Approximate reliability and resilience indices of over-year reservoirs fed by AR(1) Gamma and normal flows

RICHARD M. VOGEL
Department of Civil & Environmental Engineering, Tufts University, Medford, Massachusetts 02155, USA

T. A. McMAHON
Department of Civil & Environmental Engineering, University of Melbourne, Parkville, Victoria 3052, Australia

Abstract Approximate storage-reliability-resilience-yield (S-R-R-Y) relationships are derived for over-year water supply systems fed by autoregressive lag one Gamma and normal inflows. It is shown that a two-state Markov model may be exploited along with S-R-R-Y relationships to describe the general behaviour of over-year water supply systems. The two-state Markov model is also used to relate the probability of *n*-year no-failure operations (the concept of reliability used in the USA) to the steady-state probability of a system failure (the concept of reliability used in Australia and elsewhere) yielding a unified view of system reliability. Resiliency criteria are introduced which indicate whether or not a reservoir system is likely to return to normal operations once a failure has set in. These criteria indicate that the resilience of an over-year water supply system is generally independent of its steady-state reliability. The conditions under which finite reservoir systems behave like semi-infinite reservoir systems are also documented, and a factor is derived which describes the impact of the serial correlation of the inflows on the derived S-R-R-Y relationship.

Indices approchés de fiabilité et de résilience de réservoirs interannuels alimentés par des apports autorégressifs d'ordre 1 suivant une loi Gamma ou Normale

Résumé Des relations approximatives entre stockage, fiabilité, résilience et débit ont été établies pour des systèmes d'approvisionnement en eau interannuels alimentés par des apports autorégressifs d'ordre 1 suivant une loi Gamma ou Normale. On a démontré qu'un modèle markovien à deux états peut être utilisé parallèlement aux relations entre stockage, fiabilité, résilience et débit afin de décrire le comportement global de systèmes d'alimentation en eau interannuels. Le modèle Markovien à deux états a également été utilisé pour exprimer la probabilité d'absence d'échec en n années (c'est la notion de fiabilité utilisée aux Etats-Unis) en fonction de la probabilité d'échec du système en régime permanent (c'est la notion de fiabilité utilisée en Australie et dans certains autres pays) fournissant ainsi une conception unifiée de la fiabilité. Des critères de résilience ont été introduits, qui indiquent si un réservoir est ou non susceptible de revenir à un mode d'exploitation normal après un échec. Ces critères indiquent que la résilience d'un système d'alimentation en eau interannuel ne dépend généralement pas de sa fiabilité en régime

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permanent. Les conditions dans lesquelles des systèmes de réservoirs finis se comportent comme des systèmes de réservoirs semi-infinis ont également été étudiées et un paramètre décrivant l'impact de la corrélation sérielle des apports sur la relation établie entre stockage, fiabilité, résilience et débit a été défini.

INTRODUCTION

Two general classes of reservoir systems exist: over-year (alternatively known as carry-over) and within-year systems. Within-year systems are characterized by reservoirs which typically refill at the end of each year. Such systems are particularly sensitive to seasonal and even daily variations in both the hydrological inflows and the system yield. Over-year systems do not typically refill at the end of each year. Here a failure is defined as the inability of a reservoir system to provide the contracted demand in a given year. Water supply failures for within-year systems tend to be short-lived in comparison with over-year systems since within-year systems tend to refill on an annual basis. Naturally, all reservoir systems exhibit some combination of over-year and within-year behaviour. However, for the moment consider two reservoir systems having an equal steady-state probability of a failure year q, one being a system dominated by exclusively over-year behaviour and the other dominated exclusively by within-year behaviour. During an n-year period, one would expect nq failures, yet for the within-year system those failure sequences will typically last only a few days or months whereas for the over-year system a typical failure may last years (if no new water is imported and demand curtailment programmes are not implemented).

A prerequisite to the proper operation, management and design of over-year reservoir systems is a thorough understanding of the likelihood, duration and magnitude of potential reservoir system failure sequences. For this purpose, the storage-reliability-yield (S-R-Y) relationship is one important ingredient. However, reliability statements alone do not convey information regarding the consequences of failure (system vulnerability) or the ability of the system to recover from a failure (system resilience). This study formulates an approximate yet general approach to understanding the overall behaviour of over-year reservoir systems focusing attention on both the S-R-Y relationship and the frequency, magnitude and duration of reservoir system failures.

Storage reservoirs tend to be large and complex systems requiring equally complex mathematical models to simulate their behaviour. Historically, one modelling approach has been replaced by or appended to another more complex one to deal with such issues as the Hurst phenomenon, model parameter uncertainty, optimal operations, spatial and temporal disaggregation schemes, etc. What is lacking are simple yet accurate "back-of-the-envelope" type methods which give insight into a wide range of reservoir storage system characteristics and reliability indices before one embarks on a complex modelling expedition. Such "back-of-the-envelope" methods would also be useful for the education of future water supply analysts.

As academics who attempt to teach future water resource engineers, the authors have found it challenging to explain the deterministic and stochastic behaviour of water supply systems similar to the manner in which flood frequency analysis and other hydrological concepts are taught. Most current textbooks recommend the simulation of water supply system behaviour using either the historical record or synthetic streamflow traces, the latter often in conjunction with the sequent peak algorithm. Yet such exercises do not always impart much knowledge of overall reservoir system behaviour other than state the desired S-R-Y relationship. What is needed are simple yet accurate algebraic expressions which can be easily manipulated to describe a host of resilience and vulnerability indices in addition to the S-R-Y relationship so that, for example, one could illustrate the frequency, magnitude and duration of reservoir system failure sequences. Otherwise, one is often lost in the myriad of computer output from more complex reservoir system simulation exercises.

The goal of this study, similar to that of Vogel & Bolognese (1995), is to develop a set of simple expressions which both enhance our understanding of the behaviour of water supply systems and provide an explanation of over-year reservoir system behaviour.

