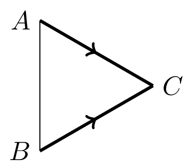
MATH 19-02: MATHEMATICS OF SOCIAL CHOICE

TUFTS UNIVERSITY DEPARTMENT OF MATHEMATICS
SPRING 2018

(1) *Explain: if a 3-candidate election has two Smith candidates out of three, then there must be a tie in the pairwise comparison graph.*



The picture should really help clarify what's going on here. We are given that there are two Smith candidates, so without loss of generality $\mathcal{S} = \{A, B\}$. But that means that the arrows above are forced in the pairwise comparison graph, so the only remaining question is the arrow between A and B. If it were not a tie—say A beats B—then A would actually be Condorcet, so there would be a smaller dominating set consisting of A alone, which is impossible since $\mathcal{S} = \{A, B\}$. Therefore it must be a tie!

(2) *Find a preference schedule and system for which there is a spoiler who is a Smith candidate. In your example, are they a winning spoiler? Losing spoiler? Weak spoiler?*

A reasonable place to go looking for this is a situation where everybody is Smith! Our easiest example is the basic Condorcet cycle

×1	×1	×1
A	B	C
B	C	A
C	A	B

. Here, the graph is a cycle and everybody is Smith. But also, everybody spoils! For instance, without candidate C, we see that B beats A head-to-head in any system that is two-way-fair, such as plurality. Thus if considering C as a spoiler we have $\mathcal{W}_{\text{plur}} = \{A, B, C\}$ but $\mathcal{W}'_{\text{plur}} = \{B\}$. This worked—$C$ is a strong spoiler.

They are also a winning spoiler, but not a losing spoiler or a weak spoiler.

(3) *If there are $n = 10$ candidates, how many consolidations do you have to consider to run each of these methods?: plurality, runoff, elimination, Coombs, Borda, Smith ($\mathcal{W} = \mathcal{S}$), Smithified plurality, pairwise comparison, sequential, and dictatorship.*

Plurality: 0 (just read off top row)

Runoff: 1 (consolidate down to top two first-place vote-getters)

Elimination: 8 (consolidate down to 9, 8, 7, ... , 2 by eliminating bottom first-place vote-getter at each stage)

note: I can see that you could count that as 9 by counting the "consolidation" down to 1. Either way is fine, just explain your reasoning.

Coombs: 8 (just like Elimination but with a different way to choose who's dropped at each stage)

Borda: 0, because you just award points based on the original preference schedule

Smith: this one's a little tricky. I'd say you need to build the whole PWCG. Since that graph has 45 edges (a graph with n candidates has $n(n-1)/2$ edges), you'd need that many pairwise runoffs. Then you can find the Smith set just from the graph.

Smithified plurality: I'd accept a few different answers here!

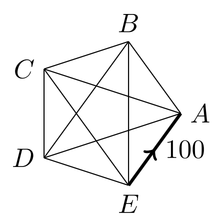
- same reasoning as Smith, need all 45 runoffs in the graph.
- actually you need 46 consolidations: first build the graph, then consolidate down to the Smith set.

- actually you need 0 consolidations. Weak candidates have NO first-place votes, so getting rid of them doesn't change the top row of the preference schedule. The Smithified plurality winner is the regular plurality winner!

PWC: still need the graph, so 45.
Sequential: 9 (that's the number of head-to-heads if you arrange 10 candidates in a sequence)
Dictatorship: 0 (just read off the first choice of the Dictator)

(4) *What has to be true about a pairwise comparison graph for it to be helpful in checking whether a system is unanimity-fair? Supposing $N = 100$ voters and $n = 5$ candidates, draw an example of a PWCG with one unanimous preference, and explain what you would have to check about $\mathcal{W}$ if considering unanimity-fairness of your system.*

Unanimity-fair means that *if all voters prefer $X > Y$, then $Y \notin \mathcal{W}$.* You can only use an example to test this if there is some unanimous preference in your example! Otherwise it does not help. On the graph, a unanimous preference would appear as an arrow with a margin of 100. (Because the preference is 100 : 0.) So you'd have to see some edge with a 100 and check that the loser of that head-to-head is not in $\mathcal{W}$.



Check if $A \in \mathcal{W}$.

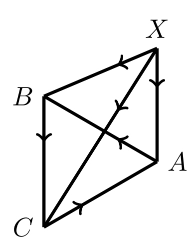
How do you use that info? If $A \in \mathcal{W}$, then you've shown that the system is NOT unanimity-fair, because everyone prefers E to A, but A is still in the winner set.

On the other hand, if $A \notin \mathcal{W}$, we've learned nothing conclusive—this example did respect the unanimous preference, but some other example might not.

(5) *Build a preference schedule with a Condorcet candidate and a cycle.*

That's not hard—we'll just start with a cycle and put candidate X on top!

$\times 4$	$\times 7$	$\times 5$
X	X	X
A	B	C
B	C	A
C	A	B



(6) *Build a preference schedule where nobody has 40% of the first-place votes, but there is some consolidation which produces a majority candidate.*

No problem—just make a divided vote in a three-way race. Then any consolidation down to two candidates will produce someone with a considerable majority.

$\times 35$	$\times 35$	$\times 30$
A	B	C
B	C	A
C	A	B

$\longrightarrow$

$\times 65$	$\times 35$
A	B
B	A

(7) *Explain: the runoff method is unanimity-fair.*

If there is a unanimous preference (say $X > Y$) then X appears above Y in every single column in the preference schedule. Now the runoff method takes the top two first-place vote-getters. Candidate Y has no first-place votes at all, because everyone prefers X to Y. So Y can't be one of the candidates in the runoff, so $Y \notin \mathcal{W}$.