WATER RESOURCES RESEARCH, VOL. 31, NO. 3, PAGES 645–654, MARCH 1995

Storage-reliability-resilience-yield relations for over-year water supply systems

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Abstract. An approximate yet general approach for describing the overall behavior of water supply systems dominated by carry-over storage is introduced. Generalized relationships among reservoir system storage, yield, reliability, and resilience are introduced for water supply systems fed by autoregressive normal and lognormal annual inflows. Relationships for reservoir system resilience are derived which represent the likelihood that a system will recover from a failure, once a failure has occurred. Monte Carlo experiments document that a two-state Markov model can reproduce the relationships between resilience and reliability for a wide class of water supply systems. A two-state Markov model combined with some existing analytical relationships among storage, reliability, and yield provides a very general theoretical foundation for understanding the trade-offs among reservoir system storage, yield, reliability, and resilience.

Introduction

Two general classes of reservoir systems exist: over-year and within-year systems. Within-year systems are characterized by reservoirs which typically refill at the end of each year. Such systems are particularly sensitive to seasonal, monthly, and even daily variations in both the hydrologic inflows and the system yield. Over-year systems do not usually refill at the end of each year and, such systems are particularly prone to water supply failures (empty reservoirs) during periods of drought that extend over several years. Here we define a failure as the inability of a reservoir system to provide the contracted demand during a given year. Water supply failures for within-year systems tend to be short-lived, in comparison with over-year systems, since within-year systems tend to refill on an annual basis. Naturally, all reservoir systems exhibit some combination of over-year and within-year behavior. However, for the moment, consider two reservoir systems having an equal steady state probability of a failure, q, in a given year, one being a system dominated by exclusively over-year behavior and the other dominated exclusively by within-year behavior. During an N-year period, one would expect Nq failures. However, for the within-year system those failure sequences will typically last only a few days or months, whereas for the over-year system, a typical failure may last years (if no new water is imported and demand curtailment programs are not implemented).

A prerequisite to the proper operation, management, and design of over-year reservoir systems is a thorough understanding of the likelihood, duration, and magnitude of potential reservoir system failure sequences. For this purpose, the storage-reliability-yield (SRY) relationship is one important ingredient. However, reliability statements alone do not convey information regarding the consequences of failure (system vulnerability) or the ability of a system to recover from failure (system resilience). This study formulates an approximate, yet general approach for understanding the overall behavior of over-year reservoir systems focusing attention on both the SRY relationship and the frequency, magnitude, and duration of reservoir system failures.

Paper number 94WR02972.
0043-1397/95/94WR-02972$05.00

Storage reservoirs tend to be large and complex systems requiring equally complex mathematical models to simulate their behavior. Historically, one modeling approach has been succeeded by, or appended to, another more complex one to deal with such issues as the Hurst phenomenon, model parameter uncertainty, optimal operations, spatial and temporal disaggregation schemes, etc. What is lacking are simple, reasonably accurate "back-of-the-envelope type" methods which give insight into a wide range of reservoir storage system characteristics and reliability indices before resorting to a complex modeling expedition. Such back-of-the-envelope methods are also useful for the education of water supply analysts.

Most current textbooks in the United States recommend the simulation of water supply system behavior using either the historical record or synthetic streamflow traces in conjunction with the sequent peak algorithm [e.g., *Loucks et al.*, 1981]. Such exercises provide definitive results but do not impart much knowledge about overall reservoir system behavior other than the desired SRY relationship. What is needed are simple, yet accurate expressions which can easily be exploited to describe the resilience of water supply systems in addition to the SRY relationship. Such simple methods could enhance our overall understanding of the behavior of a water supply system prior to the design and implementation of more complex and more definitive reservoir system simulation studies.

The goal of this study is to develop simple expressions which both enhance our understanding of the behavior of water supply systems and provide an explanation of over-year reservoir system behavior. A related study by *Vogel* [1987] uses a two-state Markov model of reservoir storage states to derive and validate relationships among N-year no-failure reliability, p, and steady state probability of failure, q, for reservoir systems dominated by within-year behavior. However, that study does not relate reliability and resilience indices to other system parameters such as storage capacity, yield, or streamflow statistics as is done here, nor does it deal with over-year reservoir systems. *Hashimoto et al.* [1982] describe the use and importance of reliability, vulnerability, and resilience indices for exposing the consequence of reservoir system failures.

Definition of Some Water Supply System Performance Indices

System Reliability

Two schools of thought exist regarding the reliability of water supply systems. In the United States, system reliability is usually defined as the probability of no-failure reservoir operations, p, over an N-year planning period. This is the interpretation of reliability which results when one applies the sequent peak algorithm in conjunction with a stochastic streamflow model [see *Vogel*, 1987]. Outside the United States, system reliability is usually defined in terms of the steady state probability of a system failure, q, where a failure is defined as the inability of the system to deliver the desired yield or demand. These reliability definitions may be related mathematically, as is shown later on.

System Resilience

Hazen [1914], followed by *Sudler* [1927], *Hurst* [1951], and others, introduced one of the most useful indices of reservoir system performance, which we term the resilience index

$$m = \frac{(1-\alpha)\mu}{\sigma} = \frac{(1-\alpha)}{C_v}, \tag{1}$$

where α is the annual system demand or yield as a fraction of the mean annual inflow, μ, and σ is the standard deviation of the annual inflows and C_v is the coefficient of variation of the annual inflows. *Perrens and Howell* [1972] termed m the standardized inflow. After its use by *Hurst* [1951] the nondimensional index m has subsequently found use in both analytic investigations in "water storage theory" [*Pegram et al.*, 1980; *Buchberger and Maidment*, 1989] and in Monte Carlo investigations of the storage-reliability-yield relationship [*Perrens and Howell*, 1972; *Vogel and Stedinger*, 1987]. *Vogel and Stedinger* [1987] suggested that as long as $0 \leq m \leq 1$, the system is dominated by over-year behavior, whereas if $m > 1$, the system is dominated by within-year behavior. Actually, $m = 1$ is an arbitrary maximum for over-year behavior because systems with $m > 1$ may exhibit a small degree of over-year behavior. However, systems with $0 \leq m \leq 1$ are dominated by over-year behavior and this study is limited to systems in that range.

The concept of resilience was introduced to the water resources literature by *Matalas and Fiering* [1977] and has subsequently been discussed and defined in a number of different ways. *Hashimoto et al.* [1982] define resilience as the probability that the system will recover from a failure once a failure has set in. We exploit that definition here. Many other possible definitions exist and have been discussed in the literature [i.e., *Fiering*, 1982; *Moy et al.*, 1986].

