Worksheet 6

Mathematics of Social Choice
Duchin, Spring 2021

First we'll define two kinds of changes (or "moves") on preference schedules. A move *favorable to X* is one in which some voters change their preferences so that X is raised, while the relative order of the others stays the same. (Example: FOXY → FXOY or XFOY.) Let's say a move *neutral to X* is one in which every voter keeps X in the same position in their preferences, switching others around but never moving them past X, so the ones above stay above and the ones below stay below. (Example: the only move of FOXY that is neutral to X changes it to OFXY.)

A voting system is called *monotonic* if it can never happen that some moves favorable to X switch them from a winner to a loser. *Strongly monotonic* means that it can never happen that some combination of moves favorable to and neutral to X switch them from a winner to a loser. And *Pareto efficient* means that if every single voter ranks a candidate first, that candidate will be the (only) winner.

Problem 1. Explain why ① Borda count and ② plurality are monotonic.
③ Elimination is not monotonic. Discuss what kind of preference schedule would be needed to show this, and try to find one!

① Borda: A move favorable to X can only move X up, so their Borda points can only go up. Meanwhile, they may displace some other candidates, so others' Borda points can only stay the same or go down. Therefore if nobody had more Borda points before ($X \in W$), then certainly nobody can have more after ($X \in W'$). So Borda is monotonic. □

② Plurality: moving X up may sometimes move them into first place on some ballots. If so, it displaces someone else, so others' first-place votes can only go down.
Same as before! If X was at least tied for most first-place votes before ($X \in W$) then that's still true after ($X \in W'$)! So plurality is monotonic. □

[③ see last page]

Problem 2. Show that Dictatorship is Pareto-efficient, strongly monotonic, and always has just one winner. (Even though this sounds pretty basic, we are about to prove that it is the ONLY system—out of all the infinitely many systems in the world—that has all three of these properties.)

Recall dictatorship of voter #k means that $W = \{\text{voter k's first choice}\}$.
So by def, it has a single winner. ✓

[Pareto efficiency]: suppose there's a total consensus and some candidate X is everyone's top choice. That includes the dictator! So $W = \{X\}$, which confirms Pareto efficiency. ✓

[strongly monotonic]: Suppose $W_{\text{dictatorship of } k^{th} \text{ voter}} = \{X\}$.

That means voter #k ranks X at the top of their ballot.
FAVORABLE moves can only move X up, so X stays at the top of k's ballot.
NEUTRAL moves leave X in place on each ballot, so X stays at the top of k's ballot.

No matter how many of these moves you make, therefore, $W'_{\text{dictatorship of } \#k} = \{X\}$. ✓

Problem 3. So far, the (somewhat reasonable) basic voting systems we have studied are plurality, Borda, runoff, elimination, Coombs, pairwise comparison, and beatpath. (Then there are also Smithified versions of those, which you can think of as upgrades to get rid of weak spoilers.) We can fix a candidate sequence and use sequential order to break ties, making them all into single-winner systems. For instance, if the chosen sequence is ALBERT, then a tie between L and T would be resolved in favor of L (since they came first in the sequence).

Plurality, Borda, and pairwise comparison all work by essentially awarding each candidate a number of points—you can use these points to rank *all* the candidates. For the preference schedule below, first name a sequential order, then give the voters' collective ranking of the candidates using these three systems.

Bonus: come up with ideas for deriving rankings from some of the other systems as well, and explain your work. (There are many reasonable ways to do this!)

$n=4$
$N=24$

×9	×6	×2	×7
A	C	C	B
C	B	A	A
B	A	D	D
D	D	B	C

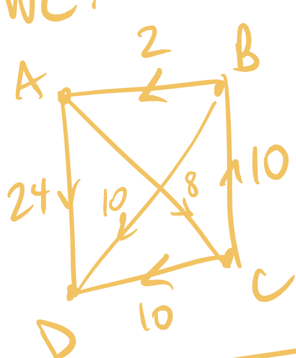
I'll fix the sequence $\boxed{C, B, A, D}$.
(for my tie breakers)

PLURALITY:
first place votes
A: 9, B: 7, C: 8, D: 0 $\rightsquigarrow \begin{pmatrix} A \\ C \\ B \\ D \end{pmatrix}$ in order of first-place votes.

BORDA:
$A: 9\cdot4 + 9\cdot3 + 6\cdot2 = 63 + 12 = 75$ pts
$B: 7\cdot4 + 6\cdot3 + 9\cdot2 + 2\cdot1 = 28 + 18 + 20 = 66$ pts
$C: 8\cdot4 + 9\cdot3 + 7\cdot1 = 32 + 27 + 7 = 66$ pts
$D: 9\cdot2 + 15\cdot1 = 33$ pts

Tie: but C comes first in sequence $\rightsquigarrow \begin{pmatrix} A \\ C \\ B \\ D \end{pmatrix}$

PWC:



H2H wins
A: 2
B: 2
C: 2
D: 0

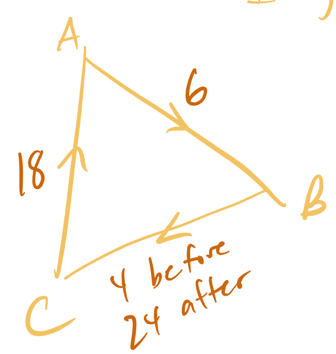
tie - but C before B before A in sequence $\rightsquigarrow \begin{pmatrix} C \\ B \\ A \\ D \end{pmatrix}$

BACK to PROBLEM 1

(3) Elimination is not monotonic.

There are lots of examples, but they all work by changing which two are the last candidates standing.

I just messed around till I found one.



let's arrange so it changes from B vs C to B vs A via move favorable to B.

($N=78$, $n=3$)

x15	x7	x18	x8	x10	x20
A	A	B	B	C	C
B	C	C	A	B	A
C	B	A	C	A	B

x10
B
C
A

let's have these 10 voters bump B to the top!

original:

A: 22 first-place votes
B: 26
C: 30

B vs C

now, first-place votes

A: 22
B: 36
C: 20

so it's A vs B

$W_{elim}=\{B\}$

$W'_{elim}=\{A\}$