Worksheet 9
Mathematics of Social Choice
Duchin, Spring 2021

Problem 1. A classic compactness metric is the *Polsby-Popper score* of a planar region $\mathbf{R}$, defined as the ratio $\mathsf{PoPo}(\mathbf{R}) = 4\pi A/P^2$, where P is the perimeter and A is the area of the region. The idea is that this score depends only on shape and not on size. We'll explore that here.

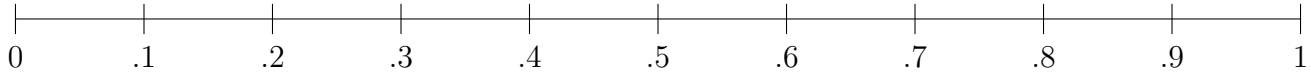
(a) Verify that the Polsby-Popper score of a circle of radius 10 is the same a for a circle of radius 3. Going further, verify that PoPo of a circle of radius r does not depend on r.

(b) Verify that PoPo of a square with a side of length s does not depend on s. Going further, suppose that a rectangle has length ℓ and width w. Show that the PoPo score of the rectangle only depends on the ratio $x = \ell/w$. (That is, come up with a formula for PoPo of a rectangle that only depends on x but not on ℓ and w individually.)

Problem 2. For any region $\mathbf{R}$, let P be its perimeter and A be its area. Derive a formula for *the ratio of the area of a region* $\mathbf{R}$ *to the area of a circle with the same perimeter*. (Your answer should depend only on P and A and constants.) Compare this formula to the formula for PoPo.

Problem 3. Let **H** be a regular hexagon, and let **H**′ be a hexagon with vertices $(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)$. Let **O** a regular octagon and **O**′ an octagon with vertices $(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2),(1,-2),(2,-1)$. Sketch these shapes and find their PoPo scores. (We've included some grid lines to help with some of those shapes.)

At the bottom of the page, place all the regions we've considered on the PoPo scale. Make some observations and conjectures about which polygons are the most "compact."



PoPo scale

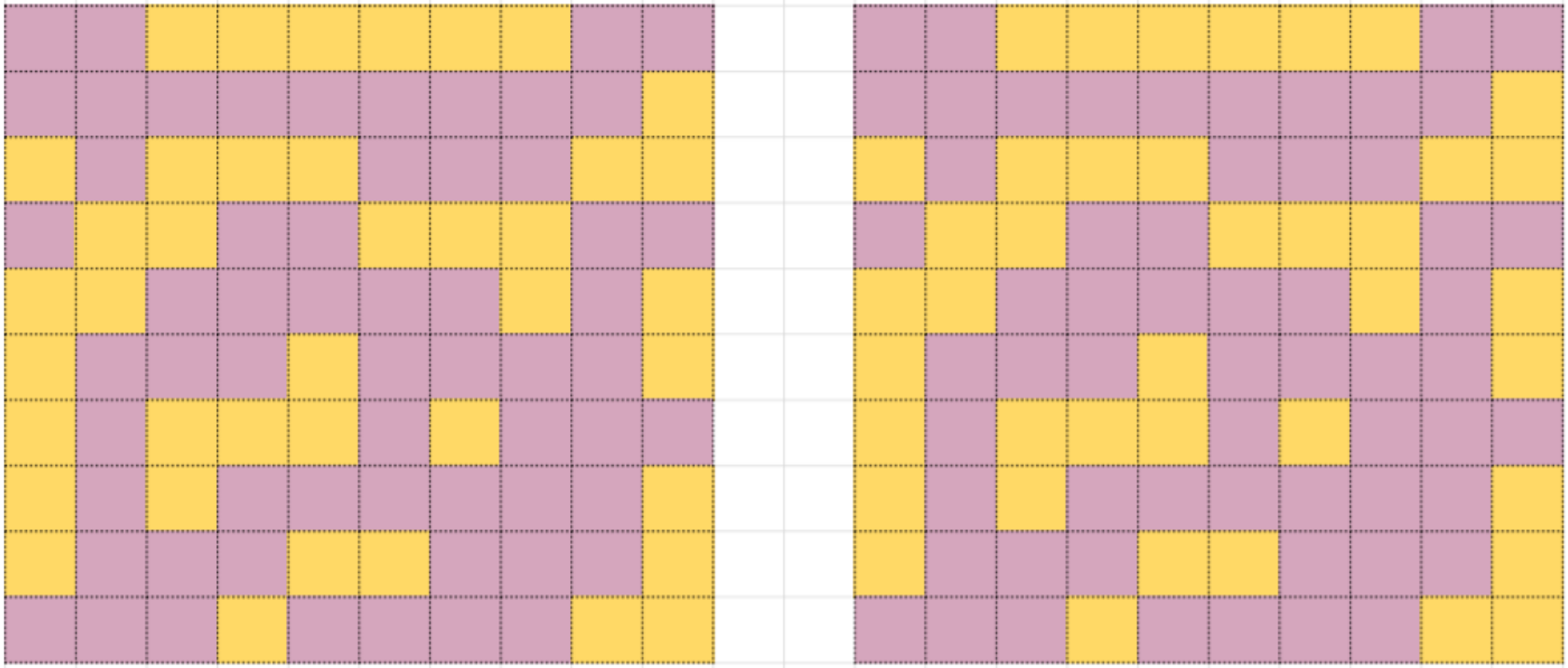
Problem 4. You have a 10×10 grid with 40 orange squares (lighter gray on printout) and 60 pink (darker gray). You want to divide it into 10 districts.