This study is similar to Vogel & Bolognese (1995), yet the approach taken here is quite different. Vogel & Bolognese (1995) derive S-R-R-Y relations for AR(1) normal and lognormal inflows, whereas this study derives S-R-R-Y relations for AR(1) normal and Gamma inflows using an entirely different approach. This study employs a very simple storage-reliability-yield relationship introduced by Gould (1964), as opposed to the multivariate regression equations developed by Vogel & Stedinger (1987) which are used by Vogel & Bolognese (1995). A similar study by Vogel (1987) uses a two-state Markov model of the states of reservoir system to derive relationships among *n*-year no-failure reliability and steady-state reliability for reservoir systems dominated by within-year behaviour. However, that study does not connect reliability indices to other system parameters such as storage capacity, yield or streamflow statistics as is done here, nor does it deal with over-year systems. Reliability indices such as the average recurrence interval of a reservoir system failure, derived by Vogel (1987), are useful for describing the likelihood of future reservoir system failures; however, they do not expose the consequences of that failure. Hashimoto *et al.* (1982) describe the use of vulnerability and resilience indices for exposing the consequence of reservoir system failures.

A REVIEW OF GENERAL STORAGE-RELIABILITY-YIELD RELATIONSHIPS

What is needed for a unified understanding of the behaviour of storage reservoir systems is a model which relates system storage to system reliability, yield, inflow characteristics and release rule. McMahon & Mein (1986), Klemeš (1987), Vogel & Stedinger (1987), Votruba & Broza (1989) and Phatarford (1989) provide a review of the literature relating to the development of general S-R-Y relationships. Essentially, two schools of thought exist regarding the development of S-R-Y relationships. In the USA, (S-R-Y) relationships are usually based on an interpretation of reliability which depends upon the most critical drawdown period of a reservoir over its planning horizon. Basically, these methods utilize the automated equivalent of Rippl's (1883) mass curve approach, known as the sequent peak algorithm (see Loucks *et al.*, 1981, Section 5.3.2), in conjunction with stochastic streamflow models to obtain the probability of no-failure reservoir operations, p, corresponding to a specific reservoir capacity-yield combination. Alternatively, in Australia and elsewhere, a common approach to estimating the S-R-Y relationship is to determine the steady-state probability of failure, q, corresponding to a specific reservoir capacity-yield combination (McMahon & Mein, 1986; Votruba & Broza, 1989).

A two-state Markov model of reservoir system states is adopted in order to be able to relate system storage, reliability and yield to the frequency, magnitude and duration of reservoir system failures. In addition, the two-state Markov model allows one to relate steady-state reliability, $1 - q$, to n-year no-failure reliability, p. Others have successfully exploited a two-state Markov model for representing sequences of reservoir surplus and failures (Hirsch, 1979; Stedinger *et al.*, 1983; Vogel, 1987; Vogel & Bolognese, 1995). However, those studies have not provided a direct link between the two-state Markov model and a reservoir system model.

A MEASURE OF THE RESILIENCE OF RESERVOIR SYSTEMS

Hazen (1914), followed by Sudler (1927) and Hurst (1951), introduced one of the most useful indices of reservoir system performance, here defined as the resiliency index:

$$m = \frac{(1-\alpha)\mu}{\sigma} = \frac{(1-\alpha)}{C_v} \tag{1}$$

where α is the annual yield as a fraction of the mean annual inflow μ, σ is the standard deviation of the annual inflows, and C_v is the coefficient of variation of the annual streamflows. Figure 1 shows the relationships among m, α and C_v given in equation (1). Perrens & Howell (1972) termed m the standardized inflow. After its use by Hurst (1951), the non-dimensional index m has subsequently found use in both analytical investigations in "water storage theory" (Pegram *et al.*, 1980; Buchberger & Maidment, 1989) and in Monte Carlo investigations of the storage-reliability-yield relationship (Perrens & Howell, 1972; Vogel & Stedinger, 1987). Vogel & Stedinger (1987) suggest that as long as $0 \leq m \leq 1$, the system is dominated by over-year behaviour, whereas if $m > 1$, the system is dominated by within-year behaviour. It is shown later on that m is related to the probability that a storage reservoir will recover from a failure and hence it is an ideal measure of reservoir system resiliency. Over-year reservoirs with values of m near 0 are less likely to recover from a failure than reservoirs with values of m near unity. Systems with low resiliency (m near zero) are characterized by having either very large values of C_v or α or both (see Fig. 1). Similarly, reservoirs with values of m near or above unity are more likely to refill once empty and hence such systems are more likely to exhibit within-year rather than over-year behaviour. When $m > 1$, the theory presented here does not apply and the reservoir exhibits predominantly within-year behaviour, that is, the reservoir refills in most years.

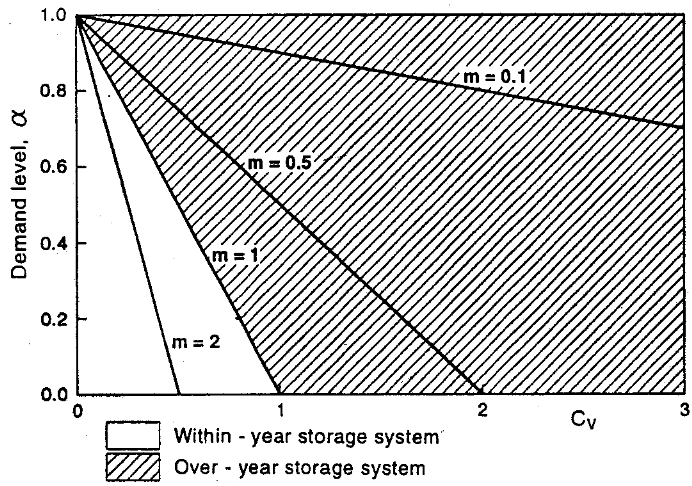

Fig. 1 The demand level α as a function of the resiliency index m and the coefficient of variation of the inflows C_v.

Since resilient reservoir systems (large resiliency index m) tend to have either small demand levels α or small coefficients of variation, one expects that regions with low streamflow variability will contain more resilient reservoir systems than regions with high streamflow variability, for a fixed demand level. Similarly, demand levels generally increase over time, and so one expects a general reduction in the overall resiliency of reservoir systems over time.

APPLICABILITY OF GENERAL STORAGE-RELIABILITY-YIELD RELATIONSHIPS

Most general analytical S-R-Y relationships are inadequate for design purposes because they cannot be general and at the same time account for complexities such as the seasonal nature of evaporation, precipitation, streamflow and operating rules. Phatarford (1989) recommends using Monte Carlo simulation methods for handling specific reservoir design problems and using general analytical S-R-Y relationships for obtaining qualitative results and for obtaining insight into the mathematics of reservoir operations. Monte Carlo simulations of reservoir systems using monthly or even daily time steps are so detailed that it is easy to miss general yet important features of the reservoir operations. For example, significant attention has been devoted in the literature to the development and application of monthly stochastic streamflow models for use in reservoir operations studies, yet few studies have evaluated the general relationships among reservoir system reliability, resilience and vulnerability.

