

The utility of L-moment ratio diagrams for selecting a regional probability distribution

MURRAY C. PEEL

Department of Civil and Environmental Engineering, The University of Melbourne, Victoria 3010, Australia

e-mail: mpeel@civag.unimelb.edu.au

Q. J. WANG

Agriculture Victoria, Tatura, Victoria 3616, Australia

e-mail: qj.wang@nre.vic.gov.au

RICHARD M. VOGEL

Department of Civil and Environmental Engineering, Tufts University, Medford, Massachusetts 02155, USA

e-mail: rvogel@emerald.tufts.edu

THOMAS A. McMAHON

Department of Civil & Environmental Engineering, The University of Melbourne, Victoria 3010, Australia

e-mail: t.mcmahon@civag.unimelb.edu.au

Abstract L-moment ratio diagrams are increasingly being used in the literature for selecting a probability distribution function for regional frequency analysis. Two graphical methods are often used in the distribution selection process, the sample average and a line of best-fit through the sample L-moment ratios. Examples of homogeneous and heterogeneous regional samples are simulated to illustrate the utility of the two distribution selection methods. Distribution selection for homogeneous regional data is best based on the sample average and not on a line of best-fit through the data points. For very heterogeneous regional data, exhibiting a large range in the distributions shape parameter, the line of best-fit is useful for distribution selection. These results emphasize the importance of using heterogeneity tests in conjunction with L-moment ratio diagrams.

Key words L-moment ratio diagrams; regional probability distribution; regional frequency analysis; selection of distribution; heterogeneity testing

Utilité des diagrammes de rapports de L-moments pour le choix d'une distribution régionale de probabilité

Résumé Les diagrammes de rapports de L-moments sont de plus en plus utilisés dans la littérature afin de choisir une fonction de distribution de probabilité pour l'analyse de fréquence régionale. Deux méthodes graphiques sont souvent utilisées pour le choix d'une distribution, la moyenne de l'échantillon et la droite du meilleur ajustement des rapports de L-moments de l'échantillon. Des échantillons régionaux homogènes et hétérogènes ont été simulés pour mettre en évidence l'utilité de ces deux méthodes de choix de distribution. Pour des données régionales homogènes, le meilleur choix de distribution est obtenu à partir de la moyenne de l'échantillon. Pour des données régionales très hétérogènes, présentant une large gamme de paramètres de forme, la droite du meilleur ajustement se révèle la plus utile pour le choix de la distribution. Ces résultats soulignent l'importance de l'utilisation de tests d'hétérogénéité parallèlement aux diagrammes de rapports de L-moments.

Mots clefs diagrammes de taux de L-moment; distribution régionale de probabilité; analyse de fréquence régionale; choix de distribution; essais d'hétérogénéité

NOTATION

GEV	generalized extreme value distribution
GP	generalized Pareto distribution
LN3	three parameter lognormal distribution
LOWESS	locally-weighted scatterplot smoothing
κ	GEV and GP shape parameter

INTRODUCTION

The method of L-moments introduced by Hosking (1990) is increasingly being used by hydrologists. Hosking (1990) noted the benefits of L-moment ratios over product moment ratios in that the former are more robust in the presence of extreme values and do not have sample size related bounds. This has led to the recommendation that L-moment ratio diagrams should always be used in preference to product moment ratio diagrams in hydrological analysis (Vogel & Fennessey 1993). L-moment ratio diagrams have been suggested as a useful tool for discriminating between candidate distributions to describe regional data (Hosking, 1990; Stedinger *et al.*, 1993; Hosking & Wallis, 1997). Numerous authors (for example Schaefer, 1990; Pearson, 1993; Vogel *et al.*, 1993a,b; Chow & Watt, 1994; Önöz & Bayazit, 1995; Vogel & Wilson, 1996) have used L-moment ratio diagrams as part of their distribution selection process for regional data.

Generally the distribution selection process, using L-moment ratio diagrams, involves plotting the sample L-moment ratios as a scatterplot and comparing them with theoretical L-moment ratio curves of candidate distributions. Two graphical tools used to assist in distribution selection are the sample average and a line of best-fit through the sample L-moment ratios. Numerous authors (for example Vogel *et al.*, 1993a; Chow & Watt, 1994; Hosking & Wallis, 1995) have used the sample average, while the line of best-fit method was introduced by Vogel & Wilson (1996).

