

Worksheet 7
Mathematics of Social Choice
Duchin, Spring 2021



Problem 1. Here's an initial ballot in which V is a Pareto candidate (consensus first choice).

#1	#2	#3	#4	#5
V	V	V	V	V
O	O	O	O	O
T	T	T	T	T
E	E	E	E	E
R	R	R	R	R

For any Pareto-efficient voting method of your choice, you can follow the strategy from the impossibility proof in this week's lecture and move R into first place column by column. The winner can change at each step. By the end of this process, R is a Pareto candidate. That means there has to be some column k that makes R the winner for the first time.

What is k for (a) Plurality with alphabetical tiebreaker? (b) Beatpath with alphabetical tiebreaker? (c) Borda with alphabetical tiebreaker?

Problem 2. Consider this preference schedule:

$\times 3$	$\times 2$	$\times 3$		$\times 3$	$\times 1$	$\times 3$	$\times 1$
C	B	A	\mapsto	C	B	A	A
B	A	C		B	A	C	B
A	C	B		A	C	B	C

Use this to show that Borda count is not strategy-proof. Make sure you explain which voter was being “strategic” and how.

Problem 3. The impossibility theorems are stated on the next page.

(a) Using the theorems, explain why the Sequential system (with earlier-in-order tiebreakers) must be vulnerable to strategic voting.

(b) Consider a Sequential (O, C, S, M) election with the preference schedule below. Who wins?

$\times 8$	$\times 12$	$\times 10$	$\times 10$	$\times 4$
M	C	O	S	S
O	S	M	M	O
S	O	C	C	C
C	M	S	O	M

(c) Suppose that you are one of the voters in the first column. What would the outcome be if you voted the *opposite* of your true preferences instead? (That is, you reverse your $MOSC$ ballot to a $CSOM$ ballot.) Is this a successful strategic vote?

(d) Was that change ($MOSC \rightarrow CSOM$) a move favorable to candidate C ? Why or why not?

The Impossibility Theorems

MÜLLER-SATTERTHWAITE THEOREM.

The only single-winner system that is Pareto-efficient and strongly monotonic is Dictatorship.

GIBBARD-SATTERTHWAITE THEOREM.

The only single-winner system that is Pareto-efficient and strategy-proof is Dictatorship.

ARROW'S IMPOSSIBILITY THEOREM.

The only ranking system that is unanimity-fair and where *the relative ranking of two candidates depends only on their two-way comparison by each voter* is Dictatorship.