Many investigators dispense with general over-year S-R-Y relationships immediately since they are thought to be too simple to capture the overall complexity of real water supply systems. To the contrary, it is shown later on that as long as the resiliency index m in equation (1) is in the range $0 \leq m \leq 1$, the seasonal behaviour of the system is damped out. For example, Vogel & Hellstrom (1988) showed that for the Quabbin Reservoir system, which provides the water supply for much of eastern Massachusetts, USA, an annual simulation of the system was almost indistinguishable from a monthly simulation of the system. This is expected, since the firm yield corresponds to $\alpha = 0.915$ and $C_v = 0.34$, hence $m = 0.25$. As long as m remains in the range $0 \leq m \leq 1$, the system will be dominated by over-year behaviour, and seasonal variability of the operations and hydrological processes will be masked in terms of the overall reservoir system behaviour.

In the following sections, simple yet general S-R-Y relationships are developed for reservoir systems dominated by over-year behaviour and fed by autoregressive lag one (AR(1)) normal or Gamma inflows. Although the derived relationships are approximate, the results can be applied to a wide class of real reservoir systems.

STORAGE-RELIABILITY-YIELD RELATIONSHIPS BASED ON THE STEADY-STATE PROBABILITY OF A FAILURE

S-R-Y relationships for independent normal and Gamma inflows

Perhaps the simplest approach to estimating the S-R-Y relationships for systems dominated by over-year behaviour was first introduced by Kritsky & Menkel (1932) (see Votruba & Broza, 1989, pp 196-198, for an English translation of their work). Independently, Gould (1964) suggested a similar procedure (see also McMahon & Mein, 1986) for a description of Gould's method). As Kritsky & Menkel (1932) published their work in Russian, Gould (1964) did not cite it.

Assuming annual streamflows are independent and arise from either a normal or Gamma distribution, then the sum of n annual flows, each with mean

μ and standard deviation σ, will follow either a normal or Gamma distribution respectively, with mean $n\mu$ and variance $n\sigma^2$. Note that one could also invoke the central limit theorem to generalize this approach to any underlying flow distribution. If the n-year flows (the sum of annual flows) are termed $\Sigma x(n)$, then $\Sigma x_q(n) = n\mu + t_q \sigma n^{1/2}$ is the qth percentile of those annual sums, where t_q is the qth percentile of a standardized random variable from either a normal or Gamma distribution. Assuming one wishes to deliver a total yield of $\alpha n\mu$ during an n-year period with reliability $1 - q$, then a reservoir of capacity:

$$\begin{aligned} S_q(n) &= \alpha n\mu - \Sigma x_q(n) \\ &= \alpha n\mu - n\mu - t_q \sigma n^{1/2} \end{aligned} \tag{2}$$

is required. Now the critical drought will correspond to the largest value of $S_q(n)$ among all possible capacities. Kritsky & Menkel (1932) suggested using iterative procedures to find the length of the critical drawdown sequence n, which maximizes $S_q(n)$. Instead, one may follow Gould (1964) who found the length of the critical drawdown sequence n^* by differentiating equation (2) with respect to n and equating to zero, which leads to:

$$n^* = \left[\frac{t_q \sigma}{2(1-\alpha)\mu} \right]^2 \tag{3}$$

Combining equations (1) and (3) produces the simple result:

$$n^* = \frac{t_q^2}{\ } \tag{4}$$

which is much simpler, yet equivalent to the iterative procedures suggested by Kritsky & Menkel (1932).

For example, if flows are normally distributed and the annual reliability is 98%, then $q = 0.02$, $t_{0.02} \approx -2$ and the critical reservoir drawdown period n^* is equal to $1/m^2$. As m approaches one, the drawdown period approaches one year as expected for within-year systems.

Substitution of n^* from equation (4) into equation (2) leads to the desired reservoir capacity of:

$$S_q = \frac{t_q^2 \sigma}{4m} \tag{5}$$

which may be termed Gould's method. Another interpretation of S_q is as follows. In a given year, the yield is $\alpha\mu$, hence $(1 - \alpha)\mu$ is available to fill the reservoir. Thus during the critical drawdown period the reservoir capacity must be $S_q = n^*(1 - \alpha)\mu$ or simply $S_q = n^* m\sigma$, which is identical to equation (5).

When annual inflows are normally distributed, the simple estimate of $t_q = z_q$ may be used (Tukey, 1960):

$$z_q = 4.91\ (q^{0.141} - (1-q)^{0.141}) \tag{6}$$

which provides a good approximation of the inverse of a standard normal variate in the range $0.01 \leq q \leq 0.99$.

When Gould (1964) derived this procedure for flows which arise from a Gamma distribution, he developed an approximation for t_q which used the fact that the difference between the lower quartiles of a standard normal and a standard Gamma variate are approximately fixed for a given value of q over a range of coefficients of variation. However, that approximation, which Gould developed using a slide rule, is only useful over a limited range of q and C_v, hence it is better to use a direct approximation to the inverse of a standardized Gamma random variable using $t_q = w_q$ where w_q is given by:

$$w_q = \frac{2}{\gamma}\left[\left\{1 - \left[\frac{\gamma}{6}\right]^2 + \left[\frac{z_q\gamma}{6}\right]\right\}^3 - 1\right] \tag{7}$$

with $\gamma = 2C_v$ when inflows follow a Gamma distribution. Equation (7), introduced by Wilson & Hilferty (1931), was shown by Chowdhury & Stedinger (1991) to have less than 1% error in the range $|\gamma| \leq 3$.

Teoh & McMahon (1982) and McMahon & Mein (1986) used 35 Australian streams and 12 Malaysian rivers to show that Gould's approach in equation (5) yields a satisfactory approximation to the required capacity of a storage reservoir obtained using the more complex probability matrix methods. Of all the preliminary design procedures investigated, McMahon & Mein (1986) recommend Gould's approach. That approach (equation (5)) is also suggested for the preliminary design of storage reservoirs in Maidment (1992). Monte Carlo simulation experiments described in a later section of this study summarize the validity of Gould's approach.