These two graphical methods are subjective and are not a replacement for more objective and complex methods like those of Chowdhury *et al.* (1991); Hosking & Wallis (1993); Chow & Watt (1994); and Fill & Stedinger (1995), which take into account the sampling variability related to the sample size of the regional data. However, they do provide a quick visual assessment of which distribution may provide a good fit to the data.

The proximity of the sample average (for regions with equal periods of record) or the record length weighted average (for regions with unequal periods of record) to a particular candidate distributions theoretical curve or point in L-skewness-L-kurtosis space has been interpreted as an indication of the appropriateness of that distribution to describe the regional data (Vogel *et al.*, 1993a; Hosking & Wallis, 1995). The similarity of a line of best-fit to the theoretical curve of a particular candidate distribution in L-skewness-L-kurtosis space has been interpreted as an indication of the appropriateness of that distribution to describe the regional data (Vogel & Wilson, 1996). This paper attempts to identify when the two graphical methods are useful or misleading for distribution selection via simulation of homogeneous and heterogeneous regional samples. Conclusions drawn from L-moment ratio diagrams are also

applicable to product moment ratio diagrams.

HOMOGENEOUS DATA

Homogeneous data are derived from a parent distribution with a fixed shape parameter and therefore fixed L-skewness and L-kurtosis. Simulations were carried out to assess the utility of the two graphical methods for distribution selection for homogeneous regional data. Ten thousand samples of size 30 were generated from a GEV distribution with $\kappa = -0.2$ (Stedinger *et al.* (1993) provide details of the GEV distribution). The L-skewness and L-kurtosis of each sample were calculated using unbiased L-moment estimators and are shown in Fig. 1 (for reason of clarity, only 1000 points are shown but the pattern holds for 10 000 points).

As shown in the Fig. 1, the sample L-moment ratios do not follow the theoretical curve for the GEV distribution. In general, points of low L-skewness tend to fall below the GEV curve, whereas points of high L-skewness tend to go above the GEV curve.

Also shown in Fig. 1 is a modified LOWESS (Cleveland, 1979) smooth which is a line of best-fit to the sample points suggested by Vogel & Wilson (1996). The modified LOWESS smooth was calculated using the 10 000 sample points, with a smoothing parameter value of 0.3 (which remains constant for all the following smooths). The modification to the LOWESS of Cleveland (1979) takes into account the variance in both L-skewness and L-kurtosis, not just L-kurtosis.

The average position of the sample points is close to the population value as shown in Fig. 1. The small discrepancy is due to bias in the sample estimators of L-moment ratios (Hosking & Wallis, 1995). The sample average provides a good indication to the parent distribution in this case. The LOWESS smooth is not similar to the GEV theoretical curve and does not provide a good indication to the parent distribution in this case.

Similar conclusions can be drawn for other examples of homogeneous GEV data. In Fig. 2, the GEV distribution curve is plotted along with four LOWESS smooths. Each smooth was obtained from 10 000 samples of size 30 generated from a GEV

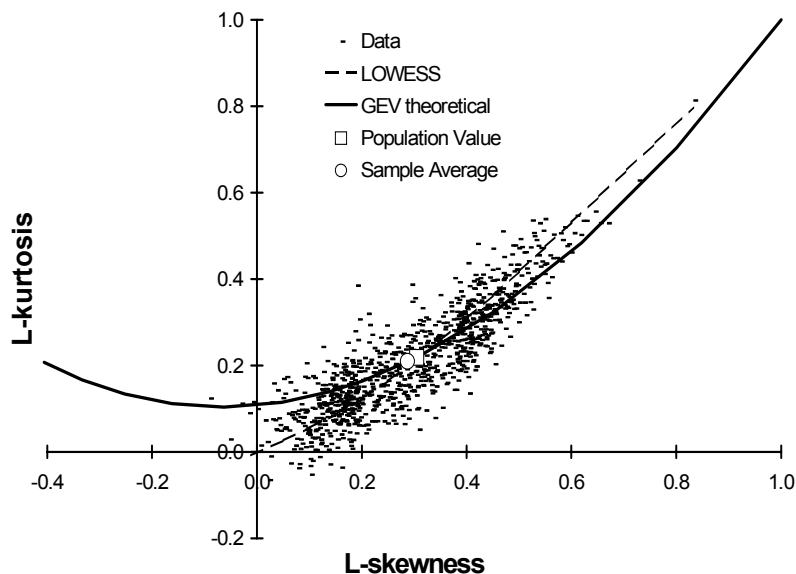


Fig. 1 Theoretical GEV curve compared with sample trend of a GEV distribution.

