permeated the teachers' work in these assignments.

Because the concomitant use of multiple representations became more prevalent in the lesson plans for the last two units, it might be argued that the topics of Units 3 and 4 lend themselves to a more geometric approach than those of Units 1 and 2. However, the change was gradual rather than abrupt. And many teachers stated in their evaluations of the courses that, as they advanced in the program, they became more comfortable with exploring different representations in their lessons.

Another internal analysis focused on *videotaped lessons* collected by the program's research team at the start of Course 1 (lessons by 31 teachers) and at the end of Course 2 (lessons by 24 teachers), on a regular school day. The decrease in numbers was due to scheduling problems. This analysis once more showed that, from the start to the end of the two courses, in addition to using more representations for the same problem, teachers posed more questions requiring students' reflective thinking, and were more likely to treat equations as involving a comparison of two functions. Because they were the teachers who responded to our invitation to be videotaped, we may want to treat this second set of data with some caution.

Changes in Student Achievement

Data Collection

We compared student performance on state mandated tests in the five target districts in Massachusetts to that of students in similar districts, from the Spring term of 2011 to the Spring of 2014. By this last test's date, 33% of teachers in grades 5 to 8 in the five target districts had completed the three courses. They constituted the first and the second cohort of teachers in the program.

We focused on changes in the percentage of proficient and advanced students of grades 5 to 8 in the Massachusetts Comprehensive Assessment System (MCAS) test, the standardized, state-mandated measure of mathematical proficiency at the time. At the end of each academic year the test was given to students in grades 5 to 8 (students from grade 9 are not included in this analysis because they do not take the MCAS in high school till grade 10). Each of the target districts was matched, as we will describe below, to five comparison districts.

The MCAS test is a statewide standards-based assessment created by the Massachusetts Department of Elementary and Secondary Education (ESE) and used throughout the state, from 1997 through 2014, to measure student performance in a variety of disciplines, including mathematics. Until the end of the 2013-2014 academic year, students in all public schools were scheduled to take the MCAS test, typically in early May. After 2014, with the implementation of the Common Core State Standards Initiative, a new test (the Partnership for Assessment of Readiness for College and Careers - PARCC) replaced the MCAS in some Massachusetts districts.

The mathematics portion of the MCAS test included (a) multiple-choice questions for which students select an answer from four alternatives, (b) short-answer questions where students generate a brief response, usually a numerical solution or a brief statement, and (c) open-response questions to be answered by presenting, in writing, one or two paragraphs in the form of a narrative, a chart, table, diagram, illustration, or graph, using a variety of strategies and approaches. Each year, the ESE developed new problems for each new test.

The ESE does not provide researchers with data on individual students. This prevented us from analyzing how students' performance may have varied according to the practice of their teachers. However, by carefully selecting comparison districts with similar populations and with similar initial MCAS test results, we could compare the evolving performance of students in the target districts to that of students from districts not involved in the program.

Performance on the MCAS is reported in terms of percentage of students falling into four categories, ordered from highest to lowest as: *Advanced*, *Proficient*, *Needs Improvement*, and *Warning/Failing.* In the custom of MCAS summary reports, we will focus on the percentage of students assessed as *Advanced and Proficient* and to changes in these percentages across the years.

The first cohort of Massachusetts teachers in the program began course 1 in the Spring of 2011 and ended course 3 in the Spring of 2012. The second cohort started in the Fall of 2012 and completed the courses in the Fall of 2013. Table 3 shows the timeline of courses taken by the first two cohorts of teachers and MCAS tests taken by their students. A third cohort of teachers started the program in Spring 2014 and concluded the three courses in the Spring of 2015. Data from this third cohort are not included in this analysis because, for the year 2014-2015, student performance in two target districts and in 11 comparison districts was then evaluated by the new PARCC assessment.