We show later that m is related to the probability that a storage reservoir will recover from a failure, hence m is a measure of reservoir system resilience. That is, reservoirs with values of m near 0 require more time to recover from a failure than reservoirs with values of m near unity. Systems with low resilience (m near 0) are characterized by having large values of C_v, large values of α, or both. Reservoirs with values of m near or above unity require less time to refill once empty. Therefore such systems are more likely to exhibit within-year rather than over-year behavior.

Resilient reservoir systems (large resilience index m) tend to have either small demand levels, α, or small coefficients of variation, C_v. Therefore for a fixed demand level one expects regions with low streamflow variability to contain more resilient reservoir systems than regions with high streamflow variability. Because demands levels generally increase over time, one expects a general reduction in the overall resilience of existing reservoir systems over time.

General Storage-Reliability-Yield Relationships

When one attempts to develop the SRY relationship for an actual reservoir system, stochastic streamflow models are often employed in combination with a reservoir simulation model developed for the system in question. For reservoir systems dominated by over-year storage requirements a variety of generalized analytical SRY relationships are available for providing a preliminary estimate of the SRY relationship. *Klemes* [1987], *Vogel and Stedinger* [1987], *Votruba and Broza* [1989], *Phatarfod* [1989], and *Buchberger and Maidment* [1989] provide recent reviews of the literature relating to the development of analytic SRY relationships.

Storage-Reliability-Yield Relationships for Normal Annual Inflows

Buchberger and Maidment [1989] show that for independent normal annual inflows the relationship between the steady state probability of failure q, the storage ratio K, and the resilience index m is given by

$$q = \theta_1(m, K) + \theta_2(m, K) \tag{2}$$

See *Buchberger and Maidment* [1989] for a description of the functions θ_1 and θ_2. The storage ratio K is the ratio of the reservoir capacity S to the standard deviation of the inflows σ.

Vogel [1985] developed analytic approximations to the relationship among probability of no-failure operations over an N-year planning period p, resilience index m, storage ratio K, and the lag 1 serial correlation of annual flows ρ, for AR(1) normal inflows (AR denotes autoregressive). Those multivariate regression relationships take the form

$$K = f(m, p, \rho, N) \tag{3}$$

and are reported in Appendix A. *Pegram* [1980] reports relations among K, m, and q in tabular form for independent normal inflows.

Storage-Reliability-Yield Relationships for Lognormal Annual Inflows

Vogel and Stedinger [1987] developed approximate multivariate relationships for lognormal annual inflows of the form

$$K = g(m, \rho, C_v, p, N), \tag{4}$$

where C_v equals the coefficient of variation of annual inflows equal σ/μ; p equals the probability of no-failure reservoir operations over an N-year period; N equals the planning period with K, m, and ρ defined earlier. *Vogel and Stedinger* [1987] describe the function g in (4), which is too complex to reproduce here. *Pegram* [1980] provides a tabular summary of the SRY relationship for correlated and uncorrelated lognormal annual inflows.

Applicability of General Storage-Reliability-Yield Relationships

General analytic SRY relationships are inadequate for design purposes because they cannot be general and at the same time account for complexities such as the seasonal nature of evaporation, precipitation, streamflow, and operating rules. *Phatarfod* [1989] recommends using Monte Carlo simulation methods for handling specific reservoir design problems and using general analytical SRY relationships for obtaining qualitative results and for obtaining insight into the mathematics of reservoir operations. However, Monte Carlo simulation of reservoir systems using monthly or even daily time steps are often so detailed that it is easy to miss general, yet important features of the reservoir system behavior. For example, significant attention in the literature has been devoted to the development and application of monthly stochastic streamflow models for use in reservoir operations studies, yet few studies have evaluated the general relationships among reservoir system reliability, resilience, and vulnerability. Few studies have addressed which definition of reliability to use and, more importantly, what level of reliability is suitable for the proper design and/or operation of a reservoir system.

Many investigators dispense with general over-year SRY relationships immediately since they are thought to be too simplistic to capture the overall complexity of real water supply systems. To the contrary, we suggest that as long as the resilience index m in (1) is in the range $0 \le m \le 1$, the seasonal behavior of the system is effectively damped out. For example, *Vogel and Hellstrom* [1988] showed that for the Quabbin–Wachusett reservoir system, which provides the water supply for much of eastern Massachusetts, an annual simulation of the system was almost indistinguishable from a monthly simulation of the system. This is expected since the quoted firm yield of 300 mgd (13, 140 L/s) for this system corresponds to $\alpha = 0.915$ and $C_v = 0.34$, resulting in a resilience index, m, of 0.25. As long as m remains in the range $0 \le m \le 1$, the system will be dominated by over-year behavior and seasonal variability of operations and hydrologic processes become moot in terms of the overall long-term reservoir system behavior.

Two-state Markov Model of Reservoir System States

SRY relationships are useful for describing the likelihood of a reservoir system failure, yet such relationships are unable to describe the ability of a system to recover from a failure. For this purpose we consider a two-state Markov model.

Combining a two-state Markov model with SRY relationships allows us to relate system storage, reliability, and yield to the frequency, magnitude, and duration of reservoir system failures. In addition, the two-state Markov model allows us to relate steady state reliability, $1 - q$, to the N-year no-failure system reliability, p. Another advantage of the two-state Markov model is its simplicity and therefore its ease of manipulation. Others have successfully exploited a two-state Markov model for representing sequences of reservoir surplus and failures [see *Klemes*, 1967; *Jackson*, 1975; *Hirsch*, 1979; *Stedinger et al.*, 1983; *Vogel*, 1987]. However, none of those studies provide a direct link between the two-state Markov model and the storage-reliability-yield relationship.

Klemes [1969] employed a multistate Markov chain model in an effort to describe the complex structure of sequences of

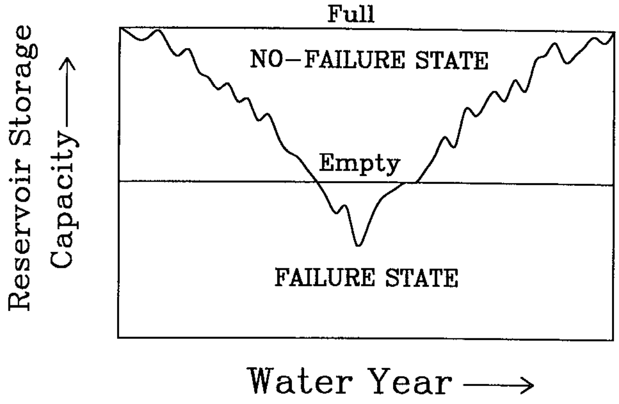

Figure 1. Reservoir system states for the two-state Markov model.

reservoir surplus and failures that arise from reasonable assumptions regarding the character of inflow and demand processes. Since a primary objective of this study is to derive relatively simple expressions to aid in the understanding of reservoir system behavior, the multistate Markov chain model formulation employed by *Moran* [1954], *Klemes* [1969], and others must be simplified considerably at the potential expense of misrepresenting the complexity of reservoir surplus and failure sequences. *Vogel* [1987] documents that a two-state Markov model can accurately represent most within-year reservoir systems. We extend those results here to show that a two-state Markov model can also capture the behavior of some over-year reservoir systems.

