THE BROOKLYN COLLEGE

JOURNAL

of

SCIENCE

1963
# The Brooklyn College JOURNAL OF SCIENCE

## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Title</th>
<th>Author</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>THE NATURE AND ORIGIN OF CORAL ATOLLS</td>
<td>R. A. Levich</td>
<td>5</td>
</tr>
<tr>
<td>ESSAY ON EMBRYOLOGY</td>
<td>Ronald Andiman</td>
<td>7</td>
</tr>
<tr>
<td>PARALLELYSIS</td>
<td>Shep. H. Greenberg</td>
<td>8</td>
</tr>
<tr>
<td>FLUORESCENCE IN GEMS AND MINERALS</td>
<td>Joel Arem</td>
<td>9</td>
</tr>
<tr>
<td>FOURTH STATE OF MATTER</td>
<td>Jack Pinchuk</td>
<td>12</td>
</tr>
<tr>
<td>CLOSURE AS A FUNCTION OF RECENCY AND FREQUENCY OF HYPOTHESIS</td>
<td>Harvey N. Switzky</td>
<td>17</td>
</tr>
<tr>
<td>THE RELATION OF PIGMENTS TO PHOTOSYNTHESIS</td>
<td>Jack Pinchuk</td>
<td>23</td>
</tr>
<tr>
<td>A POSSIBLE AURORAL GLOW ON VENUS</td>
<td>Joel Stewart Levine</td>
<td>25</td>
</tr>
<tr>
<td>SOLVENT EXTRACTION OF COMPLEXED METAL CATION KOJATES</td>
<td>Harvey S. Nager</td>
<td>28</td>
</tr>
<tr>
<td>SUPERCONDUCTIVITY</td>
<td>Allen J. Cohen</td>
<td>34</td>
</tr>
<tr>
<td>RELATIVISTIC KINEMATICS OF A SPINNING ELECTION</td>
<td>Al Sweedler</td>
<td>38</td>
</tr>
<tr>
<td>THE GAMMA FUNCTION AND STIRLING'S APPROXIMATION</td>
<td>Sheldon Krinsky</td>
<td>44</td>
</tr>
</tbody>
</table>
THE GAMMA FUNCTION AND
STIRLING’S APPROXIMATION
by Sheldon Krimsky

My aim in this paper is to connect the solution of two apparently different problems arising in Physics with a useful function, the Gamma Function. I shall state the problems, develop the Gamma Function and some of its properties and then return to solve the problems.

The first problem arises in the development of Maxwell-Boltzmann Statistics where it is necessary to find an expression for the distribution of N particles among P energy cells, where one doesn’t distinguish between distributions in the same cell. The expression obtained for W the number of distinct distributions is:

(A) \[ W = \frac{N!}{N_1! N_2! \ldots N_i! \ldots N_p!} \]

where \( N_i \) is the number of particles in the \( i^{th} \) cell. The same expression can be written in terms of the summation of a running variable if we take the log of both sides of (A). We now have an expression for \( \ln W \) as seen by equation (B). The object of this development is to arrive at an expression for the number of particles at a particular energy level. To accomplish this out we need a convenient approximation for \( N_i! \), where \( N_i \) is a large number. The approximation for \( N! \) for large \( N \) is known as Stirling’s Approximation. The motivation for this development of the approximation for \( N! \) came from an exercise in Wilson’s Advanced Calculus, where he hints that the application of a certain transformation to the Gamma Function will be useful in establishing Stirling’s Approximation. The use of this transformation enables the reader with a knowledge of intermediate calculus to understand the development.

A second problem merely requires that one find the solution to a certain form of definite integral. One such integral arises in Quantum Statistics; namely the expression for the specific heat of a metal as shown in equation (C). \( y \) is the specific heat; \( \theta \) is a constant for a particular metal; \( X \) equals \( hw/3.14kt \), where \( w \) is the frequency of thermal vibrators and \( t \) is the absolute temperature of the metal. It is desirable to simplify such expressions for limiting values, such as when \( t \) approaches zero.

(B) \[ \ln W = \ln N! - \sum_{i=1}^{p} \ln N_i! \]
\( C_y = 9R \left( \frac{T}{\delta} \right)^3 \int_0^x \frac{x^4 e^x}{(e^x - 1)^2} \, dx \)

The solution to both problems is based upon the Gamma Function which I shall now develop.