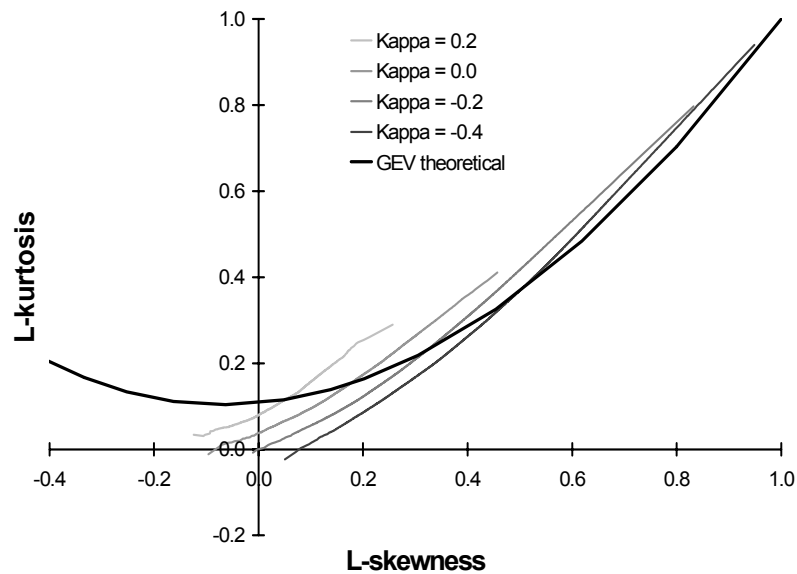


Fig. 2 Theoretical GEV curve compared with sample trends of four GEV distributions.

distribution. The four smooths are for $\kappa = 0.2, 0.0, -0.2$ and -0.4 respectively. The difference between the GEV theoretical curve and the LOWESS smooths is clearly shown, while the sample averages, not plotted here, are similar to the population values.

The same conclusions can be drawn when other distributions are considered. Figure 3 is constructed in the same way as Fig. 2 but using the LN3 distribution. The four LOWESS smooths are for samples generated with four different parameter settings.

A potentially misleading conclusion drawn from use of the line of best-fit method is illustrated with a comparison of the GP, LN3 and GEV theoretical curves with two LOWESS smooths shown in Fig. 4. The smooths are for samples generated from a GEV distribution with $\kappa = -0.4$ and a LN3 distribution respectively. The two smooths are similar to the GP distribution, when in fact the data have come from the GEV and

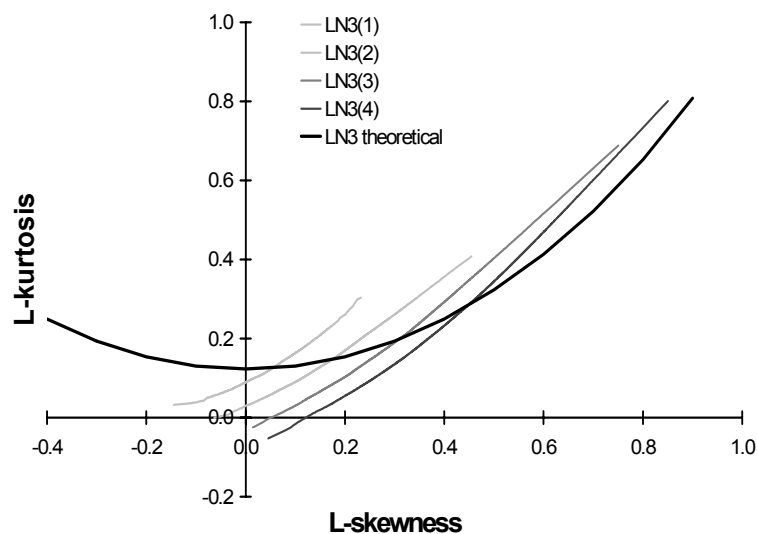


Fig. 3 Theoretical LN3 curve compared with sample trends of four LN3 distributions.

