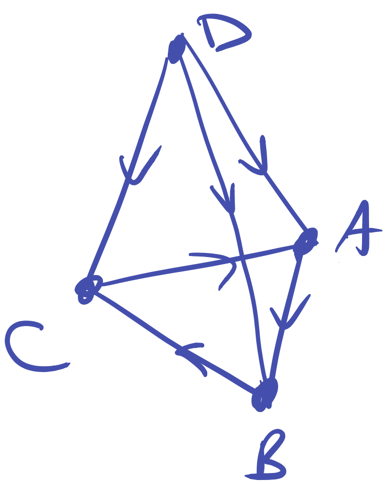
Worksheet 2
Mathematics of Social Choice
Duchin, Spring 2021

Problem 1. Come up with a preference schedule that has a Condorcet candidate but whose pairwise comparison graph contains a Condorcet cycle. (Start with a graph but be sure you build a corresponding preference schedule.)



×100	×100	×102
D	D	D
A	B	C
B	C	A
C	A	B

this part is cyclical

N=302 voters

n=4 candidates

D beats everyone unanimously!

but A, B, C form a cycle.

Problem 2. (a) Explain why no election can have more than one Condorcet candidate. (b) Explain why a majority candidate is always a Condorcet candidate.

(a) A Condorcet candidate, by definition, beats everyone head to head. If A and B are both Condorcet, we get a contradiction (A beats B head to head but B beats A head to head!) ⚡

(b) A majority candidate has more than N/2 first-place votes — this can only go up when you consolidate the preferences by eliminating other candidates! So they win any head-to-head consolidation — the def. of Condorcet candidate!

Problem 3. Which of these fairness criteria implies the other? First mark each implication as true or false. Then make a Venn diagram with bubbles for all three of these fairness conditions.

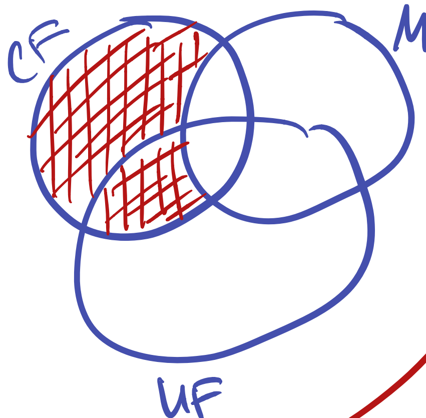
1. Condorcet-fair $\overset{?}{\Longrightarrow}$ majority-fair True
2. Unanimity-fair $\overset{?}{\Longrightarrow}$ majority fair False
3. Majority-fair $\overset{?}{\Longrightarrow}$ unanimity-fair False

see p3

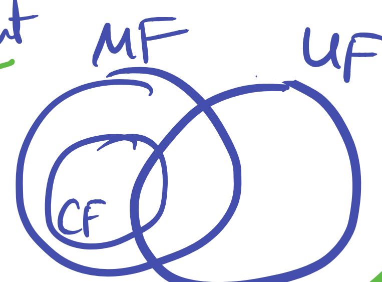
Suppose a system is Condorcet-fair. Then any Condorcet candidate will win (★)
Now suppose there's a majority candidate. They're also Condorcet by 2(b). So they win by ★!

This means everything CF is MF.

Strategy 1: use # shading to mark a Venn region that's impossible



Strategy 2: use bubble placement to communicate relationship

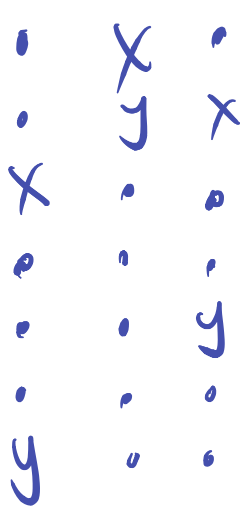


Problem 4. Explain why Borda count satisfies the unanimity criterion.

Suppose some election is conducted and everyone prefers $X > Y$. I must show $\underline{Y \not\in W}$.

This means X is ranked above Y on every ballot! So X gets more Borda points than Y from each voter, which means they accumulate at least N (the number of voters) more points overall.

So Y can't win! $\square$



Problem 3, continued.

Claim: not all UF systems are MF.

Proof: I need an example that's UF but not MF!
How about Borda. It is UF (Problem 4).

	x51	x49
8	A	B
7	B	C
6	C	D
5	D	E
4	E	F
3	F	G
2	G	H
1	H	A

A has $8 \cdot 51 + 49 = 457$ pts
B has $7 \cdot 51 + 8 \cdot 49 = 749$ pts $\Rightarrow W_{Borda} = \{B\}$.

So this example shows it's not MF
(the majority candidate loses!).

Claim: not all MF systems are UF.

Proof: Let's make up a weird system — if there's a majority candidate, they win! If not, the winner is the candidate with the most last-place votes. Call it System S. It's majority-fair by design!

But when faced with this election

x10	x10
A	B
B	A
C	C

it gives $W_S = \{C\}$, so it violates the unanimity criterion.