When Kritsky & Menkel (1932) introduced their approach they did not limit it to independent normal or Gamma inflows, since it is possible to use graphical flow duration curve procedures to estimate $\Sigma x_q(n)$ from a sample of annual streamflows derived from any probability density function (*pdf*). The normal and Gamma *pdfs* are introduced here simply so that general analytical results can be formulated.

S-R-Y relationships for serially correlated normal and Gamma inflows

In the derivation of equation (5), it was assumed that, for independent inflows, the variance of the sum of n annual inflows is equal to the sum of the variances and hence the annual flow sums, $\Sigma x(n)$, had variance equal to $n\sigma^2$. However, it is well known that, for serially correlated flow processes, the variance of a flow sum is inflated in proportion to the serial correlation of the process. For example, for a first-order autoregressive process, Loucks *et al.* (1981) showed that:

$$\mathrm{Var}(\Sigma x) = \mathrm{Var}(nx) = n\sigma^2\left[1 + \frac{2\rho}{n}\left\{\frac{n}{(1-\rho)} - \frac{1-\rho^n}{(1-\rho)^2}\right\}\right] \tag{8}$$

where ρ is the lag-one serial correlation of the annual flows. For most situations encountered in practice, the term $(1 - \rho^n)/(1 - \rho)^2$ is negligible in comparison to $n/(1 - \rho)$, in which case equation (8) can be simplified after some algebra to yield:

$$\mathrm{Var}(\Sigma x) = n\sigma^2\frac{(1+\rho)}{(1-\rho)} \tag{9}$$

Substitution of the square root of equation (9) into equation (2) (instead of just $\sigma n^{1/2}$) and repeating the derivation described by equations (3), (4) and (5) leads to:

$$n^* = \frac{t_q^2}{4m^2}\left[\frac{1+\rho}{1-\rho}\right] \tag{10}$$

$$S_q = \frac{t_q^2\sigma}{4m}\left[\frac{1+\rho}{1-\rho}\right] \tag{11}$$

Equation (11) can be used with either equation (6) or equation (7) to obtain the relationships among required storage, yield and reliability when inflows follow either a first-order autoregressive process and are normally distributed, AR(1)-N, or Gamma distributed, AR(1)-G, respectively. If flows are AR(1)-G, then one must substitute γ_o into equation (7) instead of γ, where γ_o is given by:

$$\gamma_o = \gamma\left[\frac{1+\rho^3}{(1-\rho^2)^{3/2}}\right] \tag{12}$$

in order to preserve the skewness of the generated flows (Fiering, 1967).

Interestingly, Phatarford (1986) derived an identical factor, $R = (1 + \rho)/(1 - \rho)$, in equation (11) using an entirely different approach. Phatarford terms R a correction factor which is the ratio of the reservoir size with ρ as the annual serial correlation to the size when ρ is assumed equal to zero.

Figure 2 illustrates the relationships among S/σ, m, ρ and C_v for AR(1)-N and AR(1)-G inflows, corresponding to a fixed reliability $1 - q$ equal to 99%. Here the subscript q from S is dropped, so that $S = S_q$. Figure 2(a) compares the use of equation (11) for estimating S/σ for AR(1)-N and AR(1)-G inflows when C_v = 0.25, 0.5 and 1.0. The impact of the distributional assumption is most significant for systems with m near zero. Figure 2(b) illustrates the impact of serial correlation on the required capacity of a reservoir system. Comparing Figs 2(a) and 2(b), one observes that the impact of serial correlation on the required capacity of a storage reservoir can be as important as the distributional assumption.

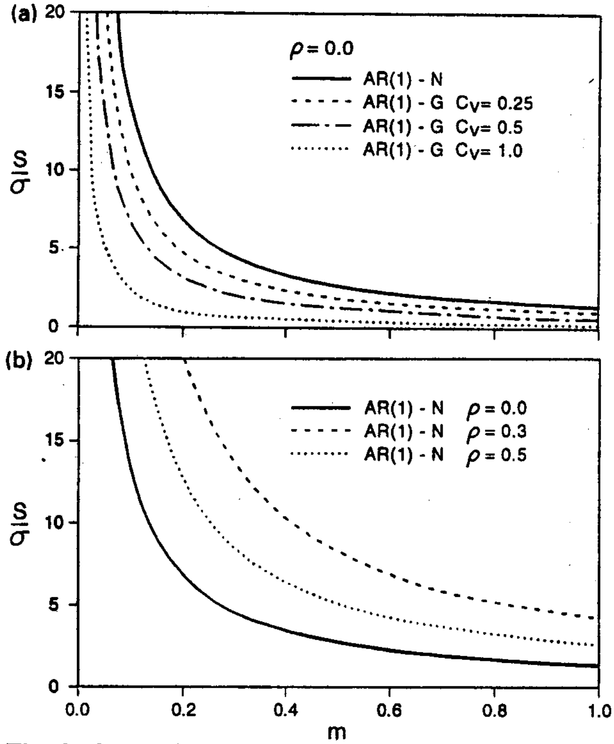

Fig. 2 Comparison of (a) the impact of different inflow distributions, and (b) different values of serial correlation coefficient, on the relationship between reservoir capacity S/σ and resiliency index for a fixed reliability $(1 - q)$ equal to 99%.

A TWO-STATE MARKOV MODEL OF RESERVOIR SYSTEM STATES

Equations (1) to (12) provide a comprehensive description of reservoir system behaviour in terms of system storage, yield and reliability, yet such relationships are unable to describe the system resilience and vulnerability in terms of the duration and magnitude of reservoir system failures. For this purpose, a two-state Markov model is considered here.

Klemeš (1969) employed an s-state Markov chain model in an effort to describe the complex structure of sequences of reservoir surpluses and failures that arise from reasonable assumptions regarding the character of inflow and demand processes. Here s denotes the number of discrete states of the reservoir. In the study, s is assumed to be equal to 2. Since a primary objective of this study is to derive simple "back-of-the-envelope" expressions to aid understanding the S-R-Y relationship, the s-state Markov chain model formulation employed by Moran (1954), Klemeš (1969) and others must be simplified considerably at the potential expense of misrepresenting the complexity of reservoir surplus and failure sequences. Klemeš (1967), Jackson (1975), Hirsch (1979), Stedinger *et al.* (1983) and Vogel (1987) used a two-state Markov chain model to characterize sequences of water supply system surpluses and failures. Vogel (1987) found that a two-state Markov model gave a satisfactory representation of the complex structure of sequences of within-year surpluses and failures derived from the Pacific Northwest hydroelectric power system. Recently, Vogel & Bolognese (1995) have shown that a two-state Markov model can accurately represent over-year reservoir systems.