Klemes [1977] showed that the number of discrete storage states required to assess the reliability of a storage reservoir with a desired level of accuracy is usually well above two states. Usually it is infeasible for an over-year reservoir system to pass from full to empty in one year. Hence most investigators have employed more than two states to model reservoir state transitions. However, if one defines one state as the failure state and another as the no-failure state, we show that such a two-state Markov model of reservoir state transitions provides an approximate description of the frequency and magnitude of reservoir system failure durations for systems with relatively high resilience indices ($m > 0.2$). Systems with very low resilience ($m < 0.2$) tend to take several years or even decades to refill once empty; such systems require more than two states to approximate their behavior.

Model Development

Let the row vector $\mathbf{Y}_t = (y_{1t}, y_{2t})$ specify the probability that a reservoir system is either in (1) the failure state or (2) the regular (no-failure) state in year t. Figure 1 illustrates the two states in the Markov model. A failure state occurs when the water in storage plus the inflow during year t are less than the contracted demand $\alpha\mu$. We assume that the states associated with $\mathbf{Y}_t$, $t = 1, \cdots, N$ form a Markov chain with the transition probability matrix

$$\mathbf{A} = \begin{pmatrix} 1-r & r \\ f & 1-f \end{pmatrix}, \qquad (5)$$

where f is the probability that a failure year follows a regular year, and r is the probability that a regular year follows a

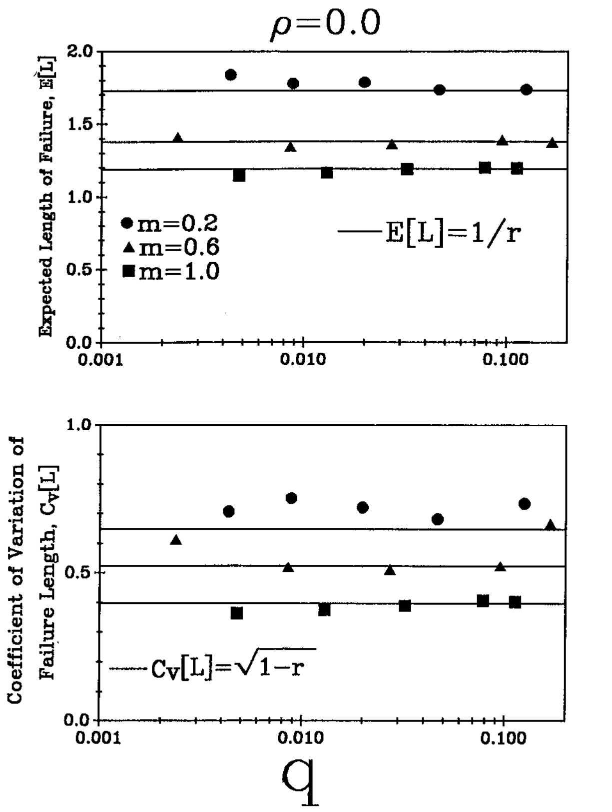

Figure 2. A comparison of theoretical and simulated average failure duration and coefficient of variation of failure duration as a function of failure probability q, and resilience index m, for independent normal inflows.

failure year. Now the probability distribution of reservoir system states follows

$$\mathbf{Y}_{t+1} = \mathbf{Y}_t\mathbf{A} \tag{6}$$

As t increases, $\mathbf{Y}_t$ reaches a steady state, and the solution to (6) becomes [*Jackson*, 1975]

$$\lim_{t\to\infty} \mathbf{Y}_t = (f/(r+f),\ r/(r+f)) \tag{7}$$

The steady state probability that the reservoir will be in the failure or regular states are $f/(r+f)$ and $r/(r+f)$, respectively, regardless of the initial state of the reservoir system. The steady state system reliability, $1 - q$, can be related to the two-state Markov model using $1 - q = r/(r+f)$ or

$$q = f/(r+f) \tag{8}$$

Equation (8) provides the link between the two-state Markov model and SRY relationships based upon a steady steady probability of failure.

Estimation of System Resilience Indices

To fully specify the two-state Markov model, we require estimates of r and f in (8). Estimation of the transition probability r is accomplished by recalling its definition as the probability that the reservoir system transfers from the failure (empty) state to the normal (nonempty) state. The failure state is defined as the condition when the water in storage plus the annual inflow for that year, Q_t, is less than the annual demand, $\alpha\mu$. Once a failure has occurred, the reservoir empties and r becomes the conditional probability

$$r = P(Q_{t+1} \geq \alpha\mu | Q_t < \alpha\mu) \tag{9}$$

As long as the annual inflows are independent ($\rho = 0$), the conditional probability statement in (9) becomes

$$r = P(Q \geq \alpha\mu) \tag{10}$$

which reduces to $r = \Phi(m)$ for independent normal inflows where Φ denotes the cumulative probability distribution of a normally distributed variable. Similarly, for independent lognormal inflows, (10) reduces to $r = 1 - \Phi\{[\ln(\alpha\mu) - \mu_y]/\sigma_y\}$ where $y = \ln Q$ and μ_y and σ_y are the mean and standard deviation of y, respectively.

