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Disaggregation Procedures for Generating Serially Correlated Flow Vectors

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The structure of disaggregation models places severe constraints on the feasible values of the lagged covariance of generated flow vectors. A new and simple class of disaggregation models is presented which employ the Valencia-Schaake disaggregation model structure but allow the models' innovations to be serially correlated. These models can reproduce (1) the covariance matrix of the disaggregated flows, (2) their covariance with the upper level flows, and (3) reasonable approximations to the lag one covariance of the disaggregated flow vectors given the constraints imposed by a disaggregation approach. The Mejia-Rousselle disaggregation model is shown, in general, to fail to reproduce the anticipated variances and covariances of the disaggregated flows because the model and its parameter estimators are not self-consistent. The paper closes with a discussion of practical modeling considerations and of staged disaggregation procedures which reduce the size of multisite multiseason models.

INTRODUCTION

Since their introduction by *Valencia and Schaake* [1972, 1973], disaggregation models have been recognized as a reasonable way to divide annual flows into seasonal flows [*Mejia and Rousselle*, 1976; *Tao and Delleur*, 1976; *Srikanthan*, 1978; *Lane*, 1979; *Todini*, 1980; *Salas et al.*, 1980] and to divide aggregate basin flows (monthly or annual) into flows at individual sites [*Loucks et al.*, 1981; *Lane*, 1979, 1982; *Salas et al.*, 1980]. An important contribution was the suggestion, by *Mejia and Rousselle* [1976] that the Valencia and Schaake model structure could be extended to allow reproduction of the correlation among monthly flows in different water years, i.e., the correlation between disaggregated flow volumes in different time units.

The Mejia-Rousselle model has been employed by *Lane* [1979, 1982] and *Salas et al.* [1980] in their "staged" disaggregation procedures. In the staged disaggregation procedure described by *Loucks et al.* [1981], the Mejia-Rousselle model was employed to reproduce the period-to-period correlation of flows at individual sites obtained by disaggregation of the aggregate basin flow for each period. *Lane* [1979, 1980, 1983] documents a general purpose computer program which can be obtained upon request and which implements most of these procedures.

Other disaggregation models have been proposed by *Svanidze* [1980], by *Harms and Campbell* [1967], and by *Hoshi and Burges* [1979], though problems were identified with the third model [*Hoshi and Burges*, 1980]. *Lane* [1979, 1982], *Salas et al.*, [1980], and *Pei and Stedinger* [1982] consider models similar to that proposed by Valencia and Schaake but which have fewer parameters.

The focus of this paper is on disaggregation techniques for the situation where the flow vectors $\mathbf{X}_t$, generated by the disaggregation of annual flows to seasonal flows or of aggregate basin flows to those at individual sites, have serial correlation not captured by the upper level flow model coupled with the Valencia-Schaake disaggregation procedure. As mentioned above, this problem was addressed by *Mejia and Rousselle* [1976]. However, as demonstrated in the first section of this paper and by *Lane* [1980, 1982], their model fails to perform as anticipated. Examination of the causes of that failure reveal the constraints imposed by the disaggregation framework on the lagged covariances of the disaggregated flows. The second section of the paper presents a class of disaggregation models which, within the disaggregation framework, can reproduce the covariances of the disaggregated flow vectors $\mathbf{X}_t$, their covariances with the upper level flows, and reasonable approximations to the lagged covariances of the $\mathbf{X}_t$ series.

MEJIA AND ROUSSELLE MODEL

This section discusses, as does *Lane* [1980, pp. V-14, V-16, 1982], why the Mejia-Rousselle modification of the Valencia and Schaake disaggregation model generally fails to generate synthetic flows which have the desired or anticipated variances and covariances. This illustrates the difficulties faced by an algorithm which attempts to generate disaggregated flow vectors with a specified lag one covariance matrix. First, some notation is necessary.

The values of generated random variables such as the higher level annual or aggregate monthly flow vectors $\mathbf{Z}_t$ will be represented by upper case letters, while the historical (observed) values will be represented by lowercase letters $\mathbf{z}_t$; here t may refer to a year, season, or a month, depending on whether $\mathbf{Z}_t$ represents an annual, seasonal, or monthly flow vector, respectively. Following the notation of *Loucks et al.* [1981, pp. 302–306], the generated $\mathbf{Z}_t$ will be an $n \times 1$ vector representing the higher level flows, while $\mathbf{X}_t$ is the $m \times 1$ vector of flows generated by the disaggregation procedure; $\mathbf{x}_t$ are the historical values. All random variables are taken to have zero mean; the historical flow series are assumed to have had their average subtracted, perhaps after some normalizing transformation. Covariances such as

$$E[\mathbf{X}_t\mathbf{X}_t^T] \tag{1}$$

will represent the true variance and covariances of the generated values, whereas

$$\hat{E}[\mathbf{x}_t\mathbf{x}_t^T] = \frac{1}{N-k}\sum_{t=1}^{N}(\mathbf{x}_t\mathbf{x}_t^T) \tag{2}$$

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represents (for $k = 0$ or 1) an estimator of the covariance matrix of the historical flows or some transformation thereof. (*Stedinger* [1981] discusses the advantage of reproducing the covariance of the transformed flows.) In general, we assume that the generated higher-level flows $\mathbf{Z}_t$ satisfy

$$E[\mathbf{Z}_t\mathbf{Z}_t^T] = E[\mathbf{z}_t\mathbf{z}_t^T] \quad (3)$$

Hence, in expectation, they have a covariance matrix equal to the sample estimate.

The basic Mejia-Rousselle model

$$\mathbf{X}_t = A_1\mathbf{Z}_t + A_2\mathbf{X}_{t-1} + \mathbf{V}_t \quad (4)$$

employs the n-dimensional $\mathbf{Z}_t$ vectors along with $m \times n$ and $m \times m$ matrices A_1 and A_2 and a sequence of independent m-dimensional innovation vectors $\mathbf{V}_t$ to generate an $\mathbf{X}_t$ series. The A_1 and A_2 matrices along with the covariance matrix of $\mathbf{V}_t$ contain

$$m \times n + m \times m + m(m+1)/2 \quad (5)$$

parameters. One could attempt to choose these values to reproduce the three covariance matrices

$$E[\mathbf{x}_t\mathbf{z}_t^T] \qquad E[\mathbf{x}_t\mathbf{x}_{t-1}^T] \qquad E[\mathbf{x}_t\mathbf{x}_t^T] \quad (6)$$

At first glance the number of statistics to be reproduced matches the number of available parameters so that the task seems feasible. However, the disaggregation model's structure places severe constraints upon the sets of variances and covariances which can be reproduced, as we now show.

