Worksheet 1 – Solutions

Mathematics of Social Choice
Duchin, Spring 2021

Problem 1. Practice all the voting systems on this simple reduced preference schedule. (Plurality, PWC, Borda, Runoff, Elimination, Sequential, Coombs, Secondality)

1	2	3	1
A	B	D	A
B	C	A	C
C	A	C	B
D	D	B	D

(Some useful consolidations)

$\times 2$	$\times 2$	$\times 3$
A	B	D
B	A	A
D	D	B

$\times 1$	$\times 2$	$\times 4$
A	B	A
B	C	C
C	A	B

Plurality – $\mathcal{W}_{\text{plurality}} = \{D\}$ because D has the most first-place votes, with 3.

PWC – Let's do the pairwise consolidations. A beats B 5–2; A beats C 5–2; A beats D 4–3;
...and we can stop there, because A beats all other candidates head-to-head, which is the best you can do in a PWC election. $\mathcal{W}_{\text{PWC}} = \{A\}$.

Runoff – The top two first-place vote-getters are D and then either A or B. If we use an alphabetical tiebreaker, say, then A and D face off, and A wins. With a different tie-breaker, it could have been B.

Elimination – C has the fewest first-place votes, so we consolidate to $\{A, B, D\}$. (See above.)
Now either A or B must be eliminated. If we keep using the alphabetical tiebreaker, it saves A's bacon, and we eliminate B. Then A beats D, so $\mathcal{W}_{\text{elim}} = \{A\}$. But if we had a different tiebreaker, B could have won this one!

Sequential – we can save ourselves some time here and not even do any calculations! Since A beats everyone head-to-head, it won't matter where they come in the sequence. They will beat all competitors and win. $\mathcal{W}_{\text{sequential}} = \{A\}$.

Coombs – this is the one where we drop the candidate with the most last-place votes, which is D. So we consolidate to $\{A, B, C\}$. (See above.)
Now B has the most last-place votes, so we are down to A vs. C head-to-head, and A wins.

Secondality – C and A have the most second-place votes, with 3 apiece, so it's all down to the tie-breaker and either one could win.

Summary: Most of these methods lead to electing A, but any of the candidates can win under a favorable set of rules and tie-breaking procedure!

Problem 2. Suppose an election has 15 voters and 3 candidates. Show that for *any* preference schedule, the sum of the Borda scores for all the candidates must be 90. What about for 18 voters and 4 candidates? **Challenge: Come up with a formula for the sum of Borda scores when there are N voters and n candidates.**

For each of the 15 voters, their first choice gets 3 points, their second choice gets 2, and their last choice gets 1. That means each voter's ballot is good for a total of 6 points distributed among the candidates. So the total is $15 \cdot 6 = 90$ points.

Challenge: How does this generalize? For each of N voters, the ballot is good for

$$n + (n-1) + \cdots + 2 + 1$$

points.

So the total number of points in a general Borda election with our simple point scheme is $N\,(n + \cdots + 1)$.

It happens that there's a nice formula for that sum, which is $n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$, making the Borda total $N \cdot \frac{n(n+1)}{2}$ if you're looking for a neat expression.

Problem 3. Show that the total number of pairwise comparison points in any election involving 4 candidates is 6. What is the total number of points for 5 candidates? Your solution should also work for when there are ties in head-to-head competitions. **Challenge: Come up with a formula for the total number of pairwise comparison points in an election with n candidates.**

If four candidates are named A, B, C, D, then there are six comparisons: A vs B, A vs C, A vs D, B vs C, B vs D, and C vs. D.

That is, there are four people, so A has to face 3 people, then B has 2 new opponents, then C has one. So the total number of head-to-heads is $3 + 2 + 1 = 6$.

This is the number of points in the PWC election, because every head-to-head either gives a point to its winner or gives a half-point to both in the case of a tie.

So now we can do the same thing for a five-candidate race: the number of head-to-heads is $4 + 3 + 2 + 1 = 10$, so that's the total number of pairwise comparison points.

Challenge: In general, we have

$$(n-1) + (n-2) + \cdots + 2 + 1$$

points overall, and if we write that up succinctly we get $\frac{n(n-1)}{2}$.