For serially correlated ($\rho > 0$) normal annual inflows, the conditional probability in (9) reduces to

$$r = \frac{\displaystyle\int_m^{\infty}\int_{-\infty}^{m} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left[-\frac{(v^2 - 2\rho v w + w^2)}{2(1-\rho^2)}\right] dv\, dw}{\Phi(-m)} \tag{11}$$

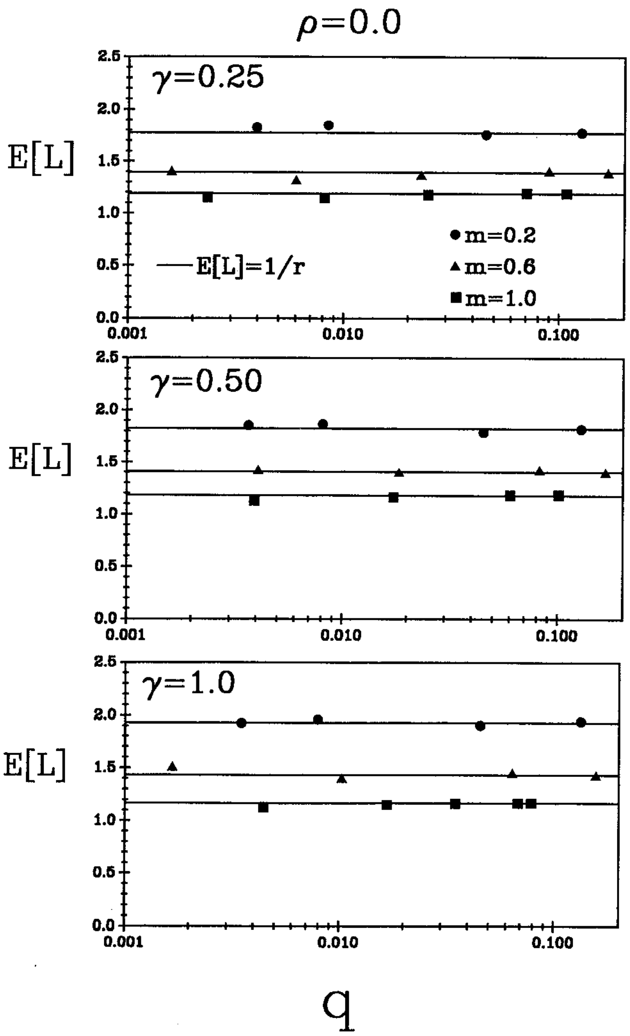

Figure 3. A comparison of theoretical and simulated average failure duration as a function of failure probability q, and resilience index m, for independent lognormal inflows.

as shown in Appendix B. An excellent approximation to r in (11) is

$$r = \Phi\left[\frac{m - \dfrac{\rho(2\pi)^{-1/2}}{\Phi(-m)\exp(m^2/2)}}{(1-\rho^2)^{1/2}}\right] \quad (12)$$

Note that both (11) and (12) reduce to $r = \Phi(m)$ when $\rho = 0$.

For serially correlated lognormal annual inflows, the conditional probability r in (9) reduces to

$$r = \frac{\displaystyle\int_{-\infty}^{B}\int_{B}^{\infty}\left(\frac{1}{2\pi(1-\rho_y^2)^{1/2}}\exp\left(-\frac{(v^2 - 2\rho_y vw + w^2)}{2(1-\rho_y^2)}\right)dv\,dw\right)}{\Phi\left(\dfrac{\ln(\alpha\mu) - \mu_y}{\sigma_y}\right)}, \quad (13)$$

where $B = (\ln(\alpha\mu) - \mu_y)/\sigma_y$ and $\rho_y = \ln[1 + \rho(\exp(\sigma_y^2) - 1)]/\sigma_y^2$ and μ_y, σ_y, and ρ_y denote the mean, standard deviation and serial correlation, respectively, of the logarithms of the annual flows. Equation (13) was derived using a procedure analogous to that outlined in Appendix B for normally distributed inflows. Either index r or m may be considered representative of the resilience of a reservoir system. However, the index r is preferred because it integrates the impact of the probability distribution and serial correlation of the inflows which the index m does not. Once r is determined, f is found by rearranging (8) to obtain

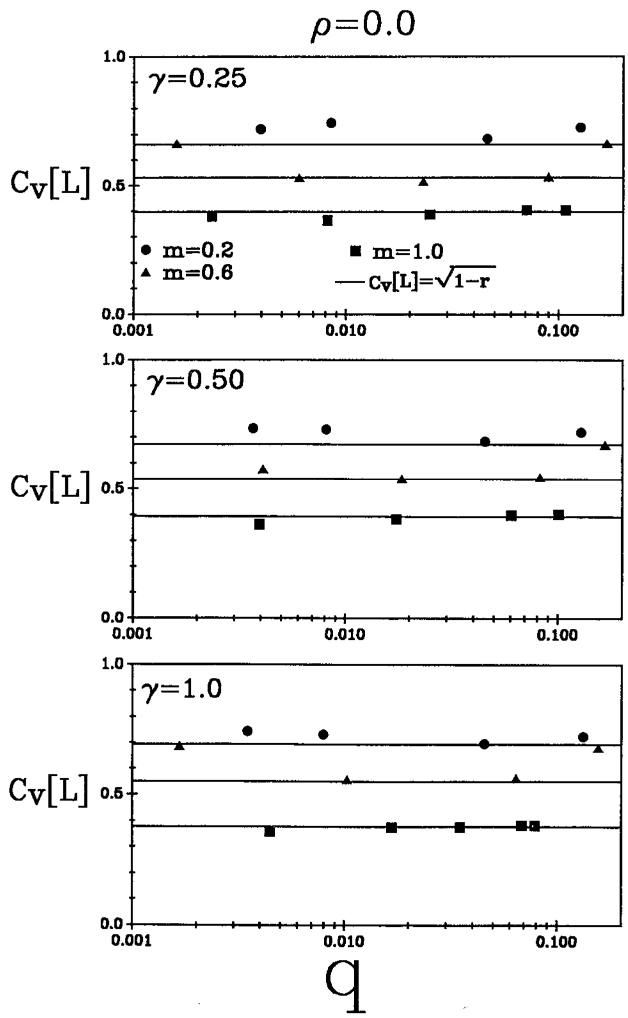

Figure 4. A comparison of theoretical and simulated coefficient of variation of failure duration as a function of failure probability q, and resilience index m, for independent lognormal inflows.

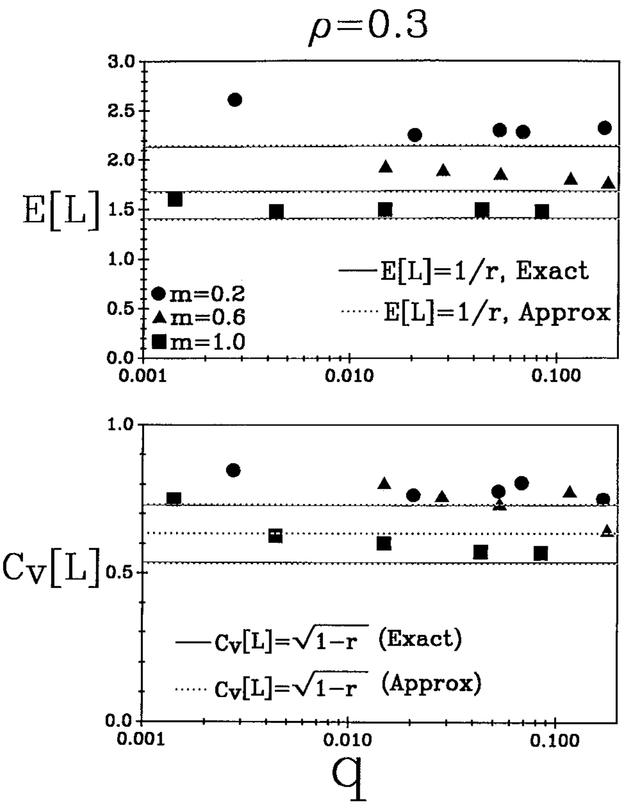

Figure 5. A comparison of theoretical (exact, (11) and approximate, (12)) and simulated average failure duration and coefficient of variation of failure duration as a function of failure probability q, and resilience index m, for serially correlated normal inflows.