Consider the function:

\[ e^{-xu}; \quad x, u > 0 \quad \int_0^\infty e^{-xu} \, du = -\frac{1}{x} e^{-xu} \bigg|_{u=0}^{u=\infty} = \frac{1}{x} \]

Integrating the function with respect to \( u \) between the limits of zero and infinity we get (D).

\( \frac{1}{x} = \int_0^\infty e^{-xu} \, du \)

A useful theorem at this point can be found in “Advanced Calculus” by Buck on p. 153.

a. if \( \int_0^\infty f(x, u) \, du \) converges for all \( a \leq x \leq b \)

b. if \( f(x, u) \) is continuous, \( a \leq x \leq b \), \( 0 \leq u < \infty \)

c. if \( \int_0^\infty f(x, u) \, du \) is uniformly convergent for \( a \leq x \leq b \), then:

\[ \frac{d}{dx} \int_0^\infty f(x, u) \, du = \int_0^\infty f_x(x, u) \, du \]

A function which satisfies the theorem for all \( x \) and \( u \) greater than zero is the exponential:

\[ e^{-xu} \]

Take \( d/dx \) of equation (D), repeat the differentiation and generalize the results for \( n \) derivatives.

\[ \frac{1}{x^n} = \int_0^\infty u^n e^{-xu} \, du; \frac{1}{x^n} = \int_0^\infty u^2 e^{-xu} \, du; \frac{N!}{x^N} = \int_0^\infty u^N e^{-xu} \, du \]

If we set \( x \) equal to 1 we have an integral form called a Gamma Function. If \( N \) is an integer we have an expression for \( N! \) and if \( N \) is not an integer we have a definition of factorials for non-integers, where \( N \) is greater or equal to zero.

\[ N! = \int_0^\infty u^N e^{-u} \, du; \]

Integrating the expression for Gamma of \( N \) plus 2 by parts, we get an expression for the Gamma Function of \( N \) plus 2. The first term of the integrated expression reduced to zero.
(E) We define: \[ \Gamma(N+1) = \int_0^{\infty} u^N e^{-u} \, du = N! \]
Since \[ \Gamma(N+1) = \int_0^{\infty} u^N e^{-u} \, du = N!; \quad \Gamma(N+2) = \int_0^{\infty} u^{N+1} e^{-u} \, du \]
u = u^{N+1}; \quad dv = e^{-u} \, du;
\[ \Gamma(N+2) = \left[ -u^{N+1} e^{-u} \right]_0^{\infty} + \int_0^{\infty} (N+1) u^N e^{-u} \, du \]
The integral reduces to:

(F) \( (N+1) \Gamma(N+1) \); \( \Gamma(N+2) = (N+1) \Gamma(N+1) \)
and since \( \Gamma(N+1) = N! \); \( \Gamma(N+2) = (N+1)(N)! = (N+1)! \)

We shall now return to the problems introduced. First we want to find an approximation for \( N! \) for large \( N \). We can now express \( N! \) as an integral:

\[ N! = \Gamma(N+1) = \int_0^{\infty} u^N e^{-u} \, du \]

The development is based upon the following transformation:

\[ u = (N + \frac{1}{2} N y) \]; \( du = \frac{1}{2} N \, dy \); \( a = \frac{1}{2} \)

The following are a few manipulations:

(G) \( N! = \int_a^{\infty} (N + \frac{1}{2} N y)^N \, e^{-N + \frac{1}{2} N y} \sqrt{2N} \, dy \)
\( N + \frac{1}{2} N y = N (1 + \frac{1}{2} \frac{N}{y}) \); \( (N + \frac{1}{2} N y)^N = N^N (1 + \frac{1}{2} \frac{N}{y})^N \)
\( (1 + \frac{1}{2} \frac{N}{y})^N = e^{N \ln (1 + \frac{1}{2} \frac{N}{y})} \)

By McCLaurin's Formula for large \( N \), neglecting third order terms we get:

(H) \( N! = e^{-N \frac{1}{2} N} N^N \int_a^{\infty} e^{N \ln (1 + \frac{1}{2} \frac{N}{y})} e^{-\frac{1}{2} \frac{N}{y}} \, dy \)
\( \ln (1 + \frac{1}{2} \frac{N}{y}) = \frac{1}{2} \frac{N}{y} - \frac{1}{4} \frac{N}{y} y^2 + \ldots \approx \frac{1}{2} \frac{N}{y} y^2 - \frac{N}{y} \)
\( N! = e^{-N \frac{1}{2} N} N^N \int_a^{\infty} e^{\frac{1}{2} N y^2} - y^2 - \frac{1}{2} N y \, dy \)
\( N! = e^{-N \frac{1}{2} N} N^N \int_a^{\infty} e^{-y^2} \, dy \)