Klemeš (1977) showed that the number of discrete storage states required to assess the reliability of a storage reservoir with a desired level of accuracy is usually well above two states. It is usually infeasible for an over-year reservoir system to pass from full (state 1) to empty (state 2) in one year, hence most investigators have employed more than two states to model reservoir state transitions. However, if one defines one state as the failure state and another as the non-failure state, Vogel & Bolognese (1995) showed that such a two-state Markov model of reservoir state transitions provides an adequate description of the frequency and magnitude of reservoir system failure durations.

Let $Y_{1,t}$, $Y_{2,t}$ be the respective probabilities that a reservoir system is in the failure state (1) and regular (non-failure) state (2) and let $Y_t = (y_{1t}, y_{2t})$. A failure state occurs when the water in storage plus the inflow during year t are less than the contracted demand $\alpha\mu$. Assume that the states associated with Y_t, $t = 1, \ldots, N$ form a Markov chain with probability transition matrix:

$$A = \begin{bmatrix} 1-r & r \\ f & 1-f \end{bmatrix} \tag{13}$$

where f = probability that a failure year follows a regular year, and r = the probability that a regular year follows a failure year. The probabilities of the states of the Markov chain are given by:

$$Y_{t+1} = Y_t A \tag{14}$$

As t increases, Y_t reaches a steady-state and the solution to equation (14) becomes:

$$\lim_{t\to\infty} Y_t = \left[\frac{f}{r+f}, \frac{r}{r+f}\right] \tag{15}$$

Jackson (1975) provided a derivation of this result. Thus, the steady-state probability that the reservoir will be in the failure or regular states are $f/(r + f)$ and $r/(r + f)$ regardless of the initial state of the reservoir system. The steady-state system reliability $(1 - q)$ can be related to the two-state Markov model using $(1 - q) = r/(r + f)$, or:

$$q = 1 - \frac{r}{r+f} \tag{16}$$

Equation (16) provides the link between the two-state Markov model and S-R-Y relationships based upon a steady-state probability of failure such as equation (11).

To specify fully the two-state Markov model, one requires an estimate of r and f in equation (16). Estimation of the transition probability r is accomplished by first recalling its definition as the probability that the reservoir system transfers from the failure (empty) state to the normal (non-empty) state. The failure state is defined as the condition when the water in storage plus the inflow for that period Q_t are less than the demand $\alpha\mu$. Once a failure has occurred, r becomes the conditional probability:

$$r = P\{Q_{t+1} \geq \alpha\mu \mid Q_t < \alpha\mu\} \tag{17}$$

which can be approximated, as shown by Vogel & Bolognese (1995), using:

$$r = \Phi\left[\frac{\dfrac{m - \rho(2\pi)^{-1/2}}{\Phi(-m)\exp(m^2/2)}}{\sqrt{1-\rho^2}}\right] \tag{18}$$

when Q follows an AR(1) normal process with Φ denoting the *cdf* of a standard normal random variable. Note that equation (18) reduces to $r = \Phi(m)$ when $\rho = 0$.

Once r is determined from equation (18), f is easily found by rearranging equation (16) to obtain:

$$f = r\left[\frac{q}{1-q}\right] \tag{19}$$

The duration of a reservoir system failure

Vogel (1987) showed that the probability function for the length of a reservoir system failure for a two-state Markov model is given by:

$$P\{L = \ell\} = r(1-r)^{\ell-1} \quad \text{for } \ell \geq 1 \tag{20}$$

where L is the length of a failure sequence. Since L is geometrically distributed, it has mean $E[L] = 1/r$, variance $\mathrm{Var}[L] = (1 - r)/r^2$, and coefficient of variation $C_v[L] = (1 - r)^{1/2}$. This theoretical description of the length of reservoir system failures has been confirmed *via* simulation by Vogel & Bolognese (1994).

VALIDATION OF THE GENERAL S-R-Y RELATIONSHIPS AND THE TWO-STATE MARKOV MODELS

Design of Monte Carlo experiments

Monte Carlo simulation experiments were performed to test the validity of the analytical S-R-Y relationships (equations (5) or (11)) and the two-state Markov model. All of the experiments follow the same general procedure. First, 30 million streamflows were simulated from the AR(1) normal model:

$$Q_{t+1} = \mu + \rho(Q_t - \mu) + \varepsilon_t \sigma \sqrt{1 - \rho^2} \tag{21}$$

with $\mu = 1$, $\sigma = 0.3$ and $\rho = 0.0$ and $\rho = 0.3$. Vogel & Bolognese (1995) and others have shown that the storage-reliability-yield relationship for systems fed by AR(1) normal inflows is independent of C_v, hence the results herein are completely general, even though it was assumed that $C_v = \sigma/\mu = 0.3$. In other words, the generalized storage-reliability-yield relationships developed for normal inflows (see Vogel & Bolognese, 1995) are standardized relationships which apply for all values of C_v. Note that large values of C_v imply that negative streamflows can occur more frequently, hence the assumption of normality applied only to basins with small values of C_v. Thirty million streamflows were required to obtain statistically stable results. Assuming system reliabilities $(1 - q)$ of 90%, 95% and 99% and values of m from 0.05 to 1.0 in steps of 0.05, equation (11) was solved to determine the required reservoir storage capacity S necessary to deliver a contracted demand of $\alpha\mu = \mu - m\sigma$ with reliability $(1 - q)$ for each of the combinations of m, ρ and q. Assuming a full reservoir equal to S at the beginning of each 30 million year simulation, the experiment proceeded by determining the amount of water in storage in each of the 30 million years. If the reservoir contents plus the inflow in a given period were less than the demand $\alpha\mu$, a failure was documented. In addition, the duration of consecutive failures L was documented. If the reservoir contents plus the inflow in a given period were greater than the storage capacity S, the excess or surplus was spilled or lost from the system. The output of the simulation experiments consisted of estimates of the system reliability as well as estimates of the mean $E[L]$ and coefficient of variation $C_v[L]$ of the failure durations.