$$f = r\left(\frac{q}{1-q}\right) \quad (14)$$

Note that systems with r near unity (m large) correspond to within-year systems because they are very likely to refill in the year following failure. Hence one may consider using the index r to distinguish between systems dominated by over-year (r small) behavior from systems dominated by within-year (r large) behavior.

Limitations of the Two-State Markov Model of Reservoir System States

The primary simplifying assumption required to relate the two-state Markov model to analytic SRY relations was

$$\begin{aligned} r &= P(Q_{t+1} \geq \alpha\mu \mid Q_t < \alpha\mu) \\ &= P(Q_{t+1} \geq \alpha\mu \mid Q_t + S_t < \alpha\mu,\ S_{t+1} = 0), \end{aligned} \quad (15)$$

where S_t is the storage at the beginning of year t. By using (9) to simplify (15) we are, in effect, assuming $S_t = 0$. Thus we are

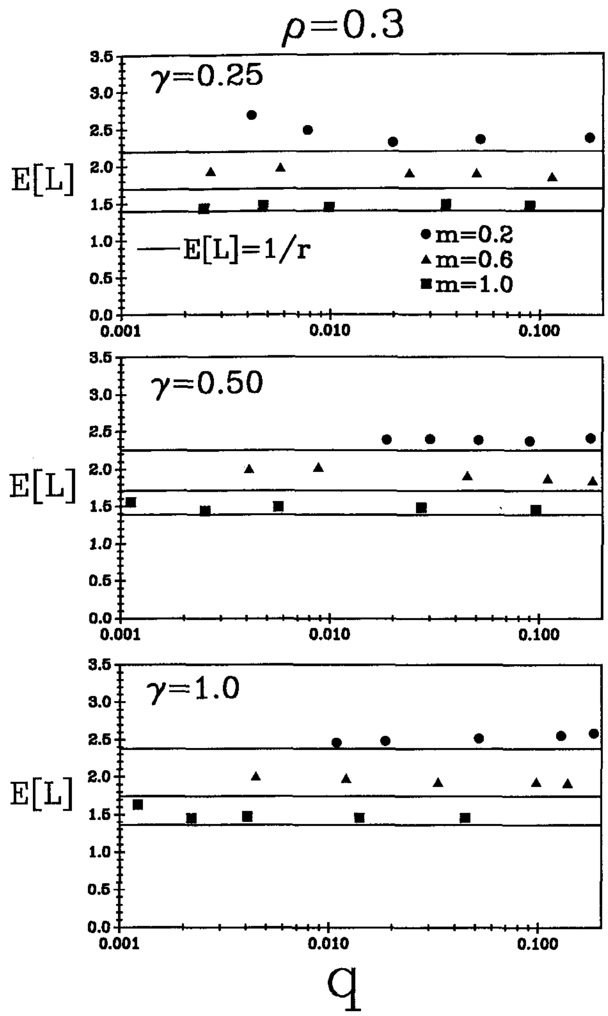

Figure 6. A comparison of theoretical and simulated average failure duration as a function of failure probability q, and resilience index m, for serially correlated lognormal inflows.

ignoring failures where the reservoir is not empty at the beginning of year t. This assumption is approximate for correlated inflows only, since for independent inflows, the inflow in year $t + 1$ is not affected by the inflow in year $t(r = P(Q_{t+1} > \alpha\mu)$ for independent inflows). For correlated inflows, this assumption will cause the expected length of failure and the coefficient of variation of failure lengths to be underestimated.

Duration of a Reservoir System Failure

The probability mass function for the length of a reservoir system failure for a two-state Markov model is given by

$$P(L = \lambda) = r(1 - r)^{\lambda - 1}; \qquad \lambda \geq 1 \tag{16}$$

where L is the length of a failure sequence [*Vogel*, 1987]. Since L is geometrically distributed, it has mean $E[L] = 1/r$, variance $\mathrm{Var}[L] = (1 - r)/r^2$, and coefficient of variation $C_v[L] = (1 - r)^{1/2}$.

Unified View of Reservoir System Reliability

In general, there are two approaches to the determination of the yield or storage capacity of a reservoir system. One approach used in the United States is to determine the no-failure yield (often called the firm or safe yield) which can be met over a particular planning period with a specified reliability. An approach used elsewhere is to determine the yield which can be delivered with a specified steady state reliability, $1 - q$. Unfortunately, these two approaches are often seen as unrelated and disconnected. Both of these schools of thought can be linked using a two-state Markov model, leading to completely consistent estimates of the reliability of reservoir systems regardless of which school of thought one happens to follow.

When the sequent peak algorithm [see *Loucks et al.*, 1981] is used to determine the smallest reservoir system design capacity, S, required to assure regular or failure-free operation over an N-year planning period with probability p, then p is a steady state probability over that planning period. This is because the sequent peak algorithm wraps the streamflow record around itself, generating the steady state solution to the problem posed. Using the two-state Markov model, the steady state probability of regular (failure-free) operation over an N-year period, p, is simply the steady state probability of normal operations in the first year, $1 - q$, multiplied by the probability that subsequent years remain free of failures.