Table 3: Timeline of courses and tests from Spring 2011 to Spring 2014

Semesters	Cohorts and Courses	MCAS
Spring 2011	Cohort 1 Course 1 (started)	MCAS 2010-2011 [baseline]
Fall 2011	Cohort 1 Course 2	
Spring 2012	Cohort 1 Course 3 (completed)	MCAS 2011-2012
Fall 2012	Cohort 2 Course 1 (started)	
Spring 2013	Cohort 2 Course 2	MCAS 2012-2013
Fall 2013	Cohort 2 Course 3 (completed)	
Spring 2014		MCAS 2013-2014

At the start of the first course offered by the program (Spring of 2011), an average of 47.9% of the students in the target five districts in Massachusetts were performing at the proficient or advanced levels in the MCAS, a performance that fell below the state average of 55.0% by 7.1 points (Table 4). We matched each of these five target districts to five other similar districts in the state. The MCAS test results in the Spring of 2011 served as the main criterion for matching districts. Once several districts with similar performance to that of each target district were identified, we chose five of them as comparison districts based upon similarity in terms of size of student population, ethnic composition, and several socio-economic and demographic variables. Table 4 also shows that the average percentage of students in the comparison districts performing at the Proficient and Advanced levels was approximately the same as that for the target districts.

Table 4: Percentage of students performing at the Proficient and Advanced Levels in the MCAS Assessment at the end of academic year 2010-2011 (baseline).

Districts	Percentage of Advanced and Proficient Students
Target	47.9
Comparison	48.0
State	55.0

Compared to the whole state of Massachusetts (see Tables 5 and 6), students from the five target districts and from the comparison districts were more likely (1) to come from minority groups; (2) to come from homes where English was not the first language; (3) to be English language learners; (4) to come from low-income populations; (5) to be diagnosed with disabilities; and (6) to be eligible for reduced lunch.

Table 5: Population characteristics and number of students in the state, target, and comparison districts in 2010-2011

Districts	Population Total	Population Density per square mile)	Average Family Income	Percent of Families Below Poverty Line	Number of K-12 Students
State	6,646,144	840	83,371	7.6	954,773
Target	225,756	5,783	66,632	7.2	23,179
Comparison	1,257,947	4,062	63,082	8.0	165,385

Table 6: Percentage of grades 5 to 8 students in each group in the state, target, and comparison districts in 2010-2011

Districts	Minority Students	First Language not English	English Language Learner	Low-Income	With Disabilities	Free Reduced Lunch
State	34.0	17.3	7.7	37.0	17.0	37.0
Target	41.9	25.1	8.3	46.7	18.8	46.7
Comparison	38.6	24.6	7.9	45.9	16.7	48.3

As Table 7 shows, teachers from the comparison districts were similar to those of the target districts according to seven criteria.

Table 7: Percentage of state, target, and comparison districts' teachers by relevant variables in 2010-2011

Districts	Licensed in Teaching Assignment	Classes with Highly Qualified	Student Teacher Ratio	Average Salary	% Retained	Exemplary*	Proficient*
State	97.5	97.7	13.9	$70,340	88.20	---	---
Target	98.58	99.02	14.2	$69,413	87.64	8.48	83.16
Comparison	98.73	97.67	14.3	$68,202	89.20	6.21	87.50

Educators are evaluated by the state and earn a Performance Rating of Exemplary, Proficient, Needs Improvement, or Unsatisfactory, based on "multiple categories of evidence, including (a) evaluator judgments based on observations and artifacts of professional practice; (b) evidence of fulfillment of both professional practice and student learning goals; and (c) multiple measures of student learning, growth, and achievement". In the Proficient category are teachers whose "performance fully and consistently meets the requirements of a standard or overall" while the Exemplary category includes those who exceed those standards (http://profiles.doe.mass.edu/statereport/educatorevaluationperformance.aspx).