Mejia and Rousselle's relationships for estimating A_1, A_2, and $E[\mathbf{V}_t\mathbf{V}_t^T]$ are easily obtained by writing their model as

$$\mathbf{X}_t = A\mathbf{Y}_t + \mathbf{V}_t \quad (7)$$

where $\mathbf{Y}_t^T = (\mathbf{Z}_t^T, \mathbf{X}_{t-1}^T)$ and $A = (A_1, A_2)$. Equations (3) and (4) of *Mejia and Rousselle* [1976] are obtained by multiplying (7) on the right by $\mathbf{Y}_t^T$ and taking expectations:

$$E[\mathbf{X}_t\mathbf{Y}_t^T] = AE[\mathbf{Y}_t\mathbf{Y}_t^T] \quad (8)$$

Assuming that $E[\mathbf{X}_t\mathbf{Y}_t^T]$ and $E[\mathbf{Y}_t\mathbf{Y}_t^T]$ equal their historical values yields Mejia and Rousselle's estimator of A:

$$A = E[\mathbf{x}_t\mathbf{y}_t^T]\,E[\mathbf{y}_t\mathbf{y}_t^T]^{-1} \quad (9)$$

By multiplying both sides of (7) by their own transpose and taking expectations, one obtains a second relationship:

$$E[\mathbf{X}_t\mathbf{X}_t^T] = AE[\mathbf{Y}_t\mathbf{Y}_t^T]A^T + E[\mathbf{V}_t\mathbf{V}_t^T] \quad (10)$$

This yields Mejia and Rousselle's estimator of $\mathbf{V}_t$'s covariance matrix:

$$E[\mathbf{V}_t\mathbf{V}_t^T] = E[\mathbf{x}_t\mathbf{x}_t^T] - AE[\mathbf{y}_t\mathbf{y}_t^T]A^T \quad (11)$$

This assumes that the moments of $\mathbf{Y}_t$ can and will equal the sample moments of the $\mathbf{Y}_t$.

The Inconsistency

In the derivation of (9) and (11) giving the values of A and the covariance of $\mathbf{V}_t$, it is assumed that the covariance matrix of the generated $\mathbf{Y}_t$ will be that of the historical $\mathbf{y}_t$ series

$$E[\mathbf{y}_t\mathbf{y}_t^T] = \begin{bmatrix} E[\mathbf{z}_t\mathbf{z}_t^T] & E[\mathbf{z}_t\mathbf{x}_{t-1}^T] \\ E[\mathbf{x}_{t-1}\mathbf{z}_t^T] & E[\mathbf{x}_{t-1}\mathbf{x}_{t-1}^T] \end{bmatrix} \quad (12)$$

However, there is no provision in this model to insure that the covariance of $\mathbf{Z}_t$ and $\mathbf{X}_{t-1}$, $E[\mathbf{Z}_t\mathbf{X}_{t-1}^T]$, will in fact equal the historical value of the off-diagonal matrices in (12), which are the transpose of one another:

$$E[\mathbf{z}_t\mathbf{x}_{t-1}^T] = E[\mathbf{x}_{t-1}\mathbf{z}_t^T]^T \quad (13)$$

Thus the derivation of estimators of A and $E[\mathbf{V}_t\mathbf{V}_t^T]$, which assumed that $E[\mathbf{Y}_t\mathbf{Y}_t^T]$ will equal $E[\mathbf{y}_t\mathbf{y}_t^T]$, incorporated an assumption which cannot be enforced and need not be true.

Actually, a very general principle can be derived from this example. There would be absolutely nothing wrong with Mejia and Rousselle's model with the specified least squares estimate of A were it used for prediction. Then one would generally want to assume that the covariance between given values of $\mathbf{Z}_t$ and $\mathbf{X}_{t-1}$ would equal the historical value $E[\mathbf{z}_t\mathbf{x}_{t-1}^T]$. The problem comes when one attempts to use the Mejia-Rousselle model to disaggregate upper level flows $\mathbf{Z}_t$ for the purpose of generating random realizations of the $\mathbf{X}_t$ series. Then the only properties that the $\mathbf{Z}_t$ and $\mathbf{X}_{t-1}$ series will have are those with which they are endowed by the modeler. Regression and time series techniques which yield good models for prediction, forecasting, and control need not provide models which are self-consistent and thus satisfactory for the purposes of stochastic simulation.

The consequences of the inconsistency in the Mejia-Rousselle model and constraints imposed by the disaggregation framework can be illustrated with an example. Assume that the upper level flows are generated by an autoregressive model of the form

$$\mathbf{Z}_{t+1} = D\mathbf{Z}_t + \mathbf{U}_t \quad (14)$$

with the method of moments estimate of D

$$D = E[\mathbf{z}_t\mathbf{z}_{t-1}^T]\,E[\mathbf{z}_t\mathbf{z}_t^T]^{-1} \quad (15)$$

and independent innovations $\mathbf{U}_t$ which insure reproduction of $E[\mathbf{z}_t\mathbf{z}_t^T]$, as assumed in (3). Multiplying (14) on the right by $\mathbf{X}_t^T$ and taking expectations yields the important relationship [*Lane*, 1980, p. V-15; 1982]

$$E[\mathbf{Z}_t\mathbf{X}_{t-1}^T] = DE[\mathbf{Z}_t\mathbf{X}_t^T] \quad (16)$$

The troublesome covariance between the generated upper level flows $\mathbf{Z}_t$ will, as a result of the model's structure, depend only upon D and the lag zero covariance of $\mathbf{Z}_t$ and $\mathbf{X}_t$. There is no reason why

$$DE[\mathbf{Z}_t\mathbf{X}_t^T] = E[\mathbf{z}_t\mathbf{z}_{t-1}^T]E[\mathbf{z}_t\mathbf{z}_t^T]^{-1}E[\mathbf{Z}_t\mathbf{X}_t^T] \quad (17)$$

should happen to equal exactly the historical lagged cross-covariance matrix $E[\mathbf{z}_t\mathbf{x}_{t-1}^T]$ when $E[\mathbf{Z}_t\mathbf{X}_t^T]$ assumes its historical value listed in (6).

The magnitude of the errors which can arise from use of (12) to represent $E[\mathbf{Y}_t\mathbf{Y}_t^T]$ when computing A and the covariance of $\mathbf{V}_t$ is illustrated in Figure 1 for $\mathbf{Z}_t$ computed using (14) with $n = m = 1$. Closed form expressions for the actual modeled variance and correlation of the X_t along with their correlation with the Z_t are derived for that case in the first section of the appendix. The third section of the appendix also provides a simple model whose population values fail to satisfy (16). *Lane* [1982] reports several multivariate examples of the failure of the Mejia-Rousselle model to reproduce the specified statistics.