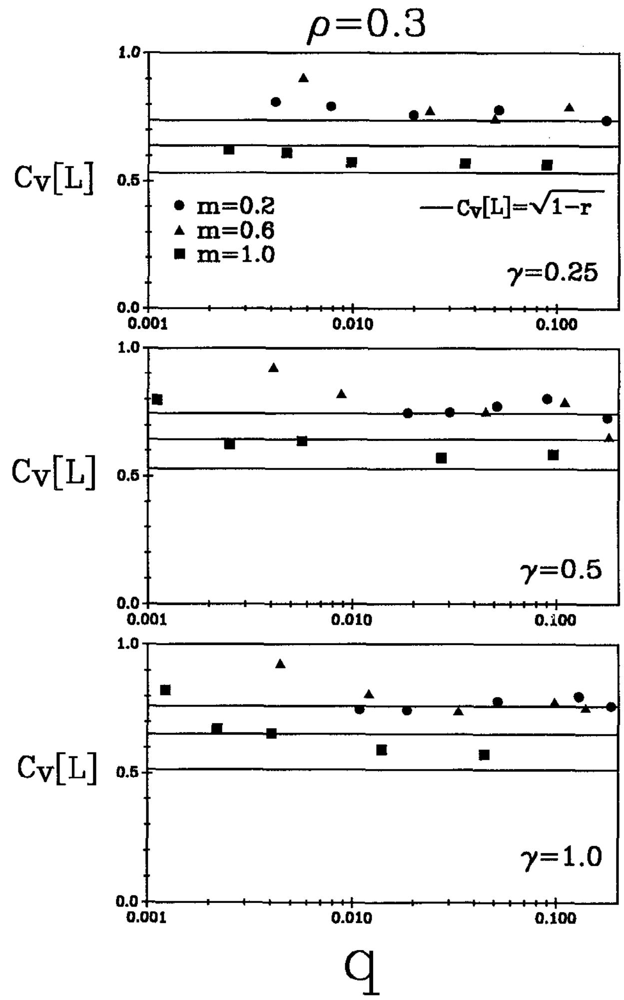

Figure 7. A comparison of theoretical and simulated coefficient of variation of failure duration as a function of failure probability q, and resilience index m, for serially correlated lognormal inflows.

$$p = (1 - q)(1 - f)^{N-1} \qquad (17)$$

Equation (17) relates the index of reliability commonly used in the United States (the probability of failure-free operation over an N-year period p) to the index of reliability commonly used elsewhere (the steady state system reliability $1 - q$). Therefore (17) can be used to compare SRY relationships developed using different interpretations of system reliability.

Monte Carlo Experiments

All of the experiments follow the same general procedure. First, 100 million normal annual inflows, with $\mu = 1$ and $\sigma = 0.2$, were generated for $\rho = 0.0$ and for $\rho = 0.3$. Similarly, 100 million lognormal inflows were generated with skewness $\gamma = 0.25$, 0.5, and 1.0. Assuming a full reservoir capacity equal to S at the beginning of each 100 m.y. simulation, the experiment proceeds by determining the amount of water in storage in each of the 100 m.y. If the reservoir contents plus the inflow in a given period are less than the required demand $\alpha\mu$, a failure is documented. If the reservoir contents plus the inflow minus the demand in a given period are greater than the storage capacity S, the excess or surplus is spilled or lost from the system.

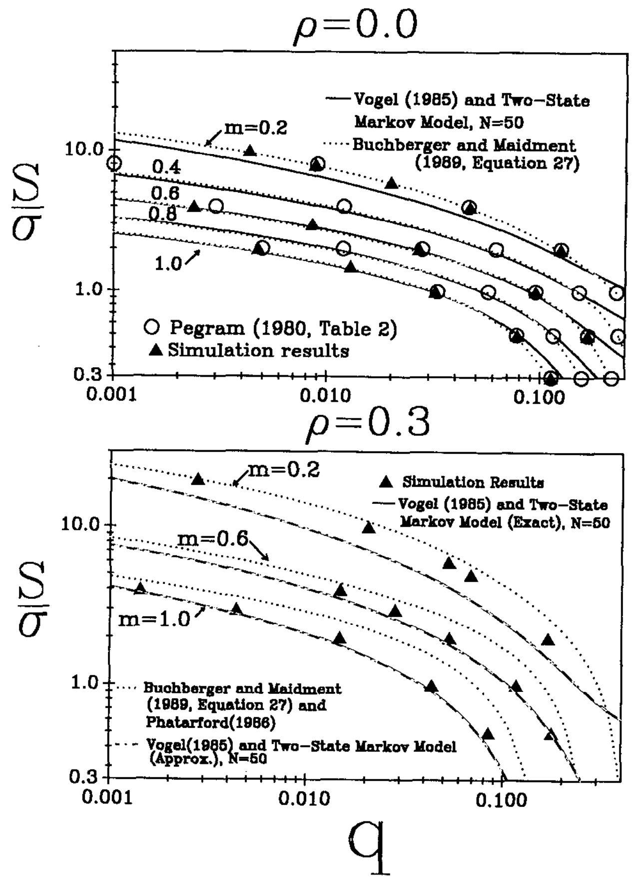

Figure 8. A comparison of the storage ratio, S/σ as a function of failure probability q, and resilience index m, for independent and serially correlated normal inflows.

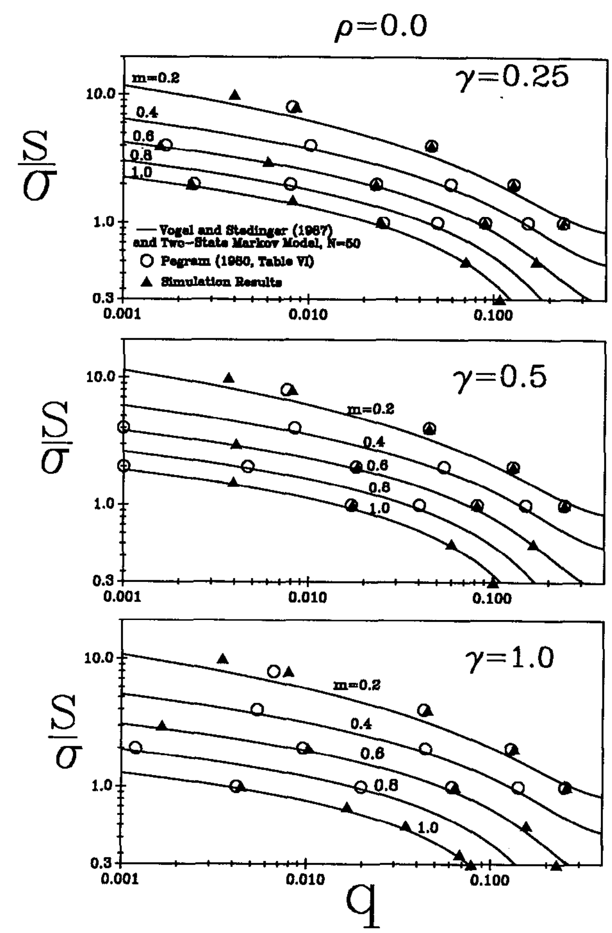

Figure 9. A comparison of the storage ratio, S/σ as a function of failure probability q, and resilience index m, for independent lognormal inflows.