In our analysis, the results of the five target districts are compared to those of the respective comparison districts and to the results for the whole state of Massachusetts. This amounts to three sources of data:

Data from the target districts: The target districts include both students who had been taught by teachers in the program and those taught by other teachers. These are the five Massachusetts districts in which part of the teachers had participated in the first two cohorts of the program. Our analysis includes data on 6,531 students in grades 5 to 8 from the Spring of 2011 to the Spring of 2014 in these districts. This represents approximately 2.3% of the students in the state. The state does not release data at the teacher-level. Therefore, target district data originate from students who had been taught by teachers in the program as well as students taught by other teachers. This provides a conservative estimate of impact, something that should be borne in mind when examining changes in student performance, below.

Data from the comparison districts: The five comparison districts were matched according to criteria listed in Tables 4, 5, and 6 to each one of the target districts, before accessing MCAS results for the years 2013-2015. Comparison districts educational goals, as stated in publicly available websites, were similar to educational goals in the target district.

Data from the State of Massachusetts: State results refer to all Massachusetts school districts (281,964 students from grades 5-8 in 2014).

Overall Results

We compared the percentages of students from grades 5 to 8 at the Advanced and Proficient levels in the target districts to (a) the percentages of students across the state and (b) the percentages of students in the 25 corresponding comparison districts (five comparison districts for each of the five target districts).

As already noted, in the Spring of 2011, when the program started, the overall percentage of students at the Proficient and Advanced levels in the target districts was nearly identical to that of comparison districts (47.9% vs. 48.0%). Three years later, the target districts had surpassed comparison districts by 4.5 points and had narrowed the gap with regard to the state from 7.1 to 4.5 percentage points (see Table 8 and Figure 4).

Table 8: Average Percentage of Students at the Advanced and Proficient Levels from 2011 to 2014.

Districts	2010-2011	2011-2012	2012-2013	2013-2014
State	55.0	55.0	57.3	55.8
Target	47.9	48.5	52.2	51.3
Comparison	48.0	46.8	47.9	46.8