Lane's Modification

The next question is how to construct a model which will reproduce as many of the statistics in (6) as is possible. When *Lane* [1982] explored this problem, he first assumed that upper-level flows would be generated using (14). When specifying $E[\mathbf{Y}_t\mathbf{Y}_t^T]$ in (8) and (10), he initially assumed that $E[\mathbf{Z}_t\mathbf{X}_{t-1}^T]$ would be given by (17), while $E[\mathbf{Z}_t\mathbf{Z}_t^T]$ and $E[\mathbf{X}_{t-1}\mathbf{X}_{t-1}^T]$ would equal their corresponding historical

If $E[\mathbf{w}_t\mathbf{w}_t^T]^{-1}$ exists (which will not be the case if $\mathbf{Z}_t = L\mathbf{X}_t$), the C that satisfies (26a) is given by

$$C = E[\mathbf{w}_t\mathbf{w}_{t-1}^T]E[\mathbf{w}_t\mathbf{w}_t^T]^{-1} \tag{26b}$$

If the indicated inverse does not exist, it only means there is more than one C which satisfies (26a) because the $\mathbf{w}_t$ are linearly dependent (i.e., there exists L' such that $L'\mathbf{w}_t \equiv 0$ for $L' \neq 0$). In such cases, values of C can be obtained by omitting redundant equations, as explained below.

Other C Estimators

In many cases, (24)–(26) will provide an adequate model of the $\mathbf{W}_t$ time series. However, sometimes one may want to use a simpler model. For example, it may be satisfactory in some cases to reproduce exactly only the historical lag one correlation of each component $(\mathbf{W}_t)_i$ series individually. Then it would suffice to let C be a diagonal matrix with nonzero diagonal elements

$$C_{ii} = \frac{E[(\mathbf{x}_t - B\mathbf{z}_t)(\mathbf{x}_{t-1} - B\mathbf{z}_{t-1})^T]_{ii}}{E[\mathbf{w}_t\mathbf{w}_t^T]_{ii}}$$

Similar restricted multivariate models have been suggested by *Matalas and Wallis* [1976], *Lane* [1979], *Salas et al.* [1980], and *Loucks et al.* [1981]. Such a restricted $\mathbf{W}_t$ model is reasonable if $\mathbf{Z}_t$ corresponds to the aggregate streamflow in one or more basins and each $(\mathbf{X}_t)_i$ is the flow in subbasin i.

If $\mathbf{Z}_t$ corresponds to an annual flow and each $(\mathbf{X}_t)_i$ to the flow in season i, then it might be reasonable to reproduce the correlation between $\mathbf{W}_t$ and $(\mathbf{W}_{t-1})_i$ for some seasons i occurring late in year $t - 1$. In particular, one could reproduce the historical value of the column vectors

$$E[(\mathbf{x}_t - B\mathbf{z}_t)(\mathbf{x}_{t-1} - B\mathbf{z}_{t-1})_i] = E[(\mathbf{x}_t - B\mathbf{z}_t)(\mathbf{x}_{t-1} - B\mathbf{z}_{t-1})^T]\mathbf{e}_i \tag{27}$$

where $\mathbf{e}_i$ is the ith unit vector. To develop a model which reproduces (27) for $m + 1 - k \leq i \leq m$, we employ the $k \times m$ matrix H_k which consists of the last k rows of an $m \times m$ identity matrix. Our new $\mathbf{W}_t$ model will be

$$\mathbf{W}_t = C'H_k\mathbf{W}_{t-1} + \mathbf{V}_t = C\mathbf{W}_{t-1} + \mathbf{V}_t \tag{28}$$

where C' is a $m \times k$ coefficient matrix. To reproduce the moments specified by (27) requires that

$$E[\mathbf{W}_t\mathbf{W}_{t-1}^T]H_k^T = C'H_kE[\mathbf{W}_{t-1}\mathbf{W}_{t-1}^T]H_k^T \tag{29}$$

As a result, C' is given by

$$C' = E[\mathbf{w}_t\mathbf{w}_{t-1}^T]H_k^T\{H_kE[\mathbf{w}_t\mathbf{w}_t^T]H_k^T\}^{-1} \tag{30}$$

provided the indicated matrix inverse exists. The $k \times k$ matrix $H_kE[\mathbf{w}_t\mathbf{w}_t^T]H_k^T$ may not have an inverse if some of the elements of $\mathbf{x}_t$ always sum to a corresponding element of $\mathbf{z}_t$; for example, 12 monthly flows sum to the annual flow and separate at-site flows sum to the aggregate basin flow. If the covariance matrix of $\mathbf{w}_t$ has rank r less than m, then an H matrix consisting of k different unit vectors with $k \leq r$ can be selected to eliminate redundant $\mathbf{w}_t$ components. This insures the invertibility of $H_kE[\mathbf{w}_t\mathbf{w}_t^T]H_k^T$. Note that estimation of $\mathbf{V}_t$'s covariance using (25) for any C will always insure that $E[\mathbf{W}_t\mathbf{W}_t^T]$ equals $E[\mathbf{w}_t\mathbf{w}_t^T]$; this insures that use of (30) to estimate C' will result in generated flows with the anticipated lagged covariances because the covariance of the generated flows $E[\mathbf{W}_{t-1}\mathbf{W}_{t-1}^T]$ which appears in (29) will equal $E[\mathbf{w}_t\mathbf{w}_t^T]$ substituted into (30).

A third way to choose C is to consider its estimation as a problem in model identification as well as parameter estimation. Letting (24) describe the basic model, one could select those coefficients C_{ij}, for each i, which yield the minimum variance or minimum square error predictor

$$\hat{W}_{it} = \sum_{j=1}^{m} C_{ij}W_{j,t-1} \tag{31}$$

of W_{it}, the ith component of $\mathbf{W}_t$. Thus it would be advantageous to set to zero those C_{ij} which are not statistically different from zero when using some reasonable criterion. Procedures for identifying minimum variance and minimum mean square error predictors accounting for parameter estimation error as well as residual unexplained variance are available [*Allen*, 1971; *Mallows*, 1973; *Akaike*, 1974; *Valdes et al.*, 1979; *Hipel*, 1981; *Cline*, 1981; *Cooper and Wood*, 1982; *Trader*, 1983].

This completes the model's development. $\mathbf{Z}_t$ is generated by some unspecified model which need only reproduce $E[\mathbf{z}_t\mathbf{z}_t^T]$. $\mathbf{X}_t$ can then be generated by using (20) where the $\mathbf{W}_t$ are generated independently of the $\mathbf{Z}_t$ by using (24). However C is determined, the appropriate covariance matrix for $\mathbf{V}_t$ should be calculated by employing (25). This insures that $\mathbf{W}_t$'s covariance matrix and hence the statistics in (21) are reproduced.

Equations (20) and (24) could be combined to yield

$$\mathbf{X}_t = B\mathbf{Z}_t + C(\mathbf{X}_{t-1} - B\mathbf{Z}_{t-1}) + \mathbf{V}_t \tag{32}$$

illustrating the dependence of $\mathbf{X}_t$ on the difference between $\mathbf{X}_{t-1}$ and $B\mathbf{Z}_{t-1}$. However, the authors recommend that (32) not be used as a basis for deriving equations to estimate B, C, and $E[\mathbf{V}_t\mathbf{V}_t^T]$ based upon the observed moments of the various flows. Simple attempts to do so may inadvertantly assume that some sample lag one covariances would be reproduced, when in fact they would not.

