THE BROOKLYN COLLEGE

JOURNAL of SCIENCE

1963

The Brooklyn College JOURNAL OF SCIENCE

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THE GAMMA FUNCTION AND STIRLING'S APPROXIMATION

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My aim in this paper is to connect the solution of two apparently different problems arising in Physics with a useful function, the Gamma Function. I shall state the problems, develop the Gamma Function and some of its properties and then return to solve the problems.

The first problem arises in the development of Maxwell-Boltzman Statistics where it is necessary to find an expression for the distribution of N particles among P energy cells, where one doesn't distinguish between distributions in the same cell. The expression obtained for W the number of distinct distributions is:

$$\text{(A)} \quad W = \frac{N!}{N_1!\,N_2!\ldots N_i!\ldots N_p!}$$

where N_i is the number of particles in the i^{th} cell. The same expression can be written in terms of the summation of a running variable if we take the log of both sides of (A). We now have an expression for ln W as seen by equation (B). The object of this development is to arrive at an expression for the number of particles at a particular energy level. To accomplish this W is usually maximized with respect to N_i and to carry this out we need a convenient approximation for $N_i!$, where N_i is a large number. The approximation for N! for large N is known as Stirling's Approximation. The motivation for this development of the approximation for N! came from an exercise in Wilson's Advanced Calculus, where he hints that the application of a certain transformation to the Gamma Function will be useful in establishing Stirling's Approximation. The use of this transformation enables the reader with a knowledge of intermediate calculus to understand the development.

A second problem merely requires that one find the solution to a certain form of definite integral. One such integral arises in Quantum Statistics; namely the expression for the specific heat of a metal as shown in equation (C). C_y is the specific heat; theta is a constant for a particular metal; X equals hw/3.14kt, where w is the frequency of thermal vibrators and t is the absolute temperature of the metal. It is desirable to simplify such expressions for limiting values, such as when t approaches zero.

$$\text{(B)} \quad \ln W = \ln N! - \sum_{i=1}^{P} \ln N_i!$$

$$\text{(C)} \quad C_V = 9R\left(\frac{T}{\Theta}\right)^3 \int_0^{\frac{\Theta}{T}} \frac{x^4 e^x}{(e^x-1)^2}\,dx$$

The solution to both problems is based upon the Gamma Function which I shall now develop.

Consider the function:

$$e^{-xu};\ x,u>0 \qquad \int_0^\infty e^{-xu}\,du = -\frac{1}{x}e^{-xu}\Big]_{u=0}^{u\to\infty} = \frac{1}{x}$$

Integrating the function with respect to u between the limits of zero and infinity we get (D).

$$\text{(D)} \quad \frac{1}{x} = \int_0^\infty e^{-xu}\,du$$

A useful theorem at this point can be found in "Advanced Calculus" by Buck on p. 153.

a. if $\int_0^\infty f(x,u)\,du$ converges for all $a \leq x \leq b$

b. if $f_x(x,u)$ is continuous; $a \leq x \leq b$; $0 \leq u < \infty$

c. if $\int_0^\infty f_x(x,u)\,du$ is uniformly convergent

for $a \leq x \leq b$ then: $\frac{d}{dx}\int_0^\infty f(x,u)\,du = \int_0^\infty f_x(x,u)\,du$

A function which satisfies the theorem for all x and u greater than zero is the exponential:

$$e^{-xu}$$

Take d/dx of equation (D), repeat the differentiation and generalize the results for n derivatives.

$$\frac{1}{x^2} = \int_0^\infty u e^{-xu}\,du;\ \frac{1\cdot 2}{x^3} = \int_0^\infty u^2 e^{-xu}\,du;\ \frac{N!}{x^{N+1}} = \int_0^\infty u^n e^{-xu}\,du$$

If we set x equal to 1 we have an integral form called a Gamma Function. If N is an integer we have an expression for N! and if N is not an integer we have a definition of factorials for non-integers, where N is greater or equal to zero.

$$N! = \int_0^\infty u^n e^{-u}\,du;$$

Integrating the expression for Gamma of N plus 2 by parts, we get an expression for the Gamma Function of N plus 2. The first term of the integrated expression reduced to zero.

(E) We define: $\Gamma(N+1) \equiv \int_0^\infty u^n e^{-u} du = N!$

Since $\Gamma(N+1) \equiv \int_0^\infty u^n e^{-u} du = N!$; $\Gamma(N+2) \equiv \int_0^\infty u^{n+1} e^{-u} du$

$u = U^{N+1}$; $dv = e^{-u} du$;

$\Gamma(N+2) = [-u^{N+1} e^{-u}]_0^\infty + \int_0^\infty (N+1)\, u^N e^{-u} du$

The integral reduces to:

(F) $(N+1)\Gamma(N+1)$; $\Gamma(N+2) = (N+1)\Gamma(N+1)$

and since $\Gamma(N+1) = N!$; $\Gamma(N+2) = (N+1)(N!) = (N+1)!$

We shall now return to the problems introduced. First we want to find an approximation for N! for large N. We can now express N! as an integral:

$$N! = \Gamma(N+1) \equiv \int_0^\infty u^N e^{-u} du$$

The development is based upon the following transformation:

$$u = (N + \sqrt{2N}\, y); \quad du = \sqrt{2N}\, dy; \quad a = \sqrt{\tfrac{N}{2}}$$