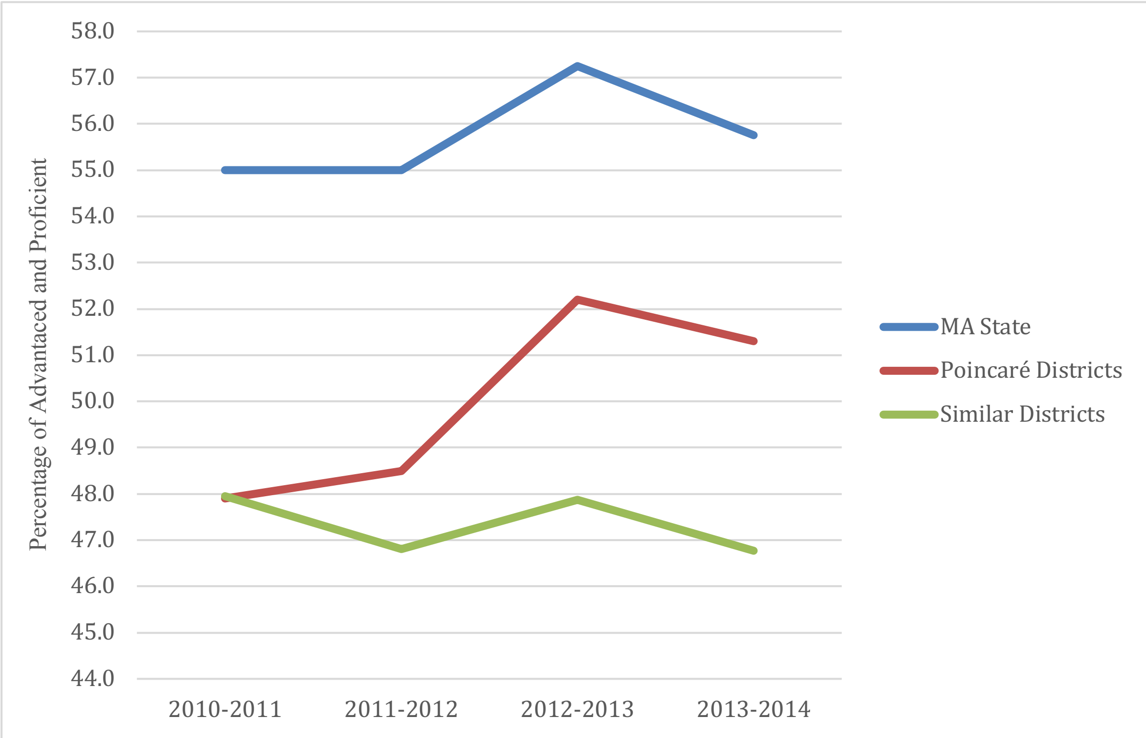

Figure 4: Percentage of students at the Advanced and Proficient levels in the MCAS test, from 2011 to 2014, in the State, Target, and Comparison Districts.

Results by District, Grade, and Percent of Teachers in the Program

Results for each grade level in each target district were compared to the average for the same grade level in each group of five corresponding comparison districts. This resulted in 20 data pairs (four grade levels for each of the five districts). Table 9 shows the differences between the target and comparison districts, along with the corresponding percentage of teachers who had taken the three courses.

Table 9. Percentage of teachers in the program and corresponding change, from 2011 to 2014, in the percentage of students at the advanced and proficient levels, for each grade, in each district.

District and Grade Level	Percentage of Teachers in Target Districts	Difference (Target - Comparison) in % of Change 2011 to 2014
A - Grade 5	0.25	10.0
A - Grade 6	0.43	6.0
A - Grade 7	0.50	8.6
A - Grade 8	0.00	-0.4
B - Grade 5	0.29	8.2
B - Grade 6	0.33	3.4
B - Grade 7	0.14	-3.8
B - Grade 8	0.17	3.2
C - Grade 5	0.13	-1.4
C - Grade 6	0.25	-1.4
C - Grade 7	0.25	-4.2
C - Grade 8	0.00	2.0
D - Grade 5	0.25	11.5
D - Grade 6	0.60	6.8
D - Grade 7	0.75	9.0
D - Grade 8	0.67	14.1
E - Grade 5	0.14	7.6
E - Grade 6	0.30	-9.4
E - Grade 7	0.60	10.4
E - Grade 8	0.56	11.4

The relative performance of target to comparison districts is determined by subtracting the comparison district change from the target district change at the same grade level. Note that in six cases, where the percentage of participating teachers was equal to 30% or lower, the differences were negative. Table 10 shows the frequency of negative and positive differences according to the percentage of teachers from the district who had enrolled in the teacher development program. The association between percentage of teachers and kind of difference (positive or negative) was significant ($p = .024$).

Table 10. Frequency of negative and positive differences in change by percentage of teachers in the program.

Percent of Teachers in Target Districts	Negative Difference in Change	Positive Difference in Change
0 to 30%	6	6
More than 30%	0	8

Fisher Exact Probability = .024

The y axis in Figure 5 displays the relative achievement of target districts vis-à-vis comparison districts, at each grade level, for school years 2010-2011 to 2013-2014. Each point corresponds to the percentage of teachers' in that grade level taking part in the teacher development program and the relative gain/loss in student achievement. If there were no program effect on the students, one would expect the plotted points to be randomly distributed above and below the x-axis. This is indeed what happens when the percentage of participating teachers is 30% or less (half of the 12 differences were positive and half were negative). When the teachers enrolled in the Institute comprised more than 30% of the teachers in a target district, student achievement in the target districts (as measured by the pre-post gains) invariably surpassed that of the comparison districts. Moreover, the greater the percentage of participating teachers in a district, the better the relative gains of the target district. This is reflected in the upward slant of the regression line and the significant Spearman correlation of r =.57, (p=.0045).