Simulation Results

Two-state Markov model for describing failure durations. Figures 2 through 7 compare the theoretical and simulated mean failure length $E[L]$ and the coefficient of variation of failure lengths $C_v[L]$ as a function of the resilience index m and failure probability, q, for independent normal (Figure 2), independent lognormal (Figures 3 and 4), serially correlated normal (Figure 5), and serially correlated lognormal (Figures 6 and 7) inflows. Overall, very good agreement is obtained between the theoretical and simulated mean and coefficient of variation of failure durations for independent normal and lognormal inflows. The agreement is less satisfactory but still acceptable for serially correlated normal and lognormal inflows. Keep in mind we are only attempting to capture, approximately, the behavior of over-year water supply systems. The agreement is less satisfactory for resilience indices, m, less than or equal to 0.2. This is due to the fact that systems with low resilience index tend to take several years or even decades to refill once empty, since they have, by definition, high demand and low resilience, therefore more than two states are required to capture their behavior. These results confirm our earlier hypothesis that for correlated inflows the expected length of failure, $E[L]$, and the coefficient of variation of failure lengths, $C_v[L]$, are underestimated using the two-state Markov model. For cases where $m > 0.2$ and $0.005 < q <$

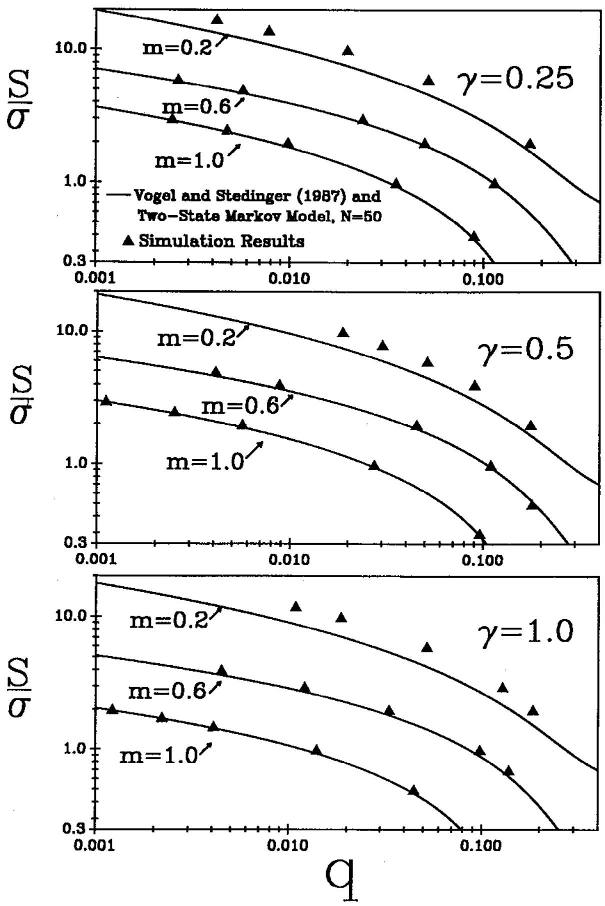

Figure 10. A comparison of the storage ratio, S/σ as a function of failure probability q, and resilience index m, for serially correlated lognormal inflows.

0.1, we conclude that the two-state Markov model provides an adequate description of the distribution of system failure durations for both normal and lognormal, correlated and uncorrelated inflows considering the approximations inherent in our two-state Markov model. These figures also document that efforts to increase system reliability by increasing reservoir storage (holding demand constant) will have little or no impact on the length of reservoir system failures. This conclusion follows from the fact that $E[L]$ and $C_v[L]$ are shown to be independent of q for fixed m in Figures 2 through 7.

General analytic storage-reliability-yield relationships and the two-state Markov model for normal inflows. Two analytic SRY models were evaluated for use when inflows are normally distributed. *Buchberger and Maidment* [1989] (equation 27) provide analytic expressions for the relationship among the storage ratio S/σ, resilience index m, and the steady state probability of failure q for systems fed by independent normal inflows. For serially correlated inflows, *Buchberger and Maidment* [1989] (equation 27) was combined with a serial correlation correction factor, $(1 + \rho)/(1 - \rho)$, derived by *Phatarfod* [1986].

Vogel [1985] developed approximate expressions which describe the relationship between S/σ, m, and the probability of no-failure operations p over N-years. These relationships are summarized in Appendix A. To allow for comparison with *Buchberger and Maidment* [1989], *Vogel*'s [1985] work was combined with the two-state Markov model to convert reliability p to reliability $1 - q$ using equation (17).

SRY relationships derived by *Buchberger and Maidment* [1989] and *Vogel* [1985] are compared in Figure 8 along with Monte Carlo simulation results and tabulated results from *Pegram* [1980]. We conclude that the two-state Markov model accurately converts no-failure reliability p over an $N = 50$-year planning period to steady state reliability, $1 - q$. Once again, the only exception is for values of m less than or equal to 0.2. We also conclude that for serially correlated inflows the expressions derived by *Vogel* [1985] are preferred to the use of *Buchberger and Maidment* [1989] (equation 27) with *Phatarfod*'s [1986] serial correlation correction factor.

General analytic storage-reliability-yield relationships and the two-state Markov model for lognormal inflows. *Vogel and Stedinger* [1987] provide analytic expressions for the relationship between standardized storage S/σ, planning period, N, skewness of the inflows, γ, serial correlation of the inflows, ρ, and the resilience index, m for AR(1) lognormal inflows. Figures 9 and 10 compare SRY relationships based on AR(1)-LN inflows for skews $\gamma = 0.25$, 0.5, and 1 with exact results from *Pegram* [1980] and our Monte Carlo simulations. Again, we conclude that the two-state Markov model is successful in converting reliability p to reliability $1 - q$ for values of m in excess of 0.2. Overall, the agreement between simulated and analytic SRY relationships [*Vogel and Stedinger*, 1987] is quite good considering the approximations inherent in our application of the two-state Markov model.

Conclusion

This study has shown that a two-state Markov model provides a satisfactory approximation to the mean and coefficient of variation of reservoir failure durations for systems dominated by over-year behavior and fed by serially correlated normal and lognormal inflows. *Vogel* [1987] found that a two-state Markov model can also accurately represent reservoir surplus and failure sequences for systems dominated by within-year behavior.

In the United States, reservoir design and operation studies often focus upon the critical drought in each inflow sequence, hence reliability is normally quoted in terms of the probability of failure-free reservoir operations over an N-year period. Another approach defines the storage-yield relationship in terms of the steady state probability of a reservoir system failure. The two-state Markov model enabled us to explain the relationship between N-year failure-free reliability p and steady state reliability $1 - q$ for over-year reservoir systems providing a unified view of system reliability. Most importantly, this study demonstrates that a two-state Markov model can adequately represent the structure of failure sequences to the extent that it can be used to convert reliability statements from one school of thought to another, providing a unified view of reservoir system reliability and resilience.