Here are some redistricting agendas you might adopt:
(1) proportional representation (4 orange seats), as compact as possible
(2) max orange representation (6 orange seats), as compact as possible
(3) competitiveness (seek districts that are 6-4 or 5-5), as compact as possible
(4) safe seats (seek 8-2, 9-1, 10-0), as compact as possible
(5) simply as compact as possible

Your assignment: try to advance each agenda as much as possible while keeping good compactness scores. For each of the five agendas, score your plan on two compactness metrics.
(A) PoPo: $4\pi A/P^2$ for each district, averaged over the districts in the plan.
(B) *cut edges*: the number of pairs of neighboring tiles that lie in *different* districts.
We'll compile the most extreme results from the whole class to investigate the efficacy of compactness metrics at detecting gerrymandering.

Gerrymandering in Squaretopia

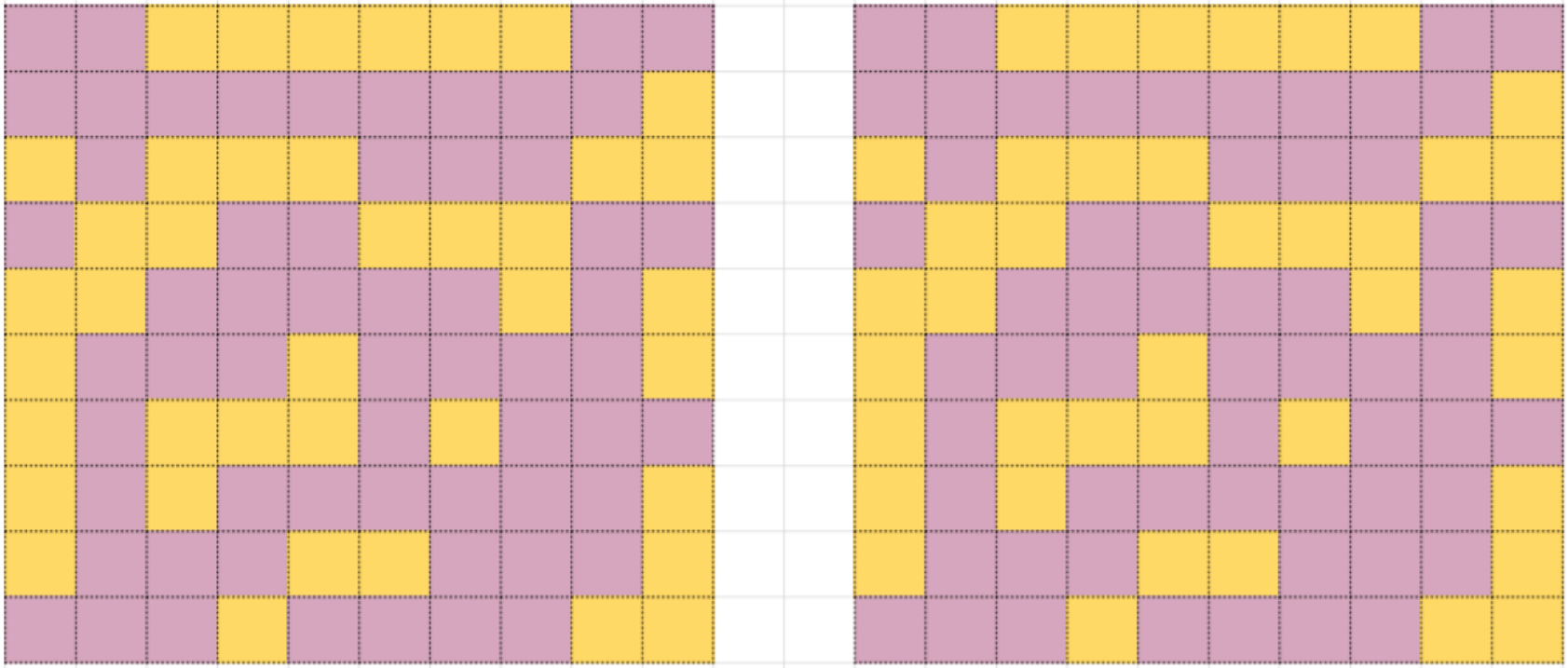
This map consists of 100 squares, 40% of which are orange.



Divide each map into ten contiguous "compact" districts of equal size such that the Orange Party wins (A) 4 districts (B) 6 districts.

Gerrymandering in Squaretopia

This map consists of 100 squares, 40% of which are orange.



Divide each map into ten contiguous "compact" districts of equal size such that the Orange Party wins (A) 4 districts (B) 6 districts.

Extra Credit

The original gerrymander! Right here in Massachusetts. This is a famous political cartoon from 1812 objecting to the shape of the South Essex district in the MA legislature, designed to favor Governor Gerry's favored candidates. Estimate its compactness score and explain how you do so.

Do the same for Pennsylvania's 7th district.