Equation (32) could be employed as the basis of generalized least squares or maximum likelihood estimates of B and C; the method of moments estimates of those matrices need not be the most statistically efficient estimators. Two features of (32) suggest that better parameter estimates might be available. First, the residuals in the regression problem that determines B (see (20)) are autocorrelated, violating the assumptions that would make each row of B in (22) the "best linear unbiased estimates" of those parameters (*Draper and Smith* [1981, pp. 153–157]; see also the discussion by *Box and Jenkins* [1976] concerning the efficient estimation of ARMA model parameters). However, as noted in the third section of the appendix, efficient parameter estimators for the prediction of future flows need not provide a satisfactory self-consistent model for stochastic streamflow generation. Second, because the covariance matrix of $\mathbf{V}_t$ and $\mathbf{W}_t$ are not diagonal (i.e., the residuals of the component "regression" models are cross-correlated), more efficient estimators of B and C may be obtainable in some cases by using generalized least squares procedures [*Johnston*, 1972; *Diaz-Granados and Bras*, 1982]. This issue is also discussed in the third section of the appendix.

Finally, we also observe that, regardless of the selected $\mathbf{Z}_t$ model, by expanding $E[\mathbf{W}_t\mathbf{W}_{t-k}^T]$ and employing

$$0 = E[\mathbf{W}_t\mathbf{Z}_{t-k}^T] \qquad 0 = E[\mathbf{Z}_t\mathbf{W}_{t-k}^T]$$

one can demonstrate that the lag k covariance of the $\mathbf{X}_t$ generated with (20) is given by

$$E[\mathbf{X}_t\mathbf{X}_{t-k}^T] = BE[\mathbf{Z}_t\mathbf{Z}_{t-k}^T]B^T + E[\mathbf{W}_t\mathbf{W}_{t-k}^T] \tag{33}$$

As with the procedure that *Lane* [1982] first attempted, one may be tempted to use (33) with $k = 0$ and $k = 1$ to derive the value of $E[\mathbf{W}_t\mathbf{W}_t^T]$ and $E[\mathbf{W}_t\mathbf{W}_{t-1}^T]$ necessary to reproduce both the covariance $E[\mathbf{X}_t\mathbf{X}_t^T]$ and lag one covariance $E[\mathbf{X}_t\mathbf{X}_{t-1}^T]$ of the $\mathbf{X}_t$ series. However, as before, there is no guarantee that a $\mathbf{W}_t$ series could have such moments, and use of (33) with $k = 1$ will, in general, not preserve linear relationships of the form $\mathbf{Z}_t = L\mathbf{X}_t$; this point is discussed in the appendix's second section.

Comparison with Alternatives and Extensions

This new model may be viewed as a reformulation of Mejia and Rousselle's disaggregation model which performs as required and works within the limitations imposed by the disaggregation structure on the covariance of $\mathbf{X}_t$ with $\mathbf{Z}_{t-1}$ and with $\mathbf{X}_{t-1}$. Assume for the moment that linear dependencies among the components of $\mathbf{Z}_t$ and $\mathbf{X}_t$ were eliminated by dropping redundant components of the $\mathbf{X}_t$ vector. (If the flows at k sites sum to $(\mathbf{Z}_t)_1$, then one of these $\mathbf{X}_t$ components can be dropped from the model and subsequently calculated as $(\mathbf{Z}_t)_1$ minus the remaining $(k - 1)$ site flows.) Then the lag one covariance matrix of $(\mathbf{X}_t^T, \mathbf{Z}_t^T)$,

$$\Sigma_1 = E\{(\mathbf{X}_t^T, \mathbf{Z}_t^T)^T, (\mathbf{X}_{t-1}^T, \mathbf{Z}_{t-1}^T)\}$$

$$= \begin{bmatrix} E[\mathbf{X}_t\mathbf{X}_{t-1}^T] & E[\mathbf{X}_t\mathbf{Z}_{t-1}^T] \\ E[\mathbf{Z}_t\mathbf{X}_{t-1}^T] & E[\mathbf{Z}_t\mathbf{Z}_{t-1}^T] \end{bmatrix}$$

will be of full rank and contains $(n + m)^2$ statistics, none of which are linearly dependent. One can say that Σ_1 has $(n + m)^2$ degrees of freedom. However, the $\mathbf{Z}_t$ model determines the n^2 lag one covariances $E[\mathbf{Z}_t\mathbf{Z}_{t-1}^T]$. Our disaggregation model's structure also implies that

$$E[\mathbf{Z}_t\mathbf{W}_{t-1}^T] = E[\mathbf{Z}_t(\mathbf{X}_{t-1} - B\mathbf{Z}_{t-1})^T] = 0 \tag{34}$$

$$E[\mathbf{W}_t\mathbf{Z}_{t-1}^T] = E[(\mathbf{X}_t - B\mathbf{Z}_t)\mathbf{Z}_{t-1}^T] = 0 \tag{35}$$

These relationships are equivalent to

$$E[\mathbf{Z}_t\mathbf{X}_{t-1}^T] = E[\mathbf{Z}_t\mathbf{Z}_{t-1}^T]B^T \tag{36}$$

$$E[\mathbf{X}_t\mathbf{Z}_{t-1}^T] = BE[\mathbf{Z}_t\mathbf{Z}_{t-1}^T] \tag{37}$$

where B depends only upon lag zero covariances of $\mathbf{X}_t$ and $\mathbf{Z}_t$. Each of (36) and (37) provide $n \times m$ additional constraints upon Σ_1. Finally, if the matrix Σ_1 is of full rank, one can allow C to be a full $m \times m$ matrix which insures that the sample lag one covariances

$$E[(\mathbf{x}_t - B\mathbf{z}_t)(\mathbf{x}_{t-1} - B\mathbf{z}_{t-1})^T] = E[\mathbf{w}_t\mathbf{w}_{t-1}^T] \tag{38}$$

are reproduced. These additional m^2 relationships show that all of the degrees of freedom in the lag one covariance matrix Σ_1 are accounted for by the model in (20), (24), and (26).

Clearly, one can view (34) and (35) as a problem in that they are somewhat arbitrary constraints imposed by the disaggregation framework. Whether they really are a problem depends on whether the lag one covariances in (36) and (37), which result from those conditions, are both practically and statistically different from the corresponding population values, given historical evidence. In such cases those constraints can be relaxed by extending the model.

One can construct a model which eliminates constraint (35) and consequently could force $E[\mathbf{X}_t\mathbf{Z}_{t-1}^T]$ to assume its historical value. This could be done by adding a term $F\mathbf{Z}_{t-1}$ to the right-hand side of (20), which defines $\mathbf{W}_t$, and making use of what would be the modeled value of $E[\mathbf{Z}_t\mathbf{Z}_{t-1}^T]$. This introduces nm addition parameters in the $m \times n$ matrix F.