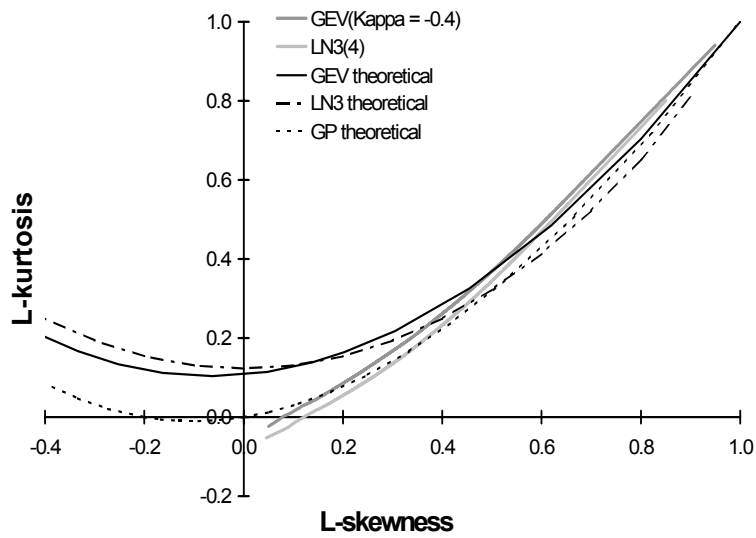


Fig. 4 Theoretical GEV, LN3 and GP curves compared with sample trends of GEV and LN3 distributions.

LN3 distributions respectively.

It is concluded from the simulations presented that the sample average is a very good indicator of the parent distribution for homogeneous data. Data homogeneity can be tested using procedures introduced by Chowdhury *et al.* (1991) and Hosking & Wallis (1993). The line of best-fit through the sample L-moment ratios is not a good indicator of the parent distribution for homogeneous data as the line of best-fit is generally dissimilar to the theoretical curve of the parent distribution function. The reason for this is that a homogeneous sample is derived from a parent represented by a point in L-skewness-L-kurtosis space. When plotted, the sample points form an ellipse in L-skewness-L-kurtosis space centred on the parent point, but not following the parent distribution curve (see also Chow & Watt (1994), Fig. 1).

HETEROGENEOUS DATA

A heterogeneous sample can be derived from either a single parent distribution with variable shape parameter values or from a combination of samples from two or more distributions. When a real sample is heterogeneous the exact nature of the heterogeneity is unknown. The first heterogeneous example (Fig. 5) is constructed from sub-samples (4×2500 samples size = 30) of the GEV distribution with shape parameter κ equal to 0.2, 0.0, -0.2 and -0.4 respectively. The LOWESS smooth better matches the GEV theoretical curve than in the homogeneous examples, although the pattern of being over for high L-skewness and under for low L-skewness persists. The sample average is close to, but above the theoretical curve indicating that the parent distribution may be GEV.

A second heterogeneous example is demonstrated in Fig. 6. The LOWESS smooth is for 3×3333 samples of size 30 generated from three GEV distributions with κ equal to 0.2, 0.1 and -0.3 respectively. This smooth deviates more from the GEV theoretical

curve than that in Fig. 5 and the sample average is again close to and above the GEV curve.

A mixed distribution heterogeneous example is presented in Fig. 7, which displays theoretical curves for the GEV, LN3 and GP distributions along with a LOWESS smooth derived from 4×2500 samples from two distributions, two parent GEV and two parent GP samples. The two GEV populations had κ equal to 0.2 and -0.2 , respectively, and the two GP populations had κ equal to 0.1 and -0.3 , respectively. The LOWESS smooth generally lies between the theoretical GEV and GP curves while the sample average would indicate either the GEV or LN3 distribution was an appropriate distribution.