46
Since \( N \) is large and since the integrand decreases very rapidly as \( y \) increases we can substitute minus infinity for the lower limit. Since the integrand is an even function:

\[
\int_{-\infty}^{\infty} e^{-y^2} dy = 2 \int_{0}^{\infty} e^{-y^2} dy
\]

\((I)\) \( N! = e^{-N} \frac{\sqrt{\pi N}}{2} N^N 2 \int_{0}^{\infty} e^{-y^2} dy \)

The method used to evaluate the integral is found on p. 376 in Sokolnikoff's "Advanced Calculus" and goes as follows:

a. let \( G = \int_{0}^{\infty} e^{-y^2} dy \)

b. let \( G = \int_{0}^{\infty} e^{-y^2} dy \)

multiply \((a)\) by \((b)\); \( G^2 = \int_{0}^{\infty} e^{-y^2} dy \int_{0}^{\infty} e^{-y^2} dy \)

The variables and the limits of the iterated integral are independent so a double integral can be formed:

\[
G^2 = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(y^2 + z^2)} dy \, dz
\]

Now we employ the following transformation:

\[
\begin{align*}
  z &= r \cos \theta; \\
  y &= r \sin \theta; \\
  dy \, dz &= r \, dr \, d\theta
\end{align*}
\]

Here we may consider our independent variables as coordinates determining the \( yz \) plane. The double integral can now be considered as an integration over the entire first quadrant. By our transformation the limits of integration become:

\[
0 \leq r < \infty; \quad 0 \leq \theta \leq \frac{\pi}{2}
\]

The square root of the result of the double integral yields a value for \( G \).

\[
G^2 = \int_{0}^{\frac{\pi}{2}} d\theta \int_{0}^{\infty} e^{-r^2} r \, dr = \left[ \frac{\pi}{2} \right] \int_{0}^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}
\]

\[
G = \left[ \int_{0}^{\infty} e^{-y^2} dy \right]^{1/2} = \frac{\sqrt{\pi}}{2}
\]

Substituting the value for \( G \) in equation \((I)\):

\[
(J) \quad N! = \sqrt{2 \pi N} \quad N^N \quad e^{-N}
\]

We now have Stirling's Approximation.

From equation \((J)\) we can easily establish the approximation for \( \ln N! \), however, there is another approximation for \( \ln N! \):
Consider: \( \ln N! = \ln 1 + \ln 2 + \ln 3 + \ldots + \ln K + \ldots + \ln N \)

As \( K \) becomes very large the difference between \( \ln K \) and \( \ln (K-1) \) approaches zero. Thus:

\[
\ln N! = (N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi}; \quad \ln N! \approx N \ln N - N
\]

Integrating by parts with \( u = \ln N \) and \( dv = dN \):

\[
(N) \ln N! = \int_{1}^{N} \ln N \, dN = N \ln N - N
\]

From the Gamma Function and from the integral approximation we get respectively:

\[
\ln N! = \sum_{i=1}^{N} \ln N_i \approx \int_{1}^{N} \ln \, dn
\]

If we neglect the \( 1/2 \) in the \( N \) plus \( 1/2 \) and if we consider the \( \ln \) of the square root of \( 2 \pi \) negligible compared to \( N \ln N \) plus \( N \) for very large \( N \) we find that the integral approximation is an approximation to Stirling's Approximation.

Finally, the second problem requires that we simplify:

\[
C_y = 9R \left( \frac{1}{\epsilon} \right)^3 \int_{0}^{\infty} \frac{x^4 e^x}{(e^x-1)^2} \, dx \quad \text{as} \quad T \to 0 \text{ or } \frac{\Omega}{T} \to \infty
\]

Since \( T \propto \frac{1}{x^2} \), if \( T \to 0 \), where \( x \gg 1, e^x \approx e^x \)

\[
C_y = 9R \left( \frac{1}{\epsilon} \right)^3 \left( \int_{0}^{\infty} \frac{x^4 e^x}{e^{2x}} \, dx = 9R \left( \frac{1}{\epsilon} \right)^3 \int_{0}^{\infty} x^4 e^{-x} \, dx = 9R \left( \frac{1}{\epsilon} \right)^3 \Gamma(5) \right)
\]

Finally: \( C_y = 9R \left( \frac{1}{\epsilon} \right)^3 4! \)

The function can now be shown to be a Gamma Function.