Likewise, constraint (34) can be relaxed allowing elimination of relationship (36). To reproduce the historical covariances $E[\mathbf{Z}_t\mathbf{X}_{t-1}^T]$, a term $G\mathbf{X}_{t-1}$ could be introduced into the generating equation for $\mathbf{Z}_t$. This introduces an additional nm parameters in the $n \times m$ matrix G. With such an addition, future upper level flows could reflect the actual value of previous lower level flows. Such a change would greatly complicate most upper level models; with such a change, flow sequences at the two levels would need to be generated jointly.

Equivalence with Lane's Modification of Mejia-Rousselle Model

Finally, we note that were upper level flows generated using (14) so that Lane's modification of the Mejia-Rousselle model could be employed, our model would be statistically indistinguishable from his if C in (24) were chosen so that (18) and (19) are satisfied; this requires reproduction of

$$E[\mathbf{w}_t\mathbf{x}_{t-1}^T] \tag{39}$$

rather than

$$E[\mathbf{w}_t\mathbf{w}_{t-1}^T] = E[\mathbf{w}_t\mathbf{x}_{t-1}^T] - E[\mathbf{w}_t\mathbf{z}_{t-1}^T]B^T \tag{40}$$

or some more restricted set of statistics. To reproduce (39), one need only let

$$\begin{aligned} C &= E[\mathbf{w}_t\mathbf{x}_{t-1}^T]E[\mathbf{w}_t\mathbf{x}_t^T]^{-1} \\ &= E[\mathbf{w}_t\mathbf{x}_{t-1}^T]\{E[\mathbf{x}_t\mathbf{x}_t^T] - BE[\mathbf{z}_t\mathbf{x}_t^T]\}^{-1} \end{aligned} \tag{41}$$

where $E[\mathbf{x}_t\mathbf{x}_t^T]$ and $E[\mathbf{z}_t\mathbf{x}_t^T]$ will also be reproduced. This demonstrates that Lane's estimating equations will yield a feasible set of parameters provided the indicated matrix inversion in (41) can be performed and the resulting value for $E[\mathbf{V}_t\mathbf{V}_t^T]$, (25), is positive semidefinite. Neither reproduction of (38) or of (39) seems to have greater merit, though we find the symmetry of (38) appealing. Both approaches will preserve linear relationships of the form $\mathbf{Z}_t = L\mathbf{X}_t$, as shown in the appendix's second section.

PRACTICAL CONSIDERATIONS

In the identification and use of stochastic streamflow models, one should consider the appropriate mathematical formulation of the model, criteria and procedures for efficient and self-consistent estimation of the specified model's parameters, and computational limitations and constraints upon parameter estimation and flow generation routines. Most of this paper examined alternative mathematical structures for the generation of serially correlated flow vectors and relationships which can be employed to develop parameter estimators consistent with those structures. This section reflects briefly upon computational limitations and practical considerations which need to be addressed in the actual use of multivariate disaggregation models.

Use of Real or Transformed Flows

An important decision in the development of disaggregation and other streamflow models is whether the modeled variables, $\mathbf{Z}_t$ and $\mathbf{X}_t$ in this case, should correspond to the real observed flows or some transformations of the flows which are well described by the normal distribution. In the latter case, the innovation vectors $\mathbf{V}_t$ can be generated from a multivariate normal distribution yielding multivariate $\mathbf{X}_t$ series, provided the specified $\mathbf{Z}_t$ are also multivariate normal.

For cases in which $\mathbf{Z}_t$ and $\mathbf{X}_t$ correspond to the real or untransformed streamflows, *Todini* [1980] describes how the B matrix in the Valencia-Schaake model and the skewness coefficient of the innovations V_{it} can be estimated, in theory, to reproduce the skewness coefficient of the X_{it} series. Such procedures suffer from two major problems.

First, with such a procedure, the required skewness coefficient for the innovation series V_{it} is often large, causing numerical and computational difficulties (S. Burges, personal communication, 1983). Second, and more important, with such an approach, only the first three moments of the flow's distribution are actually specified; there is no guarantee that the resultant marginal distributions for flows at each site or in each period will be satisfactory. This problem was illustrated by *Lettenmaier and Burges* [1977], who examined the resultant marginal distributions of annual flows generated using a model with skewed innovations. Moreover, given the frequent concern with low flows in reservoir and water supply studies, it is important that streamflow models describe the left-hand tail or lower quantiles of the marginal distribution of the flows as well as possible. Use of the skewness coefficient is a particularly poor statistic to use to achieve that objective; its use in estimation procedures generally results in relatively unreliable estimators of the lower quantiles of streamflow distributions [*Stedinger*, 1980; *Hoshi et al.*, 1983].

For these reasons, we recommend that one develop first a satisfactory model of the marginal distribution of the individual series. That selection then defines the appropriate transformation or functional relationship between the modeled normally distributed vectors $\mathbf{X}_t$ and $\mathbf{Z}_t$ and the corresponding historical and generated flows. In addition, the observed historical flows can be transformed to obtain the corresponding values of $\mathbf{x}_t$ and $\mathbf{z}_t$ which can be used to estimate the streamflow model's parameters [*Salas et al.*, 1980, pp. 73–74; *Loucks et al.*, 1981, pp. 283–285].

One reservation often expressed concerning such parameter estimation procedures is that the moments of the generated transformed flows will have means, variances, and correlations corresponding to the $\mathbf{x}_t$ and $\mathbf{z}_t$ series; as a result, the moments of the generated flows will seldom exactly equal sample estimates of those quantities calculated using the observed historical flow series (*Salas et al.* [1980, p. 73] and *Lettenmaier and Burges* [1977, p. 290] express a similar concern). However, recent work shows that reproduction of the means, variances, and covariances of the transformed values often yield substantially better estimates of the moments of the real flows [*Stedinger*, 1980, 1981]; for the lognormal distribution, the parameter estimators obtained using the transformed values are or are more nearly the maximum likelihood estimators which are often highly efficient. *Stedinger and Taylor* [1982] provide a discussion of these issues and of problems associated with streamflow model implementation.

The other problem that arises when one first generates the nonlinear transformations of the real flows occurring within a year or within a basin is that the resultant (untransformed) flows in the individual periods or subbasins generally fail to add up to the value of the annual flow or overall basin flow specified through $\mathbf{Z}_t$. This difference can be ignored or, as is more often the case [*Lane*, 1979, 1982; *Salas et al.*, 1980, p. 427], a correction can be made to eliminate the discrepancy. For example, if the real space flows Q_{it} are modeled by a three-parameter lognormal distribution, then Q_{it} and X_{it} satisfy

$$Q_{it} = \tau_i + \exp[X_{it}] \tag{42}$$

where the restriction that $\mathbf{X}_t$ have zero mean is relaxed.