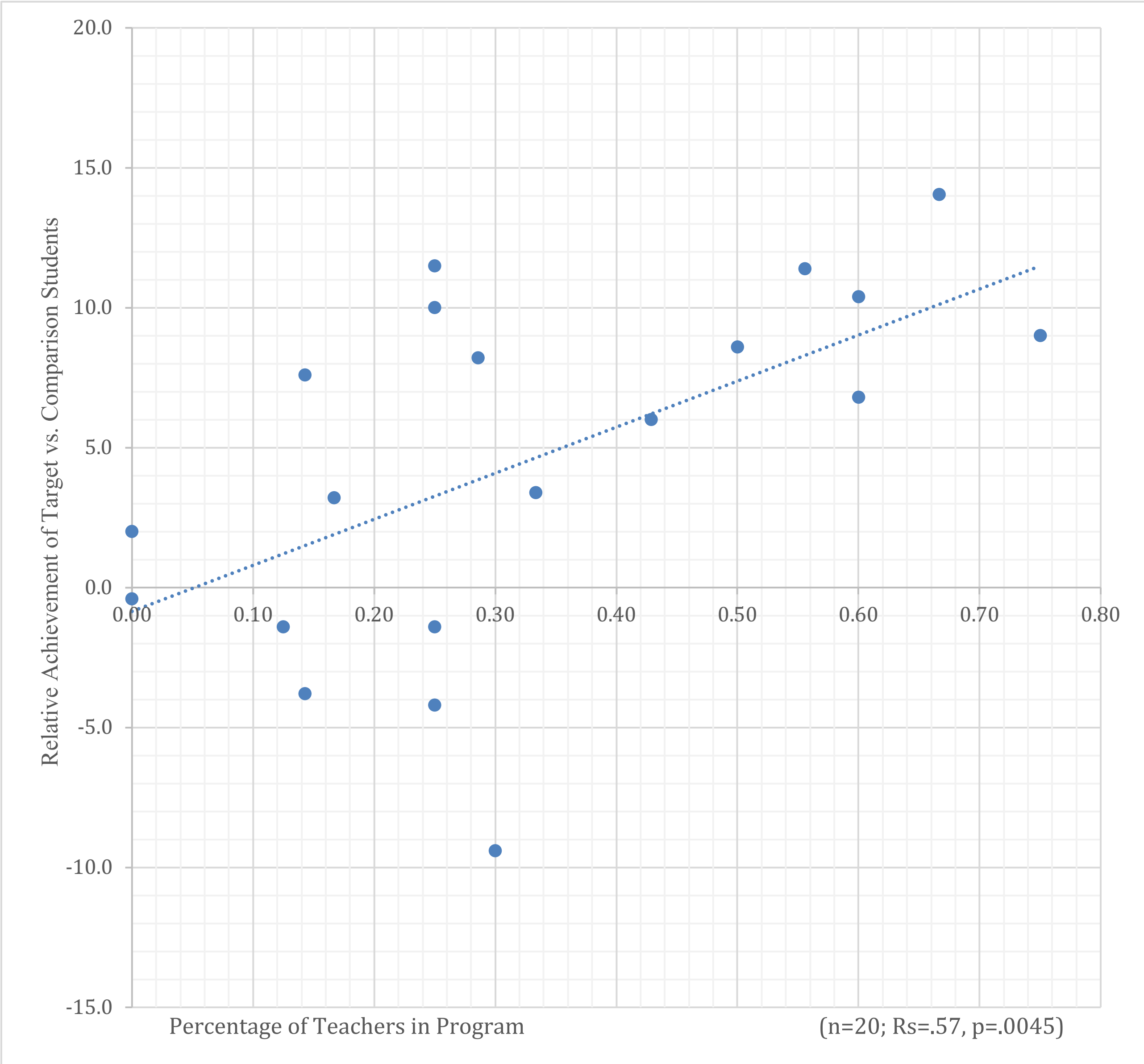

Figure 5: Relative achievement (target versus comparison district), for each grade level from 2011 to 2014, as a function of the percentage of teachers who completed the program.

Discussion

We described a teacher development program that employs functions as a lens through which key topics in the middle and early high-school mathematics curriculum, from arithmetic to algebra, are reinterpreted, studied in depth, and interconnected. We also documented changes observed in classroom teaching and in student performance on a state mandated test.

