PRACTICE PROBLEMS FOR FINAL EXAM
–SOLUTIONS–

MATH 19-02, SPRING 2018

For these practice problems, you can use a calculator. On the exam, the problems will be designed so that you don't need a calculator.

(1) *California will have a ballot initiative in the next election about whether to split up into three states, called North California (NorCal), South California (SoCal), and New California (NewCal). Currently California has a population of 39.5 million, and the populations of the new states would be 13.3 million in NorCal, 13.9 million in SoCal, and 12.3 million in NewCal.*

(a) CA has 53 Congressional representatives. Split them up to the three new states using the Huntington-Hill method.

The quotas are computed as $Q_i = (M_i/M) \cdot 53$, so they are $Q_1 = 17.845...$, $Q_2 = 18.650...$, and $Q_3 = 16.503...$, respectively.

The cutoffs for the $\sqrt{n(n+1)}$ rounding rule are $\sqrt{17 \cdot 18} = 17.492...$, $\sqrt{18 \cdot 19} = 18.493...$, and $\sqrt{16 \cdot 17} = 16.492...$, respectively. That means that the quotas all get initially rounded **up**, because they're above their cutoffs. This distributes $18 + 19 + 17 = 54$ seats, which is one too many.

So we adjust M, the population of California. We want to make it larger so that the adjusted quotas are smaller. We don't have to change it by much because the third number is so close to the cutoff. So let's try $M' = 39.55$ million.

The new quotas are $Q'_1 = 17.823...$, $Q'_2 = 18.627...$, and $Q'_3 = 16.482...$. So it worked! Now the NewCal quota is just below the cutoff of 16.492..., so the new rounding gives $m_1 = 18$, $m_2 = 19$, and $m_3 = 16$ seats to the three chunks of California.

(b) What is the Population Paradox? Explain how you would check if California has been suffering from it in the last few cycles. (Just explain what information you'd need; no need to go look it up.)

The Population Paradox is that State i might have faster population growth than State j in between Censuses, but State j gains a seat and they either stay the same or lose a seat. To check this, I'd need two kinds of data: Census population numbers from the last several cycles (to compare the growth of California to other states), and Congressional apportionment from the last several cycles.

(2) *What has a higher Polsby-Popper score ($4\pi A/P^2$), an equilateral triangle or a 3×1 rectangle?*
What about the Reock scores?
What does this have to do with redistricting?

(For the triangle, you just end up using facts about the 30-60-90 triangle repeatedly.) Equilateral triangle of side 1 has $P = 3$ and $A = \sqrt{3}/4 = .433...$, so its PoPo score is .604... The circumscribing circle has the distance from the center to a vertex of the triangle as its radius, which means $r = \sqrt{3}/3$, so its area is 1.047.... That means the Reock score is roughly $.433/1.047 = .413...$

The three by one rectangle has $P = 8$ and $A = 3$, so its PoPo score is .589... Its Reock score is its area divided by the area of a circle with diameter $\sqrt{10}$. That circle has area 7.853..., so its Reock score is the ratio $3/7.853... = .381...$

This means that it's close, but the triangle has a slightly better Polsby-Popper score and a slightly better Reock score.

PoPo and Reock are both possible ways to quantify the *compactness* of shapes, which is supposed to tell you how eccentric the shapes are. The idea is that when voting districts are very eccentrically shaped, they are more likely to have been created to advance a specific agenda.

(3) *In partisan symmetry scores, why is it considered good if the seats-votes curve goes through or near to the point* $(1/2, 1/2)$*?*

Suppose that a certain election had district-by-district vote shares of $(.38, .82, .45, .58, .6)$ *from the Republican point of view. Assuming equal turnout and uniform partisan swing: what share of the votes and what share of the seats did Republicans receive? Would Democrats have had the same share of the representation if they had had the same share of votes?*

The point $(1/2, 1/2)$ represents the situation where each party has an equal share of the votes and an equal share of the seats. If you care about *symmetry* between the two parties, then your election system should produce this outcome!

In the given voting scenario, the overall percentage of Republican votes is $\overline{V} = .566$, and the number of Republican seats is three out of five. In order to reverse that so that Ds get 56.6% of the votes, we'd need to reduce $\overline{V}$ to .434, which means that the state overall, and each district, would lose .132 from its vote share. If I apply that uniform swing to all the districts, the new totals are

$$(.248, .688, .318, .448, .468),$$

in which case there would be one R and four D seats.

So if Rs get 56.6% of the votes, they get three seats, but if Ds get 56.6% of the vote, they get four seats. Not symmetrical!

(4) *Can you come up with a shape that has...*

- *Polsby-Popper score near 0 and Reock score near 1*
- *convex hull score near 1 and Reock score near 0*
- *skew score near 1 and square-Reock score near 0*

A circle that has a long thin trench dug out from it has a Reock score near 1 but a very high perimeter, so PoPo is near zero.

A long thin rectangle has a convex hull score equal to 1 but a Reock score very close to zero.

A slightly fattened plus sign + has a skew score of 1 but a square-Reock score near zero.

(5) *Is it possible for a move to be favorable to one candidate and neutral to another candidate? If so, give an example.*

Yes, it's possible! For instance, if the preference order A, B, C is changed to A, C, B, that is favorable to C but neutral to A.

(6) *Give an example of a preference schedule with five candidates so that all of them are involved in a Condorcet cycle.*

×10	×10	×10	×10	×10
A	B	C	D	E
B	C	D	E	A
C	D	E	A	B
D	E	A	B	C
E	A	B	C	D

This has a cycle from A to B to C to D to E and back to A, with each one beating the next by a margin of 30 votes.

(7) *Make up an example of a voting system that is unanimity-fair but not Pareto efficient, or explain why this is impossible.*

It's impossible! Suppose the system is unanimity-fair, and suppose there is a Pareto candidate. Then that candidate is ranked first on every ballot, but that means they're unanimously preferred to *every other candidate*. UF says that if there's a unanimous preference, then the dispreferred candidate shouldn't win. For this to be true, everyone but the Pareto candidate would be eliminated from contention. But then the Pareto candidate must win, so the system is automatically Pareto efficient.