Suppose that the sum over i of Q_{it} should equal the annual flow or aggregate basin flow Q_t^*, which itself should equal some transformation of a component of $\mathbf{Z}_t$. Then it is appropriate to make some small adjustments in the X_{it} values to eliminate the discrepancy between Q_t^* and the sum of the Q_{it} while minimizing the likely distortion such adjustments could cause. By letting σ_i be the standard deviation of X_{it}, ideally one could choose the adjustment parameter δ so that

$$Q_t^* = \sum_{i=1}^{m} [\tau_i + \exp(X_{it} + \delta\sigma_i)] \tag{43}$$

To avoid the need to solve this nonlinear equation, it is attractive to replace $\exp(\delta\sigma_i)$ by $(1 + \delta\sigma_i)$, where the two will be nearly equal for small $\delta\sigma_i$. Equation (43) becomes

$$Q_t^* - \sum_{i=1}^{m} \tau_i = \sum_{i=1}^{m} (1 + \delta\sigma_i) \exp(X_{it}) \tag{44}$$

The required value of δ will be

$$\delta = \left[Q_t^* - \sum_{i=1}^{m} \tau_i - \sum_{i=1}^{m} \exp(X_{it})\right]\left[\sum_{i=1}^{m} \sigma_i \exp(X_{it})\right]^{-1} \tag{45}$$

Our work has shown that the adjustments $(1 + \delta\sigma_i)$ to $\exp(X_{it})$ yield reasonable results. Appropriately, the largest changes $\delta\sigma_i \exp(X_{it})$ are made to the largest flows and to those flows which are the most unstable (as reflected by σ_i). These are both reasonable and desirable properties for such an adjustment algorithm. Others have employed adjustment procedures where the change in each Q_{it} is proportional to Q_{it}'s standard deviation, to $Q_{it} - E[Q_{it}]$, or to Q_{it} itself [*Lane*, 1980, p. V-25].

Model Size

While the mathematical formulation of disaggregation models is independent of the dimension of $\mathbf{Z}_t$ and $\mathbf{X}_t$, those values have a tremendous effect on the computer storage space and processing capacity necessary to use such models. To disaggregate simultaneous annual flows at 10 gages to monthly flows would result in a 120×10 A matrix containing 1200 elements and a covariance matrix for $\mathbf{V}_t$ with $(120)(120 + 1)/2 = 7260$ distinct elements [see *Salas et al.*, 1980, pp. 448–449]. Clearly, such large models are to be avoided because of the sheer computational effort required as well as the numerical stability problems associated with the decomposition of the covariance matrix of the $\mathbf{V}_t$ so as to be able to generate those random variables [*Salas et al.*, 1980, pp. 86–87; *Loucks et al.*, 1981, pp. 311–312].

One can reduce the size of an all at once disaggregation model if one chooses not to reproduce the observed sample covariance among every monthly flow observed at every site. *Lane* [1979, 1982] and *Salas et al.* [1980], as well as *Stedinger and Pei* [1982] and *Pei and Stedinger* [1982] have developed reasonable condensed annual to monthly disaggregation models which do not attempt to reproduce the covariances among all monthly flows within a water year; the number of parameters required is reduced by 50–70% for a single-site annual to monthly disaggregation model. Several multivariate extensions of these condensed disaggregation models are possible.

Even more appealing from both a modeling and computational point of view is to use a staged disaggregation procedure, such as those discussed by *Lane* [1979, 1982] and *Salas et al.* [1980] or by *Loucks et al.* [1981, pp. 303-306]. For our 10-site example, with the staged procedure suggested by *Loucks et al.* [1981], one would first generate the aggregate annual flow for all 10 sites and then disaggregate that value to the aggregate flow in each month. Then one could divide the aggregate flow in all but the first month among the 10 sites, using a model such as that in (20) and (24); C would be chosen to capture the persistence of the flows at each site by reproducing the lag one correlation of the corresponding W_{it} series. Doing the 10-site annual to monthly disaggregation in one step resulted in a model with 8460 distinct parameters in the A and symmetric BB^T matrices. With the two-stage model, the first disaggregation (aggregate annual to aggregate monthly) requires a model with $12 \times 1 + 12 \times 13/2 = 90$ parameters; the second stage requires 11 models each of which has $10 \times 2 + 10 \times 11/2 = 75$ parameters and a model for the first month in the water year (for which we take $\mathbf{W}_t = \mathbf{V}_t$ for $C = 0$) with $10 + 55 = 65$ parameters. Thus the staged disaggregation model has only 980 parameters altogether or roughly an eighth; in this case, the number of parameters required by the 10-site all at once approach. In general, the all at once monthly disaggregation procedure requires $84n^2 + 6n$ parameters for n sites, while the two-stage procedure requires only $6n^2 + 29n + 90$.

The staged procedure is attractive and appropriate for highly interconnected water supply systems where the total volume of water available to the system in each month and where it arrives is crucial, while the persistence of either high or low flows at each particular site is less important. (See *Hirsch et al.* [1977] for an illustration of the operating issues associated with such systems). This staged disaggregation procedure specifically addresses: (1) the distribution of the total basinwide annual and monthly flows, (2) the covariance among concurrent monthly flows at individual sites, and (3) the month to month correlation of flows at each site. That should be adequate for a great many situations.

CONCLUSIONS

Disaggregation models provide a straightforward procedure for dividing annual or seasonal flows among subperiods and dividing aggregate flows among subbasins. The simple Valencia-Schaake algorithm can perform this function provided that the resultant lag one covariance matrix of the generated vectors is satisfactory. This paper considered disaggregation models which explicitly model the persistence of the disaggregated flows. We have shown that reproduction of the exact historical serial correlations of the lower level flow vectors can be an impossible task within a disaggregation framework because of the constraints imposed by that framework.

These constraints have been discussed and a new and flexible class of disaggregation models described. These models employ the basic Valencia-Schaake model structure but allow the residual vectors themselves to arise from a serially correlated stochastic process. Like the Valencia-Schaake model, these models can reproduce the covariance between concurrent upper level and lower level flows as well as the covariance of the lower level flows with themselves. In addition, the new models can reproduce reasonable approximations to the lag one covariance matrix of the lower level flow vectors. While no disaggregation model can in general reproduce the exact historical lagged covariances of the lower level flows, the new models can do as well as is possible within the constraints imposed by the disaggregation framework.

Ongoing work has found that these new disaggregation models are competitive with more general multivariate streamflow models for the purpose of generating cross-correlated and persistent multisite flow series. However, in particular situations, these disaggregation models might produce lower level flow vectors whose lag one covariance matrix is both practically and statistically different from the true population values, given historical information. In such cases, one may need to employ a more general multivarite time series model or one of the extensions of the basic disaggregation model suggested in the text.