The following are a few manipulations:

(G) $N! = \int_{-a}^\infty (N + \sqrt{2N}\, y)^N e^{-(N+\sqrt{2N} y)} \sqrt{2N}\, dy$

a. $(N + \sqrt{2N}\, y) = N(1 + \sqrt{\tfrac{2}{N}}\, y)$; $(N + \sqrt{2N}\, y)^N = N^N (1 + \sqrt{\tfrac{2}{N}}\, y)^N$

b. $(1 + \sqrt{\tfrac{2}{N}}\, y)^N = e^{N \ln. (1 + \sqrt{2}\, y/\sqrt{N})}$

By McClaurin's Formula for large N, neglecting third order terms we get:

(H) $N! = e^{-N} \sqrt{2N}\, N^N \int_{-a}^\infty e^{N \ln. (1 + \sqrt{2}\, y/\sqrt{N})} e^{-\sqrt{2N} y} dy$

$\ln (1 + \sqrt{2}\, y/\sqrt{N}) = \sqrt{\tfrac{2}{N}}\, y - \tfrac{2}{N} \tfrac{y^2}{2!} + \cdots \cong \sqrt{\tfrac{2}{N}}\, y - \tfrac{y^2}{N}$

$N! = e^{-N} \sqrt{2N}\, N^N \int_{-a}^\infty e^{\sqrt{2N} y - y^2 - \sqrt{2N} y} dy$

$N! = e^{-N} \sqrt{2N}\, N^N \int_{-a}^\infty e^{-y^2} dy$

Since N is large and since the integrand decreases very rapidly as y increases we can substitute minus infinity for the lower limit. Since the integrand is an even function:

$$\int_{-\infty}^{+\infty} e^{-y^2} dy = 2\int_0^{\infty} e^{-y^2} dy$$

$$\text{(I)} \quad N! = e^{-N}\sqrt{2N}\, N^N\, 2\int_0^{\infty} e^{-y^2} dy$$

The method used to evaluate the integral is found on p. 376 in Sokolnikoff's "Advanced Calculus" and goes as follows:

a. let $G = \int_0^{\infty} e^{-y^2} dy$ b. let $G = \int_0^{\infty} e^{-z^2} dz$

multiply (a.) by (b.); $G^2 = \int_0^{\infty} e^{-y^2} dy \int_0^{\infty} e^{-z^2} dz$

The variables and the limits of the iterated integral are independent so a double integral can be formed.

$$G^2 = \int_0^{\infty}\int_0^{\infty} e^{-(y^2+z^2)} dy\, dz$$

Now we employ the following transformation:

$$z = r\cos\theta \ ; \quad y = r\sin\theta \ ; \quad dy\, dz = r\,dr\,d\theta$$

Here we may consider our independent variables as coordinates determining the yz plane. The double integral can now be considered as an integration over the entire first quadrant. By our transformation the limits of integration become:

$$0 \le r < \infty \ ; \quad 0 \le \theta \le \frac{\pi}{2}$$

The square root of the result of the double integral yields a value for G.

$$G^2 = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} e^{-r^2} r\,dr = \int_0^{\frac{\pi}{2}} d\theta \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

$$G = \int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

Substituting the value for G in equation (I):

$$\text{(J)} \quad N! \doteq \sqrt{2\pi N}\, N^N e^{-N}$$

We now have Stirling's Approximation.

From equation (J) we can easily establish the approximation for ln N!, however, there is another approximation for ln N!:

Consider: $\ln N! = \ln 1 + \ln 2 + \ln 3 + \cdots \ln K + \cdots \ln N$

As K becomes very large the difference between ln K and ln(K-1) approaches zero. Thus:

$$\ln N! = (N+\tfrac{1}{2}) \ln N - N + \ln\sqrt{2\pi}; \quad \ln N! = N \ln N - N$$

Integrating by parts with u equal to ln N and dv equal to dN:

$$(K) \quad \ln N! = \int_1^N \ln N \, dN = N \ln N - N$$

From the Gamma Function and from the integral approximation we get respectively:

$$\ln N! = \sum_{i=1}^{N} \ln N_i \approx \int_1^N \ln dn$$

If we neglect the 1/2 in the N plus 1/2 and if we consider the ln of the square root of 2 pi negligible compared to N (ln N) plus N for very large N we find that the integral approximation is an approximation to Stirling's Approximation.

Finally, the second problem requires that we simplify:

$$C_y = 9R\left(\frac{T}{\Theta}\right)^3 \int_0^{\frac{\Theta}{T}} \frac{x^4 e^x}{(e^x-1)^2} dx \quad \text{as } T \to 0 \text{ or } \frac{\Theta}{T} \to \infty$$

Since $T \propto \frac{1}{x}$; if: $T \to 0$, where $x \gg 1, e^x - 1 \approx e^x$

$$C_y = 9R\left(\frac{T}{\Theta}\right)^3 \int_0^\infty \frac{x^4 e^x}{e^{2x}} dx = 9R\left(\frac{T}{\Theta}\right)^3 \int_0^\infty x^4 e^{-x} dx$$

$$= 9R\left(\frac{T}{\Theta}\right)^3 \Gamma(5); \text{ finally}: C_y = 9R\left(\frac{T}{\Theta}\right)^3 4!$$

The function can now be shown to be a Gamma Function.