As we noted, functions officially enter the mathematics curriculum in the United States in Grade 8 and are a major feature of present-day high-school mathematics. This is consistent with the view that functions are essentially certain types of closed-form algebraic expressions, namely those expressions for which any given input from one or more variables is associated with a single output value. However, functions are not the same as algebraic expressions. Different algebraic expressions can represent the same function, most functions cannot be represented by algebraic expressions, and there are many ways, besides algebraic expressions, that allow the representation of functions.

When one takes this to heart, it becomes clear that functions can be worked with and expressed before the arrival of algebraic notation, a theory of functions, and even a definition of the term *function*. This happened historically. And, we suggest, something similar can happen in mathematics education. Functions are already present throughout the early mathematics curriculum, arriving no later than the four basic operations of arithmetic. However, they tend to remain underexploited until and unless students are given opportunities to handle problems involving generalized variables. In our approach, we brought the use of variables to a leading role in the presentation of the mathematical content.

The evaluation model of the program rested on comparisons of (a) the teachers' performance at the start and end of the program, and (b) the performance of all students in each target district versus similar districts. There remains the question of selection bias between the target and comparison districts. Could it be that the districts that joined the teacher development project were somehow more likely to have improved on their own? Two facts argue against the claim that somehow the project selected target districts more likely to show gains. Firstly, comparison districts were closely matched to the target district in terms of prior performance on the state mathematics test, ethnic/racial breakdown, percentage of students enrolled in school lunch programs, average household income, and number of students in the district. Results showed that, at target districts grades where more than 30% of their teachers enrolled in the development program, students' performance on state-mandated tests was significantly better than that of students from the comparison districts. Moreover, there was a significant, positive correlation between the percentage of teachers in the program and the mathematical performance of students in their districts. This correlation seems difficult to explain by appealing to extraneous factors. It would be interesting to compare the performance of students of program enrollees with the remaining students from the same district. However, state norms of confidentiality prevented us from obtaining data that would enable such a comparison.

In order to conclude, with a minimum of doubt, that the teacher development program was responsible for the statistically significant results obtained, one would ideally employ an experimental design involving a random assignment of teachers, from the same schools, to the intervention and comparison groups. The present design was not experimental. We originally undertook the project as an effort to improve mathematics teaching and learning in New England, based on experience we had gained as researchers and while collaborating with mathematicians and physicists. Given the duration and intensity of the program (10-12 hours a week, extending over 18 months) we decided that participation should be voluntary. Had an experimental design been employed, teachers would not have a choice between treatment and control conditions.

Possible Reasons for the Program's Contribution

Although there is evidence that both the teachers and their students derived benefits from the development program, identifying the specific features that could be responsible for the observed gains is a far more challenging task. A multi-faceted program such as the present one leaves considerable room for interpretation and requires theoretical analysis in addition to an examination of empirical data. We recommend that the reader view the ensuing discussion as having the character of clinical judgment.

The length and intensity of the program (three 15-week terms with a workload of 8-12 hours per week) and institutional and financial support may well have been critical for teachers to engage themselves deeply in what was, for many, a novel approach to K-12 mathematics. In the online setting, teachers engaged regularly with colleagues from diverse districts and with course instructors. The teachers also met weekly with colleagues at their schools and monthly with staff from the program. In this way, teachers received a substantial amount of continuous feedback on their work and on their teaching projects that may have contributed significantly to their improvement.

The novelty of the program manifested itself not only through the mathematical content—notably, the early introduction of functions in the curriculum—but also in the way in which functions were adopted as a lens through which standard fare of mathematics could be reinterpreted. Perhaps equally important was the variety of mathematics problems teachers encountered in their weekly online discussions. The problems were directly related to the weekly readings, but teachers would not be able to simply find the answer in the text. Instead, they needed to struggle to make sense of the problems and frame them appropriately. This is likely to have contributed to their deeper understanding of mathematics content.