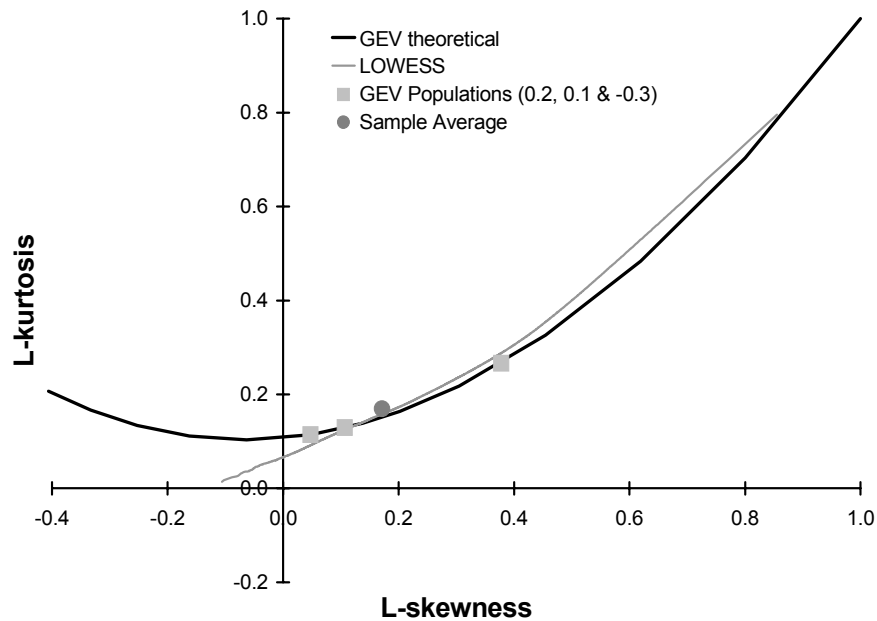


Fig. 6 Theoretical GEV curve compared with sample trend of unevenly mixed GEV distributions.

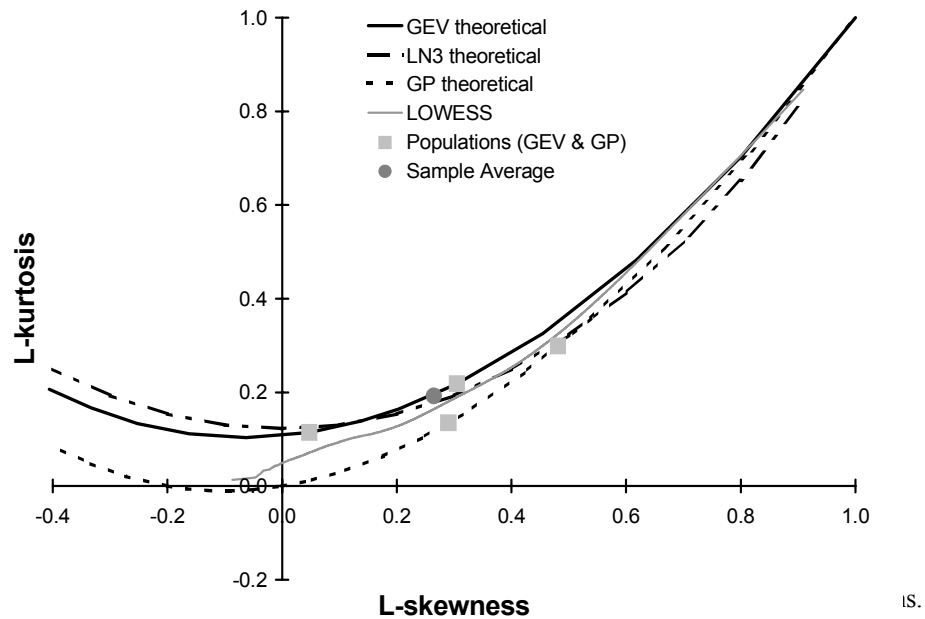


Fig. 7 Theoretical GEV, LN3 and GP curves with sample trend of mixed distribution sample.

A real world (and heterogeneous) situation is shown in Fig. 8 where the LOWESS smooth was derived from 627 L-moment ratios from annual maximum flow series of streams around the world (Peel, 1999). Sample record lengths range from 15 to 122 years. The sample L-moment ratios, LOWESS smooth and sample size weighted average are presented in Fig. 8(a). The LOWESS smooth, weighted average and candidate distribution theoretical curves are presented in Fig. 8(b). The LOWESS smooth and sample size weighted average are very similar to those in Fig. 7 and thus it appears as though no individual distribution is a good fit to this data. The smooth is consistently above the GP distribution and below the GEV and LN3 curves for low L-skewness. The sample average is close to the GEV and LN3 distribution curves. Both the sample size weighted average and LOWESS smooth are not helpful for distribution selection in this case.

From these heterogeneous examples it appears as though the line of best-fit is of some utility for distribution selection when several samples are drawn from a single distribution with a large range of shape parameter values, but are of little utility when the sample is a combination of samples from two or more distributions. The sample average was generally not useful for distribution selection in any of the heterogeneous simulations. For heterogeneous samples the sample average and the line of best-fit could only indicate what the parent distribution might be, they reveal no information about the nature of the sub-samples that were combined to form the final heterogeneous sample.