A final note of caution is appropriate. Because the disaggregation models can reproduce some lagged covariances while they either do not or cannot reproduce others, care must be exercised to insure that a model's parameters are estimated using appropriate equations. If care is not exercised, then one may implicitly assume that some historical statistics will be reproduced when that is not the case. This illustrates a difference in the requirements which should be imposed on models to be used for prediction, forecasting, and control and those to be used for stochastic streamflow generation. When developing equations for prediction, one generally assumes that historical relationships among the explanatory variables will persist. However, when generating streamflows or other stochastic sequences, the only characteristics of the real system which will be reproduced in the generated series are those reproduced by the selected generating model and its parameters. Thus, special care should be exercised to insure that models used for stochastic streamflow generation and statistics used to estimate the model's parameters are self-consistent.

APPENDIX

Derivation of Modeled Statistics of the Univariate Mejia-Rousselle Model

Here we derive closed form expressions for the modeled statistics $E[X_t^2]$, $E[X_tX_{t-1}]$, and $E[X_tZ_t]$ of the Mejia-Rousselle model for a simple univariate case ($n = m = 1$). Let the upper level flows be generated using an AR(1) model of the form

$$Z_t = DZ_{t-1} + U_t \tag{A1}$$

The basic Mejia-Rousselle model can be written

$$X_t = A_1Z_t + A_2X_{t-1} + V_t \tag{A2}$$

In this instance, the estimates of the model's parameters are easily derived by using (9), (11), and (15). When $E[x_t^2] = E[z_t^2] = 1$, one obtains

$$A_1 = \frac{E[x_tz_t] - E[z_tx_{t-1}]E[x_tx_{t-1}]}{1 - \{E[z_tx_{t-1}]\}^2} \tag{A3}$$

$$A_2 = \frac{E[x_tx_{t-1}] - E[z_tx_{t-1}]E[x_tz_t]}{1 - \{E[z_tx_{t-1}]\}^2} \tag{A4}$$

$$E[V_t^2] = 1 - A_1E[x_tz_t] - A_2E[x_tx_{t-1}] \tag{A5}$$

The modeled variance may be computed by substitution of (A3), (A4), and (A5) into (10). For this univariate example that substitution yields

$$E[X_t^2] = \frac{1}{1 - A_2^2}[A_1\{2E[X_tZ_t] - A_1\} + E[V_t^2]] \tag{A6}$$

where $E[X_tZ_t]$ is found by substitution of (A3) and (A4) into (8), which gives

$$E[X_tZ_t] = \frac{A_1}{1 - A_2E[z_tz_{t-1}]} \tag{A7}$$

as well as

$$E[X_tX_{t-1}] = \frac{(A_2 + D)A_1E[X_tZ_t] + A_2E[V_t^2]}{1 - (A_2)^2} \tag{A8}$$

We assume for the purposes of this simple demonstration that $E[Z_tZ_{t-1}] = E[z_tz_{t-1}]$. Expressions (A6), (A7), and (A8) were used in the development of Figure 1.

Preservation of Linear Relationships Among $\mathbf{z}_t$ *and* $\mathbf{x}_t$

The text discusses the desire to insure that when $\mathbf{z}_t = L\mathbf{x}_t$, one should also have

$$LE[\mathbf{X}_t\mathbf{X}_{t-1}^T] = E[\mathbf{Z}_t\mathbf{X}_{t-1}^T] \tag{A9}$$

To show that this is the case for Lane's second model (see (18)), first note that

$$LB = LE[\mathbf{x}_t\mathbf{z}_t^T]E[\mathbf{z}_t\mathbf{z}_t^T]^{-1} = I \tag{A10}$$

where I is the identity matrix. Now (18) is equivalent to (19), which requires that

$$E[(\mathbf{X}_t - B\mathbf{Z}_t)\mathbf{X}_{t-1}^T] = E[(\mathbf{x}_t - B\mathbf{z}_t)\mathbf{x}_{t-1}^T] \tag{A11}$$

Multiplying by L and noting that $Lx_t = z_t$ and $LB = I$ yields

$$LE[\mathbf{X}_t\mathbf{X}_{t-1}^T] - E[\mathbf{Z}_t\mathbf{X}_{t-1}^T] = 0 \tag{A12}$$

as required.

Now consider the case where flows are generated using (20) and (24). If C is estimated as in (26), then

$$E[(\mathbf{X}_t - B\mathbf{Z}_t)(\mathbf{X}_{t-1} - B\mathbf{Z}_{t-1})^T] = E[(\mathbf{x}_t - B\mathbf{z}_t)(\mathbf{x}_{t-1} - B\mathbf{z}_{t-1})^T] \tag{A13}$$

Again, multiplying on the left by L yields

$$LE[\mathbf{X}_t\mathbf{X}_{t-1}^T] - E[\mathbf{Z}_t\mathbf{X}_{t-1}^T] - LE[\mathbf{W}_t\mathbf{Z}_{t-1}^T]B^T = 0 \tag{A14}$$

As noted in (35), the third term vanishes, yielding the desired result.

Finally, if one uses (33) with $k = 1$ to specify the lagged covariance of $\mathbf{W}_t$ needed to reproduce $E[\mathbf{X}_t\mathbf{X}_{t-1}^T]$, they would obtain

$$E[\mathbf{W}_t\mathbf{W}_{t-1}^T] = E[\mathbf{x}_t\mathbf{x}_{t-1}^T] - BE[\mathbf{z}_t\mathbf{z}_{t-1}^T]B^T \tag{A15}$$

Multiplying by L on the left, one finds that

$$LE[\mathbf{W}_t\mathbf{W}_{t-1}^T] = E[\mathbf{z}_t\mathbf{x}_{t-1}^T] - E[\mathbf{z}_t\mathbf{z}_{t-1}^T]E[\mathbf{z}_t\mathbf{z}_t^T]^{-1}E[\mathbf{z}_t\mathbf{x}_t^T] \tag{A16}$$

If $\mathbf{Z}_t = L\mathbf{X}_t$, then for $\mathbf{W}_t = (\mathbf{X}_t - B\mathbf{Z}_t)$, $L\mathbf{W}_t \equiv 0$, and $LE[\mathbf{W}_t\mathbf{W}_{t-1}^T]$ should vanish because $E[(L\mathbf{W}_t)(L\mathbf{W}_t)^T]$ will be zero. However, as noted earlier with regard to (16) and (17), the right-hand side of (A16) need not vanish. For example, if

$$\begin{aligned} Z_t &= \alpha_1Z_{t-1} + \alpha_2X_{t-1,1} + U_t \\ X_{t,1} &= \beta Z_t + V_t \\ X_{t,2} &= (1 - \beta)Z_t - V_t \end{aligned} \tag{A17}$$

then $Z_t = X_{t,1} + X_{t,2}$ with U_t and V_t independent series. For a long historical record, sample moments will approach the population values, which are

$$\begin{aligned} E[Z_t^2] &= \sigma_z^2 \\ E[X_{t,1}^2] &= \beta^2\sigma_z^2 + \sigma_v^2 \\ E[Z_tX_{t,1}] &= \beta\sigma_z^2 \\ E[Z_tZ_{t-1}] &= E[(\alpha_1Z_{t-1} + \alpha_2X_{t-1,1} + U_t)Z_{t-1}] \\ &= (\alpha_1 + \alpha_2\beta)\sigma_z^2 \\ E[Z_tX_{t-1,1}] &= E[(\alpha_1Z_{t-1} + \alpha_2X_{t-1,1} + U_t)X_{t-1,1}] \\ &= \alpha_1(\beta\sigma_z^2) + \alpha_2(\beta^2\sigma_z^2 + \sigma_v^2) \\ &= (\alpha_1 + \alpha_2\beta)\beta\sigma_z^2 + \alpha_2\sigma_v^2 \end{aligned} \tag{A18}$$