This study has also reviewed simple analytic SRY relations [*Buchberger and Maidment*, 1989; *Vogel*, 1985; *Vogel and Stedinger*, 1987], which describe the approximate behavior of over-year reservoir systems fed by serially correlated normal and lognormal inflows. Monte Carlo experiments confirm the ability of the two-state Markov model combined with the cited SRY relations to explain the reliability and resilience of over-year water supply systems. The simple analytic annual model of reservoir systems described here provides a very general theoretical foundation for understanding the trade-offs among reservoir system storage, yield, reliability, and resilience.

Appendix A: SRY Relationships for Normal Inflows

Vogel [1985] found that for reservoirs fed by AR(1) normal inflows, the storage capacity S, required to meet a constant demand $\alpha\mu$ over N years, without failure, follows a three-parameter lognormal distribution with parameters τ, μ_y, and σ_y so that

$$K = S/\sigma = \tau + \exp(\mu_y + z_p\sigma_y), \tag{A1}$$

where

$$\mu_y = \ln\left[\frac{\mu_s - \tau}{\left(1 + \frac{\sigma_s^2}{(\mu_s - \tau)^2}\right)^{1/2}}\right] \tag{A2}$$

$$\sigma_y^2 = \ln\left(1 + \frac{\sigma_s^2}{(\mu_s - \tau)^2}\right), \tag{A3}$$

z_p is the pth quantile of the standard normal distribution (i.e., $z_p = \Phi^{-1}(p)$) and μ_y and σ_y^2 are the mean and variance of the logarithm of S. In order to determine the values of μ_y and σ_y^2, the following approximations for μ_s, σ_s, and τ are provided.

$$\mu_s = \exp(a + bm)m^{m(cp+dN)}N^{[e+f\ln(m)]}\left(\frac{1+\rho}{1-\rho}\right)^{g\ln(N)} \tag{A4}$$

$$\sigma_s^2 = \exp\left(a + \frac{bm}{N} + \frac{c(1+\rho)}{m(1-\rho)}\right)m^{(d+emN)}N^{(f+gm)}\left(\frac{1+\rho}{1-\rho}\right)^h \tag{A5}$$

$$\tau = \left[\exp\left(a + \frac{bm}{N} + \frac{c(1+\rho)}{N(1-\rho)}\right) \cdot m^{(dmN+e(1+\rho)/(1-\rho))}N^{[f+gm+(h(1+\rho)/(1-\rho))]}\right] - 5 \tag{A6}$$

The parameter estimates for a–h are given in Table 1. The regression equations for μ_s, σ_s^2, and τ in (A4) through (A6) yield excellent approximations with values of the adjusted R^2 equal to 99.92, 99.73, and 94.98, respectively [see *Vogel*, 1985]. *Vogel and Stedinger* [1987] provide similar approximations for AR(1) lognormal inflows.

Table 1. Parameter Estimates for Constants a through h

Constant	Parameter μ_s	σ_s^2	τ
a	0.153	−2.51	0.514
b	−1.32	19.4	6.00
c	−0.843	−0.0284	1.42
d	0.00694	−1.39	0.00546
e	0.385	0.0151	0.0364
f	−0.0592	0.752	0.369
g	0.100	−0.468	−0.147
h	0.0	2.00	−0.00867

These equations are valid for $0.1 \le m \le 1.0$; $20 \le N \le 100$; and $0.0 \le \rho \le 0.5$.

Appendix B: Derivation of Transition Probability r in Two-State Markov Model for Correlated Normal Annual Inflows

In this section we derive an expression for the transition probability r in (9) for serially correlated normal annual inflows. Equations (1) and (9) can be combined to produce

$$r = P(Q_{t+1} \ge \alpha\mu | Q_t < \alpha\mu)$$
$$= P\left(\frac{Q_{t+1} - \mu}{\sigma} \ge \frac{\alpha\mu - \mu}{\sigma} \middle| \frac{Q_t - \mu}{\sigma} < \frac{\alpha\mu - \mu}{\sigma}\right)$$
$$= P(Z_{t+1} \ge -m | Z_t > m), \tag{B1}$$

where Z is a standard normal random variable with zero mean and unit variance. Rewriting (B1) as

$$r = P(Z_{t+1} < m | Z_t > m) \tag{B2}$$

and noting that Z follows an AR(1) model $Z_{t+1} = \rho Z_t + V_t(1 - \rho^2)^{1/2}$ with V_t an independently distributed $N(0, 1)$ random variable, (B2) becomes

$$r = P(\rho Z_t + V_t(1 - \rho^2)^{1/2} < m | Z_t > m]. \tag{B3}$$

Using the definition of a conditional probability, (B3) can be rewritten as

$$r = \frac{P[\rho Z_t + V_t(1 - \rho^2)^{1/2} < m, Z_t > m]}{P\{Z_t > m\}} \tag{B4}$$

Defining $W = Z_t$ and $V = \rho Z_t + V_t(1 - \rho^2)^{1/2}$, results in

$$r = \frac{P(V < m, W > m)}{\Phi(-m)}, \tag{B5}$$

where V and W follow a bivariate normal distribution of the form

$$f(v, w) = \frac{1}{2\pi\sigma_v\sigma_w(1 - \rho_{vw}^2)^{1/2}} \cdot \exp\left\{-\frac{\left[\left(\frac{v - \mu_v}{\sigma_v}\right)^2 - 2\rho_{vw}\left(\frac{v - \mu_v}{\sigma_v}\right)\left(\frac{w - \mu_w}{\sigma_w}\right) + \left(\frac{w - \mu_w}{\sigma_w}\right)^2\right]}{2(1 - \rho_{vw}^2)}\right\} \tag{B6}$$

with $\mu_v = \mu_w = 0$, $\sigma_v = \sigma_w = 1$ and $\rho_{vw} = \rho$. Substitution of these values of μ_v, μ_w, σ_v, σ_w, and ρ_{vw} into (B6) and noting that the numerator in (B5) is

$$P(V < m, W > m) = \int_m^\infty \int_{-\infty}^m f(v, w)\, dv\, dw$$

leads to the expression for r given in equation (11).

Acknowledgments. We are grateful to T. A. McMahon, R. M. Phatarfod, J. R. Stedinger, and V. Klemes for their comments on a very early version of this manuscript.

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(Received September 16, 1993; revised November 9, 1994; accepted November 10, 1994.)