Validation of the two-state Markov model for describing failure durations

Figures 3 and 4 compare the theoretical and simulated mean $E[L]$ and coefficient of variation $C_v[L]$, respectively, of system failure durations as a function of the resilience index m, serial correlation ρ, and failure probability q. In all cases, approximate agreement was obtained between the theoretical and simulated mean and coefficient of variation of failure durations. It was concluded that the two-state Markov model provides an adequate description of the distribution of reservoir system failure durations.

The theoretical expressions for $P[L = \ell]$, $E[L]$ and $C_v[L]$ document that the distribution of L is independent of system reliability, $(1 - q)$. Figures 3 and 4 verify that both $E[L]$ and $C_v[L]$ are approximately independent of q. Interestingly, the ability of the system to recover from a failure is independent of its reliability. Hence, increasing the reliability of an over-year water supply system will not necessarily have an impact on its ability to recover from a failure.

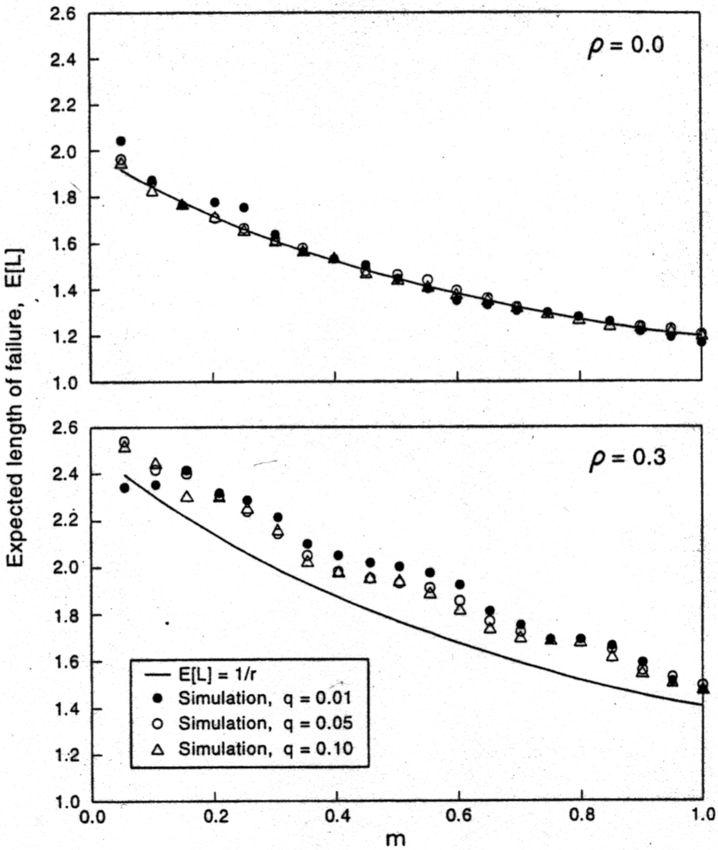

Fig. 3 A comparison of the theoretical (solid lines) and simulated average failure duration as a function of resilience index m, serial correlation ρ, and failure probability q.

Validation of derived S-R-Y relationships (equation (11))

Next, the ability of equation (11) to approximate the S-R-Y relationship was evaluated for systems dominated by over-year behaviour. Figure 5 also illustrates the simulated steady-state failure probabilities corresponding to the Monte Carlo experiments described above. Here, steady-state failure probability q is defined as the total number of years which experienced a failure divided by 30 000 000. The figure also describes the actual reliabilities one would experience had one designed a reservoir using equation (11) with the values of failure probability q, resilience index m and serial correlation ρ, shown. Overall, the agreement between theoretical and simulated failure probabilities was poor. However, the agreement was satisfactory for values of the

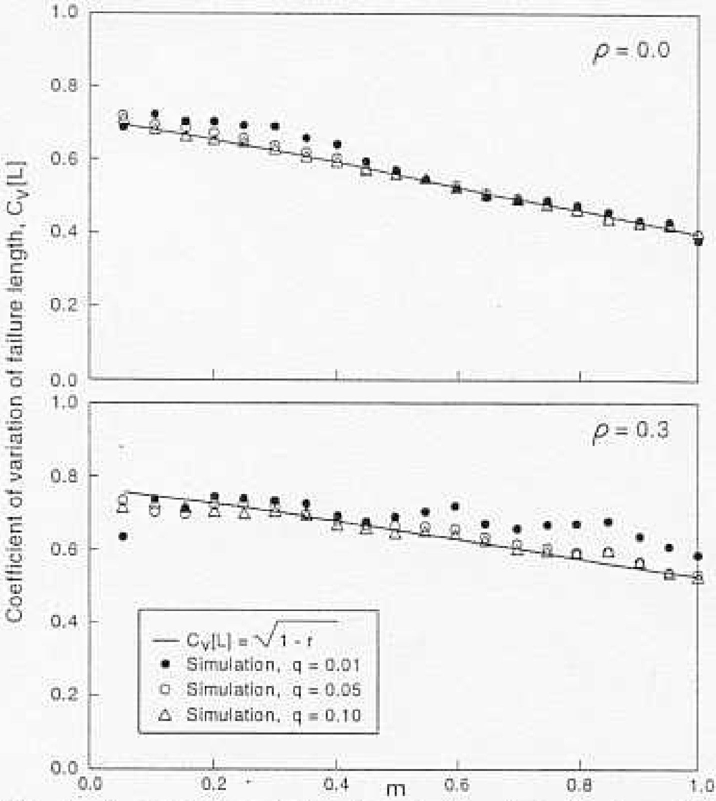

Fig. 4 A comparison of the theoretical and simulated coefficients of variation of system failure durations as a function of the resilience index m, serial correlation ρ, and failure probability q.

resilience index in the range $0.2 \leq m \leq 0.8$, values of failure probability $0.01 \leq q \leq 0.05$, and for $\rho = 0.3$. These ranges of m, q and ρ correspond to realistic values for many systems dominated by over-year storage. As shown in Fig. 2, values of $m < 0.2$ lead to reservoir capacities which approach infeasible (infinite) dimensions.