Now, the last expression for $E[Z_tX_{t-1,1}]$ will equal the second term on the right-hand side in (A16):

$$E[Z_tZ_{t-1}]E[Z_t^2]^{-1}E[Z_tX_t] = (\alpha_1 + \alpha_2\beta)\beta\sigma_z^2 \tag{A19}$$

if and only if $\alpha_2\sigma_v^2$ is identically zero.

With respect to (A15), one should also note that when $L\mathbf{w}_t \equiv 0$, it will be impossible to generate a $\mathbf{W}_t$ series which reproduces $E[\mathbf{w}_t\mathbf{w}_t^T]$ and for which $E[L\mathbf{W}_t\mathbf{W}_{t-1}^T]$ is anything other than zero. Thus use of (33) with $k = 1$ to specify $E[\mathbf{W}_t\mathbf{W}_{t-1}^T]$ can easily yield an infeasible value for that matrix.

Evaluation of Alternative Parameter Estimation Procedures

As discussed following (32), the method of moments estimates of B, C, and $\Sigma_V = E[\mathbf{V}_t\mathbf{V}_t^T]$ need not be as efficient as alternative parameter estimation procedures. First, we note that if one views estimation of B or C as m separate regression problems, such as

$$X_{it} = \sum_{j=1}^{n} b_{ij}Z_{jt} + V_{it} \tag{A20}$$

then the least squares estimators of each row of coefficients are essentially the method of moments estimators employed in (22) and (26); the only difference would be a plus or minus one change in the range of the index in (2) defining the sample covariances.

If one views estimation of B as a least squares problem, then the autocorrelation of the residuals in (20), as described by (24), implies that (22) need not be the best estimator of B [*Draper and Smith*, 1981, pp. 153–157; *Zellner*, 1971, pp. 86–97; *Johnston*, 1972]. To overcome this problem, one can work with the complete model

$$\mathbf{X}_t = \mathbf{B}Z_t + C\mathbf{W}_{t-1} + \mathbf{V}_t \tag{A21}$$

where $\mathbf{V}_t$ has covariance matrix

$$E[\mathbf{V}_t\mathbf{V}_t^T] = \Sigma_V \tag{A22}$$

but is otherwise independent from period to period.

If one used (A21), in which the residuals are independent from period to period, to estimate B and C using ordinary least squares, he would find that the estimates of B and C should satisfy

$$\begin{bmatrix} E[\mathbf{x}_t\mathbf{x}_t^T] \\ E[\mathbf{x}_t\mathbf{w}_{t-1}^T] \end{bmatrix} = [B, C]\begin{bmatrix} E[\mathbf{z}_t\mathbf{z}_t^T] & E[\mathbf{z}_t\mathbf{w}_{t-1}^T] \\ E[\mathbf{w}_{t-1}\mathbf{z}_t^T] & E[\mathbf{w}_{t-1}\mathbf{w}_{t-1}^T] \end{bmatrix} \tag{A23}$$

In practice, $\mathbf{w}_t$ cannot be computed until B has been estimated, but that problem can be surmounted; B and C can be estimated iteratively by using the current estimate of B to calculate the $\mathbf{w}_t$ used in the subsequent iteration.

Use of (A23) again raises the dilemma between models for prediction and models for stochastic flow generation. Use of (A21) and (A23) may yield statistically more efficient estimators of B and C. However, the right-hand side of (A23) contains $E[\mathbf{z}_t\mathbf{w}_{t-1}{}^T]$, the historical lag one cross correlation between upper level flows $\mathbf{z}_t$ and the previous value of the residuals $\mathbf{w}_{t-1}$; this historical value is unlikely to be reproduced, given that the corresponding cross correlation $E[\mathbf{Z}_t\mathbf{W}_{t-1}{}^T]$ of the generated flows will be zero. If one substitutes zero for $E[\mathbf{z}_t\mathbf{w}_{t-1}{}^T]$ on the right-hand side of (A23), then the resultant estimators of B and C are the moment estimators reported in (22) and (26).

The second issue raised following (32) was that generalized least squares (GLS) might provide better estimators of the coefficient matrices. *Bras et al.* [1983] consider the use of generalized least squares to estimate the coefficent matrix of a multisite streamflow model in which some of the elements of the coefficient matrix were constrained to be zero; *Diaz-Granados and Bras* [1982] demonstrate that the model with the GLS parameter estimates often achieved substantially smaller one step ahead forecast errors than were obtained with the same model using ordinary least squares parameter estimates.

We consider here briefly how the problem posed by estimation of C in (24) can be solved by GLS; our analysis reveals that the GLS estimator of this coefficient matrix is identical to the ordinary least squares estimator for the particular case considered in (26). Consider again estimation of C which appears in (24). The potential advantage of the use of GLS only arises because the covariance matrix of $\mathbf{V}_t$ in (A22) is not diagonal. A positive definite symmetric matrix such as Σ_V can always be factored such that

$$\Sigma_V = QQ^T \tag{A24}$$

Also, let $P = Q^{-1}$. Consider then the model obtained by multiplying (24) on the left by P:

$$(P\mathbf{W}_t) = (PC)\mathbf{W}_{t-1} + P\mathbf{V}_t \tag{A25}$$

which then has residuals with covariance

$$E[(P\mathbf{V}_t)(P\mathbf{V}_t)^T] = PE[\mathbf{V}_t\mathbf{V}_t{}^T]P^T = I \tag{A26}$$

Thus the concurrent residuals in the new model in (A25) are independent. Thus generalized least squares estimators reduces to ordinary least squares estimators (because the residuals of the individual regressions are independent) and one obtains, as an estimate of (PC),

$$(PC) = E[(P\mathbf{W}_t)\mathbf{W}_{t-1}{}^T]E[\mathbf{W}_{t-1}\mathbf{W}_{t-1}{}^T]^{-1} \tag{A27}$$

or upon eliminating P from both sides,

$$C = E[\mathbf{W}_t\mathbf{W}_{t-1}{}^T]E[\mathbf{W}_t\mathbf{W}_t{}^T]^{-1} \tag{A28}$$

This is exactly the same as (26*b*). However, if some components of C are constrained to be zero, as was suggested in the discussion associated with (27), then the GLS and ordinary least squares estimators of the parameter should differ.

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