MATH 19-02: HW 5

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As we've discussed, a *move favorable to* X is one in which some voters change their preferences so that X is raised, while the relative order of the others stays the same. (Example: $FOXY \to FXOY$ or $XFOY$)

Let's say a *move neutral to* X is one in which every voter keeps X in the same position in their preferences, switching others around but never moving them past X, so the ones above stay above and the ones below stay below. (Example: the only move of $FOXY$ that is neutral to X changes it to $OFXY$.)

We know that a voting system is called *monotonic* if it can never happen that some moves favorable to X switch them from a winner to a loser.

Let's say that a voting system is called *strongly monotonic* if it can never happen that some combination of moves favorable to and neutral to X switch them from a winner to a loser.

(1) (a) Suppose for some system and preference schedule, the original winner set is $\mathcal{W} = \{P\}$. After a move favorable to P, the new winner set is $\mathcal{W}' = \{P, Q\}$. This does not tell you whether or not the system is monotonic. Why not?

(b) Suppose for some system and preference schedule, the original winner set is $\mathcal{W} = \{M, X\}$. After a move neutral to X, the new winner set is $\mathcal{W}' = \{M\}$. This does not tell you whether the system is monotonic, but it is definitely not strongly monotonic. Why?

(2) Consider this preference schedule:

×31	×20	×10	×18	×40
A	X	B	B	M
B	O	O	O	A
X	M	X	M	B
M	A	M	A	O
O	B	A	X	X

(a) Starting with the original election, give a new preference schedule in which exactly 3 voters make moves favorable to A.

(b) Starting with the original election, give a new preference schedule in which exactly 3 voters make moves neutral to B.

(3) Show that Dictatorship is a Pareto-efficient and strongly monotonic single-winner system.
(In other words, it satisfies the hypotheses of the Müller-Satterthwaite theorem.)
Note there are three separate things to verify here.

(4) *This is an opinion question: your answer can be anything as long as you explain your reasoning.* How reasonable is it to insist that a voting system be single-winner? Does your answer change if the number of candidates (n) is high or low? Does it change if the number of voters (N) is high or low?

(5) Consider the following preference schedules:

$\times 3$	$\times 2$	$\times 2$
X	Y	Z
Z	X	Y
Y	Z	X

$\times 3$	$\times 2$	$\times 2$
X	Y	Y
Z	X	Z
Y	Z	X

Who wins each one, by the beatpath method? Considering those answers, does that tell you whether beatpath is strongly monotonic?

(6) (a) Following the proof of Müller-Satterthwaite, suppose you know you're working with an unknown single-winner voting system that is Pareto-efficient and strongly monotonic. Narrate the proof using the following sequence of detailed preference schedules.

①

#1	#2	#3	#4	#5
P	P	P	P	P
M	M	M	M	M
N	N	N	N	N
O	O	O	O	O

②

#1	#2	#3	#4	#5
P	P	P	P	P
O	M	M	M	M
M	N	N	N	N
N	O	O	O	O

③

#1	#2	#3	#4	#5
O	P	P	P	P
P	M	M	M	M
M	N	N	N	N
N	O	O	O	O

④

#1	#2	#3	#4	#5
O	O	O	P	P
P	P	P	O	M
M	M	M	M	N
N	N	N	N	O

⑤

#1	#2	#3	#4	#5
O	O	O	O	P
P	P	P	P	M
M	M	M	M	N
N	N	N	N	O

⑥

#1	#2	#3	#4	#5
O	O	O	O	M
M	M	M	P	N
N	N	N	M	P
P	P	P	N	O

⑦

#1	#2	#3	#4	#5
O	O	O	P	M
M	M	M	O	N
N	N	N	M	P
P	P	P	N	O

⑧

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	P
P	P	P	O	O

⑨

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	O
P	P	P	O	P

(b) If we're showing that the unknown voting system is Dictatorship of the kth voter using this proof technique, what is the value of k in the example above?

(c) The point of getting to a "pathological" preference schedule like ⑨ is that from there you can get to ANY detailed schedule in which voter k ranks candidate P first with a combination of moves neutral to and favorable to P.

Check this by filling in a preference schedule in the middle where the transitions are as described here.

⑨

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	O
P	P	P	O	P

$\xrightarrow[\text{to } P]{\text{neut}}$

⑩

#1	#2	#3	#4	#5

$\xrightarrow[\text{to } P]{\text{fav}}$

⑪

#1	#2	#3	#4	#5
P	O	N	P	O
O	P	O	O	P
M	M	M	N	M
N	N	P	M	N

Who wins in that final schedule and why?

(d) Note that ⑪ could have been *anything at all* as long as voter k liked P best! Explain why this finally proves that voter k is a "Dictator."

Extra Credit

Read Chapter 8, which is about the probability of weird outcomes in elections.

Answer any of the following questions, either by hand or with a computer program. If you use a computer program, include the code when you hand it in!

Probability of a cycle. Suppose there are $n = 3$ candidates in an election, say $\mathcal{C} = \{A, B, C\}$. Then there are $3! = 6$ ways to rank them. Let's say that N_1 is the number of people who choose the ranking $A > B > C$ and N_2 choose $A > C > B$ and so on up to N_6. Then clearly $\sum_i N_i = N$, the total number of voters. So we can store all the information about the preference schedule in the vector $(N_1, N_2, N_3, N_4, N_5, N_6)$.

- How many preference schedules are possible if $N = 10$? If $N = 45$ (the size of our class)? How about for general N?
- Let's let $PCyc(N)$ be the fraction of preference schedules that have a Condorcet cycle. Christoph's book says that this fraction is "substantial." Can you compute it for various values of N, and try to convince yourself that it doesn't go to 0 or 1 as N gets large? Even better, can you guess (or prove) the limit?

Probability of a tie. Now suppose that instead of all preference schedules being equally likely, we instead assign probability p_1 for the first ordering of the candidates, p_2 for the second, and so on up to $p_{n!}$. The only assumption on these probabilities is that they're all positive. Christoph tells us that for any fixed number of candidates n, we can fix the probabilities p_i and think about what happens as $N \to \infty$. We can write $PTie$ for the probability of a tied election ($|\mathcal{W}| > 1$). He claims that $PTie \to 0$ for beatpath and Smithified Borda but not for pairwise comparison.

- Prove or illustrate any part of that.

Etc. You can choose any other property of elections we have studied and investigate it probabilistically. If you want, you can develop this into a project that you do in place of Midterm 2. Start this week!