COMPARISON OF APPROXIMATE S-R-Y RELATIONSHIP DERIVED HERE (EQUATION (5)) WITH OTHER STUDIES

The approximate analytical S-R-Y relationship for reservoirs fed by independent normal inflows derived here (equation (5)) is compared with exact analytical expressions given by Buchberger & Maidment (1989). Those authors provided exact analytical expressions for the relationships between the storage ratio S/σ (which they termed K), resilience index m (which they termed ϵ) and the steady-state probability of a failure q (which they termed $H(0)$) for systems fed by independent normal inflows. Figure 6 compares their exact relationships with the approximate relationships derived here. It may be concluded from

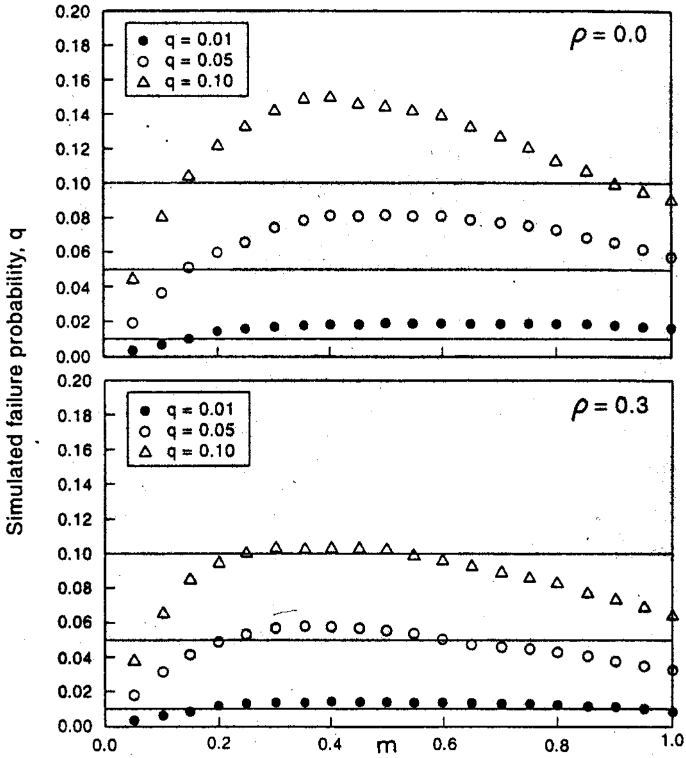

Fig. 5 A comparison of the theoretical and simulated failure probability q (using equation (11)) as a function of the resilience index m and serial correlation ρ.

Figs 5 and 6 that equation (5) provides a reasonable approximation to the overall S-R-Y relationship if one is interested in obtaining a preliminary "back-of-the-envelope" estimate of the S-R-Y relationship for a site. This view is consistent with a similar conclusion reached by Teoh & McMahon (1982).

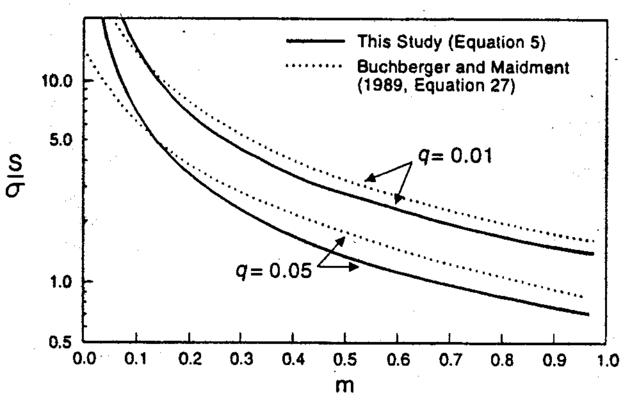

Fig. 6 A comparison of the storage ratio S/σ computed from equation (5) and from the relationships in Buchberger & Maidment (1989), as a function of steady-state failure probability q, and the resilience index m, for independent normal inflows.

A UNIFIED VIEW OF RESERVOIR SYSTEM RELIABILITY

In general, there are two approaches to the determination of the yield or storage capacity of a reservoir system. One approach used in the USA, is to determine the no-failure yield (often called the firm yield) which can be met over a particular planning period with a specified reliability. The alternative approach used in Australia and elsewhere is to determine the yield which can be delivered with a specified steady-state reliability $(1 - q)$, as is the case in this study. Unfortunately, these two approaches are often seen as unrelated and disconnected. Both of these schools of thought can be linked using a two-state Markov model, leading to completely consistent estimates of the reliability of reservoir systems, regardless of which school of thought one happens to follow.

When the sequent peak algorithm (Loucks *et al.*, 1981) is used to determine the smallest reservoir system design capacity S required to assure regular or failure-free operation over an n-year planning period with probability p, then p is a probability over that planning period. If one employs the two-state Markov model, the probability of regular (failure-free) operation over an n-year period, p, is simply the probability of normal operations in the first year $(1 - q)$, times the probability that subsequent years remain free of failures:

$$p = (1-q)(1-f)^{n-1} \tag{22}$$

Equation (22) relates the index of reliability commonly used in the USA (the probability p of failure-free operations over an n-year period) to the index of reliability commonly used in Australia and elsewhere (the steady-state system reliability $(1 - q)$. Hence one can employ the two-state Markov model to compare S-R-Y relationships developed using completely different interpretations of system reliability. Figure 7 shows such a comparison. Since the S-R-Y relationships derived in this study are only approximate (Fig. 6), the

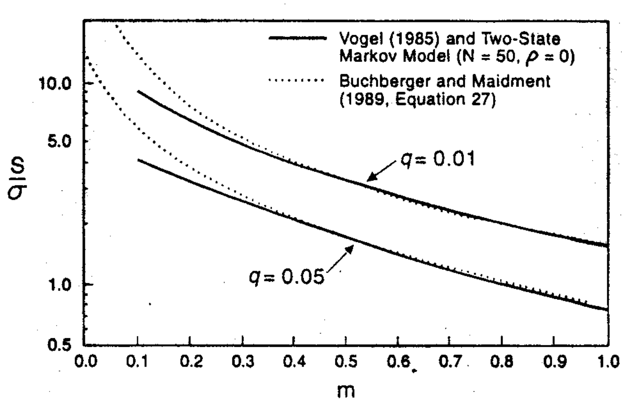

Fig. 7 A comparison of the exact relationship between the storage ratio S/σ, failure probability q and resilience index m, using Buchberger & Maidment (1989) with the approximation derived using the sequent peak algorithm (Vogel, 1985) and a two-state Markov model for independent normal annual inflows.

exact S-R-Y relationships developed by Buchberger and Maidment (1989) for independent normal inflows have been employed. For comparison, the corresponding S-R-Y relationships obtained using the sequent peak algorithm for independent normal inflows has been plotted. Here the value of no-failure reliability p is obtained from equation (22) for each value of steady-state reliability $(1 - q)$ using an $n = 50$ year planning period. The general S-R-Y relationships corresponding to the use of the sequent peak algorithm and independent normal inflows were obtained from Vogel (1985).