The present results recall those described by Chazan (1999, 2000) and Huntley et al. (2000) regarding the contribution of a functions approach to teaching mathematics in high school. The notion that functions provide important means for uniting otherwise disparate topics in mathematics is not a minor feature of the approach, nor simply one of many “factors” that may or may not play a role. Instead, as we have tried to show, this idea and a host of related ideas about mathematics underlay design decisions about the content of the courses and the sorts of activities and discussions to be engendered among teachers in the courses and in their own classrooms.

The belief that teachers need to take students’ mathematical reasoning into account is uncontroversial. However, implementing this view in classrooms requires that teachers learn to closely listen to students and engage them in productive discussions about multiple mathematical topics. Teachers need to be sufficiently attentive to students’ thinking so that they can respond, in their lesson planning and in real-time, to the variety of responses, interpretations, and questions that arise concerning any given topic.

Two resources, in particular, were employed to help teachers develop the ability to listen and respond constructively to students' thinking. Teachers either interviewed small numbers of their own students about selected topics or studied videotaped classroom episodes showing elementary and middle school students reasoning about problems entailing functions and algebraic thinking. They then analyzed, discussed with colleagues, and reported on their findings, examining the reasoning of students at a much slower pace than is typically available in a classroom. In the planning of classroom activities, teachers were encouraged to hold open-ended discussions with students, taking into account students' own ideas and ways of representing the mathematical problems.

The systematic classroom observations of cohort 1 teachers in nine participating school districts revealed that teachers increasingly encouraged students to use the multiple representations promoted by the program to solve mathematics problems. According to ratings by external evaluators, the teachers also listened more carefully to students, gave them more time to respond to questions, and more often addressed the thinking advanced by their students. This may have led students to participate more and demonstrate more interest during the lessons. As students became confident in examining and representing contextualized problems, they may have been more likely to solve standard test problems.

These changes are consistent with the open-ended nature of problems entertained (including both problems for teachers and problems designed for students), the pedagogical assignments, and the feedback by instructors. These features matched general recommendations for successful teacher development programs, such as merging content and pedagogical knowledge, collaboration among teachers, attention to student reasoning, and institutional support (see, for example, Hill; 2007; Bautista and Ortega-Ruiz, 2015; Desimone, 2009).

Closing Thoughts

It is difficult to determine, on the basis of the current results, which sorts of engagement were most important for teachers to improve their teaching and make a difference in their students' achievement on standard tests. We would hope that the particular approach through functions and their representations in the teaching of mathematics, which aimed at a deeper understanding of mathematics and closely intertwined content and pedagogical aspects, played a fundamental role in the changes we observed. In order to firmly establish such a conclusion, it would be important to know more about the advances in mathematical proficiency of both students and teachers.

Likewise, our teacher observations mostly concern matters of pedagogy. It still remains to be seen whether and how teachers and students alike have benefited from the particular approach to functions advocated by the program. And it would be useful to have measures of their reasoning about relations among variables.

Experimental studies of specific aspects of the program could contribute to clearer findings of the impact of our approach. For instance, smaller scale studies with experimental and control groups within the same schools, on topics such as fractions, ratios, and proportion, would identify how teaching these topics through the lens of functions may lead to particular results. Studies should also explore whether and how the introduction of functions in elementary and middle schools facilitates the future learning of algebra in high-school.

Another area to explore concerns the impact different resources had on teachers or students' progress. We believe that institutional and financial support contributed to the high completion rate among participants. Programs that provide no financial reward, do not fulfill state or district requirements, nor contribute to career advancement are likely to garner more limited participation. Further studies are needed to quantify how incentives play a role in a program's success.

We do not claim that our approach is the only one that will translate into students long–time gains. We also feel that many features of our program could be improved. Perhaps better preparation of pre-service mathematics teachers would make this kind of in-service development program unnecessary. We hope that the encouraging results we achieved with this rather long and intense program will help others, as well as ourselves, to further explore, in new studies, ways to improve the preparation and development of mathematics teachers and to consider curriculum changes and classroom activities that would address the challenges of promoting students' deep understanding of mathematics.

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