APPORTIONMENT NOTES

MATH 19, MATH OF SOCIAL CHOICE
SPRING 2016

Political **apportionment** is dividing up seats on a representative body. For instance, the seats in the U.S. House of Representatives need to be divided among the states. The basic principle is that each state should receive a proportion of the representation that's as close as possible to its proportion of the population.

For instance, in the 2010 Census, the population of Massachusetts was 6,547,817; the population of the U.S. was 308,156,338; and there are currently 435 seats in the House. So Massachusetts?s proportional quota is 9.2430368737... reps.

So: How do we round??

Notation. Denote the number of seats by m and the total population by M. Then we have

$$m = m_1 + m_2 + \cdots + m_s$$

and

$$M = M_1 + M_2 + \cdots + M_s,$$

where m_i is the number of seats given to the ith state, M_i is the population of the ith state, and s is the total number of states.

The *quota* of seats deserved by the ith state is

$$Q_i = \frac{M_i}{M} \cdot m = \frac{M_i}{M/m}.$$

This quota is the number of seats the district would receive under exact proportionality, but it's virtually guaranteed not to be a whole number.

We will use $\lfloor x \rfloor$ to denote the largest integer $\leq x$ (the one you'd get by rounding down) and $\lceil x \rceil$ to denote the smallest integer $\geq x$ (the one you'd get by rounding up). In this setting $\{x\}$ will denote the fractional part of x. For example, if $x = 4.3$, then $\lfloor 4.3 \rfloor = 4$, $\lceil 4.3 \rceil = 5$, and $\{4.3\} = 0.3$.

Hamilton's method. One answer to the question "How do we round?" is Hamilton's method, which works as follows:

First assign to each district $\lfloor Q_i \rfloor$ seats (its quota rounded down). Next, if there are any seats left over, give them out in order of $\{Q_i\}$. Let's look at an example to see how this works.

Example with Hamilton's method. Let $m = 100$ seats to give out and suppose there are $s = 3$ states. Suppose the states have population $M_1 = 505$, $M_2 = 492$, and $M_3 = 301$, so that the total population is $M = 505 + 492 + 301 = 1298$. Then we have

$$Q_1 = \frac{505}{1298} \cdot 100 = 38.906\ldots$$
$$Q_2 = \frac{492}{1298} \cdot 100 = 37.904\ldots$$
$$Q_3 = \frac{301}{1298} \cdot 100 = 23.18\ldots.$$

This means $\lfloor Q_1 \rfloor = 38$, $\lfloor Q_2 \rfloor = 37$, and $\lfloor Q_3 \rfloor = 23$, so we initially allocate $38 + 37 + 23 = 98$ seats, with 2 left over. Next we see that

$$\{Q_1\} = 0.906\ldots$$
$$\{Q_2\} = 0.904\ldots$$
$$\{Q_3\} = 0.18\ldots$$

so the first surplus seat is given to state 1 and the second to state 2. Thus, $m_1 = 39$, $m_2 = 38$, and $m_3 = 23$, which we can check to see that these add up to 100.

Paradoxes. While it may seem like a very reasonable way to deal with the rounding problem, Hamilton's method is not without issues. Some unexpected and (depending on your viewpoint) undesirable things can happen with this method, which we'll call paradoxes. Here are a few:

The Alabama Paradox. In 1882, there were 299 seats in the U.S. House of Representatives ($m = 299$) and Alabama was allocated 8 seats using Hamilton's method ($m_{AL} = 8$). But observe that if one seat was added to the house ($m' = 300$) and then seats were reallocated, Alabama would end up losing a seat ($m'_{AL} = 7$). This doesn't seem right: if m increases while the population proportions stay fixed, we would expect that each of the m_i would either stay the same or increase.

The Population Paradox. In 1900, Virginia's population was growing 60% faster than Maine's. However, when seats were redistributed following the census, Virginia lost a seat while Maine gained one. Intuitively, it seems that Virginia is more deserving of an extra seat. Drama!

The New States Paradox. In 1907, Oklahoma became the 46th state in the union. Before this, we had $m = 386$. By comparing to states of similar size, $m_{OK} = 5$ was assigned, and the house was enlarged accordingly to $m' = 391$. But if you reapply Hamilton's method with $m' = 391$, you get:

$$m_{NY} = 38\ ,\ {m_{NY}}' = 37\ ;$$
$$m_{ME} = 3\ ,\ {m_{ME}}' = 4.$$

The addition of the 5 seats caused New York's allocation to decrease and Maine's to increase, despite no change in population proportions. This feels wrong: Maine received one of New York's seats simply because Oklahoma showed up on the scene. No good!

Alternatives to Hamilton's method. The idea behind many alternative apportionment systems is to systematically tweak the quotas $Q_i = \frac{M_i}{M/m}$ until your favorite rounding rule gives the right number of seats. (Note, an appropriate adjusted denominator will always exist, as long as no two states have populations that are exact multiples of each other!)

Ways of rounding: suppose n is an integer such that $n < Q_i < n + 1$.

- Jefferson's method: round down ($m_i = n$)
- Adams's method: round up ($m_i = n + 1$)
- Webster's method: round to nearest integer ($m_i = n$ if $Q_i < n + \frac{1}{2}$)
- Huntington-Hill method: round by geometric mean ($m_i = n$ if $Q_i < \sqrt{n(n + 1)}$)

Amazingly, Huntington-Hill's method—adjust the denominator until rounding by geometric mean causes the m_i to add up to m exactly—has been the law of the land since 1941.

Example with Jefferson's method. Let's revisit the example we used earlier to demonstrate this new method. We have $M = 1298$ and $m = 100$, so $M/m = 12.98$. Recall that none of the Q_i were integers using this denominator. So what if we divide by 12.9 instead? Then we get

$$m_1 = \left\lfloor \frac{505}{12.9} \right\rfloor = 39$$
$$m_2 = \left\lfloor \frac{492}{12.9} \right\rfloor = 38$$
$$m_3 = \left\lfloor \frac{301}{12.9} \right\rfloor = 23$$

so $m_1 + m_2 + m_3 = 100 = m$. No leftover seats!

An impossibility theorem for apportionment.

Let's say that an apportionment method is *neutral* if the apportionments m_i depend only on the populations M_i (and not on the names of the states, or anything else).

We will say that an apportionment method satisfies the *quota rule* if

$$\lfloor Q_i \rfloor \leq m_i \leq \lceil Q_i \rceil$$

for all i. That is, every state should receive a number of seats that is within 1 of its quota. (Note, Hamilton's method satisfies the quota rule: each district is initially assigned $m_i = \lfloor Q_i \rfloor$ seats. If they then receive an extra seat, they end will end up with $m_i' = \lfloor Q_i \rfloor + 1$, which equals $\lceil Q_i \rceil$ as long as Q_i wasn't an integer already.)

Let's end this unit with bad news.

Theorem (Balinski–Young 1982): There is no neutral apportionment method that satisfies the quota rule and is free from the population paradox.

So just like there's no perfect voting method, there's also no perfect apportionment method!