WHEN DOES A FINITE STORAGE RESERVOIR BEHAVE LIKE A SEMI-INFINITE STORAGE RESERVOIR?

Buchberger & Maidment (1989) defined the index P, analogous to the Peclet number:

$$P = \frac{mS_q}{2\sigma} \tag{23}$$

for the purpose of determining when a storage reservoir of finite capacity behaves as one with a semi-infinite capacity and is defined as a reservoir that may be either topless or bottomless. They show that finite reservoirs with $P < -1$ or $P > 1$ behave as if they have no top or bottom, respectively. Essentially, the relationship between storage, reliability and yield for a semi-infinite reservoir reduces to a relationship between reliability and yield. It may be very important to understand whether or not a finite reservoir behaves like a semi-infinite reservoir. For example, if an existing finite reservoir system behaves like a semi-infinite reservoir system, then efforts to increase storage capacity will have little or no impact on the system behaviour. The use of simple indices like equation (23) could obviate the need to perform detailed simulation studies which would lead to identical conclusions.

For independent inflow, Buchberger & Maidment (1989, Tables 3, 4 and 5) showed that finite reservoirs behave like semi-infinite reservoirs only for extremely high reliabilities. However, systems fed by inflows which exhibit serial correlation will have larger values of P in equation (23) than systems fed by independent inflows. Fig. 8 illustrates the critical reliability $(1 - q^*)$ necessary for systems to achieve a value of $P = 1$ as a function of m and ρ. The solid lines correspond to the use of Buchberger & Maidment's (1989) exact S-R-Y relationships corrected for the impact of serial correlation. Here the impact of serial correlation on storage capacity was allowed for using the factor $(1 + \rho)/(1 - \rho)$ derived here (equations (8) to (11)) and by Phatarford (1986). For comparison, the critical reliabilities computed using equation (23) are depicted with the S-R-Y relationship derived here (equation (11)). It is typical for reservoir system design reliabilities to be in the range 0.95-0.99 and hence Fig. 8 illustrates that many existing finite reservoir systems probably behave like bottomless semi-infinite reservoirs. These results apply only to normally distributed inflows and, again, equation (11) provides only a rough approximation. Reservoir systems fed by inflows which exhibit skewness will result in smaller values of P than systems fed by normal inflows, since values of S_q tend to be smaller (Fig. 2).

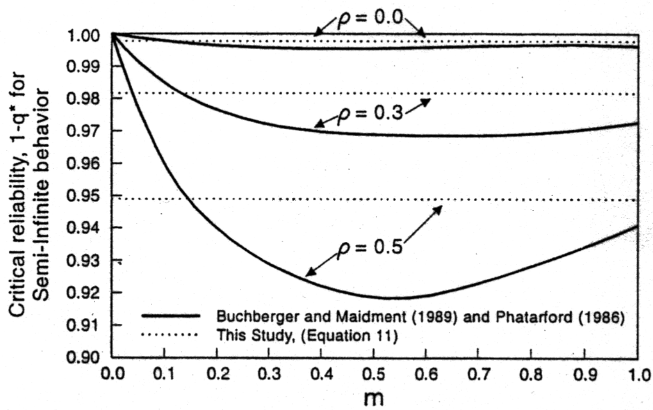

Fig. 8 The critical reliability ($1 - q^*$) for which finite reservoir systems fed by normal inflows behave like semi-infinite reservoir systems as a function of the resilience index m and serial correlation of the inflows ρ.

CRITERION FOR WITHIN-YEAR *VS* OVER-YEAR RESERVOIR OPERATIONS

In this section, a simple criterion is introduced for determining whether or not a reservoir system is dominated by over-year storage requirements. Systems dominated by over-year storage requirements are less likely to recover from a failure than systems dominated by within-year storage requirements. To determine whether a reservoir is likely to return to normal operations, one is

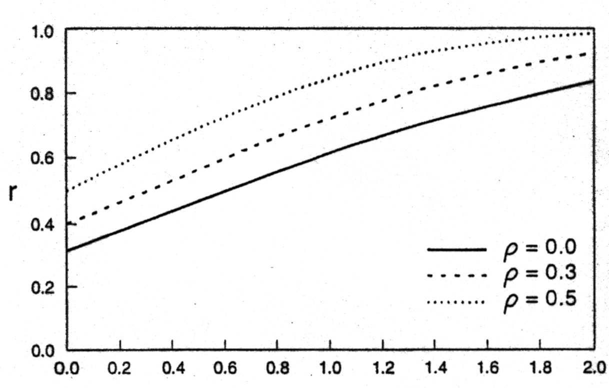

Fig. 9 The probability of returning to normal operations r, as a function of resilience index m and serial correlation ρ, corresponding to AR(1) normal inflows.

(equation (11)) which describes the approximate behaviour of over-year reservoir systems fed by AR(1) normal and AR(1) Gamma inflows. Monte Carlo experiments and comparisons with other studies revealed that equation (11) can provide a reasonable approximation to the storage capacity of an over-year reservoir system; however, it must be used cautiously when estimating system reliability. More accurate general analytical relationships among storage capacity, reliability and yield are available for over-year reservoir systems fed by independent normal inflows (Buchberger & Maidment, 1989), autoregressive normal (Vogel, 1985; Vogel & Bolognese, 1995) and autoregressive lognormal inflows (Vogel & Stedinger, 1987).

The conditions when one can expect a finite reservoir system to behave like a semi-infinite reservoir system have also been documented. For serially correlated normal inflows, the conditions when increases in the storage capacity no longer produce increases in either the system yield or the system reliability have been described.

In addition, in equation (11), a factor has been derived which accounts for the impact of serial correlation on reservoir storage capacity. This factor is identical to the factor derived by Phatarford (1986) using entirely different reasoning.

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