Worksheet 4
Mathematics of Social Choice
Duchin, Spring 2021

Problem 1. Given any preference schedule, you can find the Smith set $\mathcal{S}$ directly from those election results, no matter the winner selection method. A dominating set is *subset of candidates with all arrows pointing out*. The Smith set is the smallest domset. From now on, we will call a candidate "strong" if they are in the Smith set.

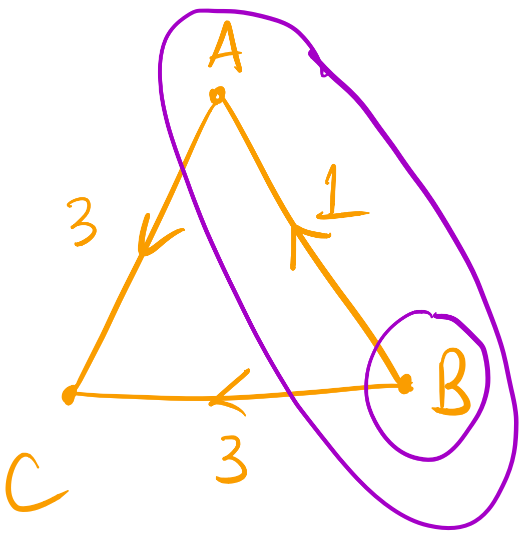
For the following preference schedule, find $\mathcal{D}_A, \mathcal{D}_B, \mathcal{D}_C$ and $\mathcal{S}$.

$\times 3$	$\times 2$	$\times 2$	$\times 1$	$\times 1$
A	B	C	A	B
B	A	B	C	C
C	C	A	B	A

×4 ×5
A B
B A

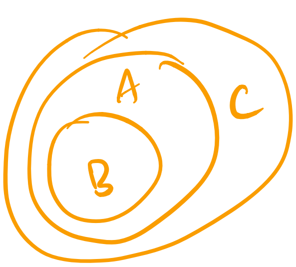
×6 ×3
A C
C A

×6 ×3
B C
C B



Condorcet candidate so $S=\{B\}$

but also, A and B dominate C.



$\mathcal{D}_A=\{A,B\}$

$\mathcal{D}_B=\{B\}$

$\mathcal{D}_C=\{A,B,C\}$

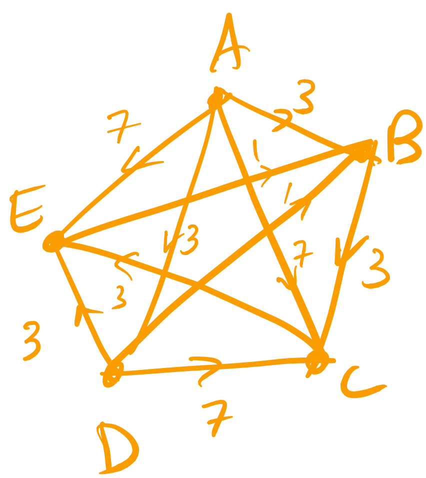
Problem 2. Suppose $|\mathcal{S}| = 1$. What can you say about the strong candidate in that case?

if $\mathcal{S} = \{X\}$ (just one element),
then X must beat all others, so X is a Condorcet candidate!

In fact, $|\mathcal{S}| = 1 \iff$ there is a Condorcet candidate.

Problem 3. The winner of a sequential tournament, no matter what order it was done in, is always in the Smith set. For the preference schedule below, make the PWC graph and show work for a sequential tournament to find a strong candidate. Draw a Venn-style diagram showing the candidate strength tiers in $\mathcal{C}$.

×3	×2	×2
A	A	D
B	E	A
D	D	C
C	B	E
E	C	B



$\mathcal{S} = \{A\}$

I asked "who didn't they beat?"

$\mathcal{D}_B = \{B, A, E, D, C\}$

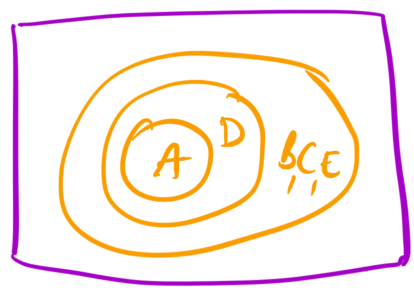
$\mathcal{D}_C = \{C, B, A, E, D\}$

$\mathcal{D}_D = \{D, A\}$

$\mathcal{D}_E = \{E, A, D, C, B\}$

Sequential: I'll choose the order EBCAD.

E vs B
↳ B vs C
↳ C vs A
↳ A vs D
↳ $\mathcal{W}_{seq} = \{A\}$
EBCAD



(OK this wasn't really necessary if I'd noticed that A is Condorcet!)