CONCLUSIONS

It has been demonstrated that using graphical methods with L-moment ratio diagrams

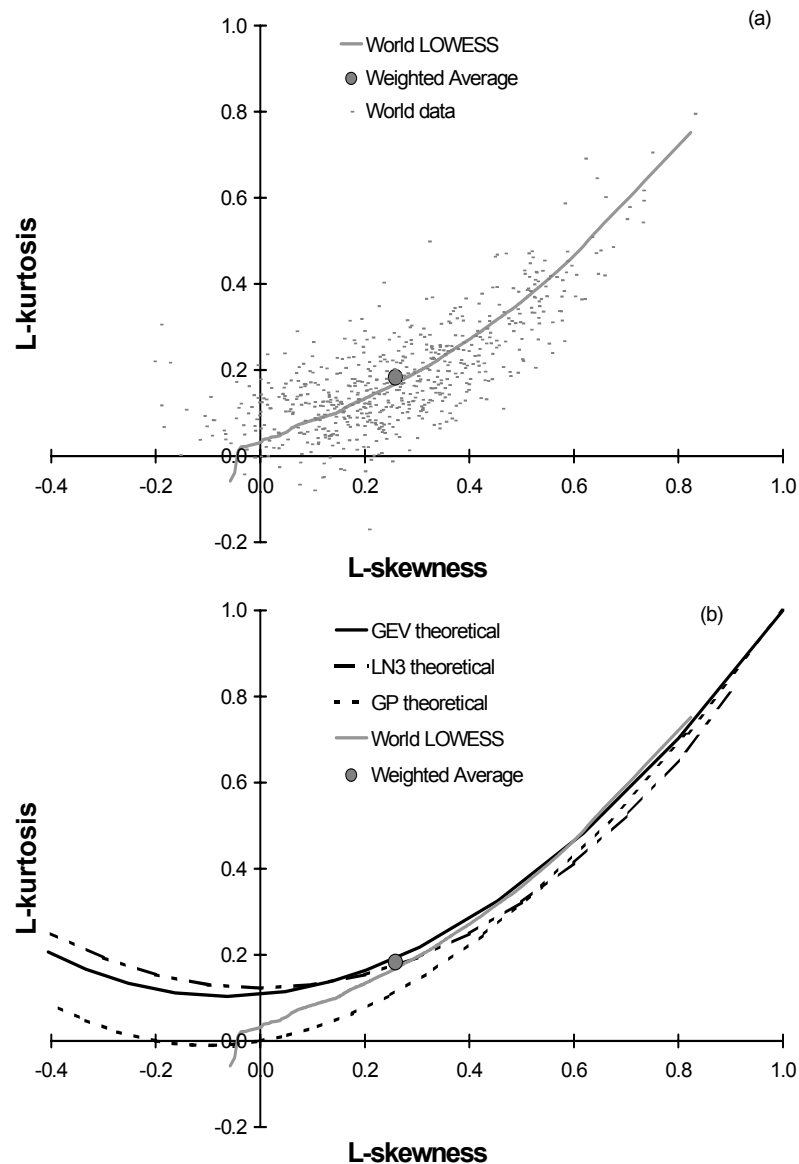


Fig. 8 (a) Weighted sample average and trend of world annual maximum flows; and (b) theoretical GEV, LN3 and GP curves and trend of world annual maximum flows.

in the distribution selection process for regional data can be misleading. Distribution selection for homogeneous data is best based on the sample average and not on a line of best-fit through the data points. For heterogeneous data, the line of best-fit is useful for distribution selection when data are drawn from a single distribution function with a large range of parent shape parameter values. In practice, however, there are no means of knowing how a real heterogeneous regional sample is constructed and whether it complies with this condition. These results emphasize the importance of using heterogeneity tests in conjunction with L-moment ratio diagrams.

Acknowledgements This research was funded by the Cooperative Research Centre for Catchment Hydrology through a postgraduate scholarship. Roberta Armitage provided

assistance translating the abstract into French. The authors are grateful to Charles Pearson and another, but anonymous, referee for very careful reviews of this paper.

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Received 28 February 2000; accepted 11 October 2000