Partisan Metrics
Mathematics of Social Choice
Duchin, Spring 2021

Seats versus votes

For decades, political scientists have been trying to develop metrics of partisan fairness—scores that indicate whether one party has received undue advantage from a districting plan. The simplest ones just take a plan and an election and spit out a numerical score. In this unit we'll explore two kinds of partisan metric: *partisan symmetry* and the *efficiency gap*.

Let's standardize some notation. Suppose that a state has districts numbered 1 through m (e.g., a state with m seats in the House of Representatives). Then an election is conducted. Simplifying assumption: we'll only pay attention to votes for the two major parties, R and D. Denote the Republican vote share in district i by V_i, so that the outcomes can be recorded with a vector of R shares

$$(V_1, V_2, \ldots, V_m)\,.$$

Then we can make some auxiliary definitions.

- $\overline{V} = \frac{1}{m}\sum_{i=1}^{m} V_i$ is the mean R vote share;
- $V_{\rm med}$ is the median R vote share—this is the middle value of the V_i, if you listed them in order;
- The number of districts with R vote shares above average is denoted $m_{\uparrow}$.

Let's also write $\overline{S}$ for the Republican seat share, which is their fraction of the m House seats. All of these numbers range from 0 to 1.

Many political scientists use the hypothesis of *uniform partisan swing* to study elections. That means that you consider what would happen if you add or subtract the same amount from the vote share in each district. You can generate the *seats-votes curve* (or SV curve for short) by varying the amount you adjust the votes by and seeing how that changes S. For instance, if you had three districts and the shares in an election were $(.48, .53, .7)$, then Republicans would win two seats. A uniform swing towards Republicans of five points would give new shares of $(.53, .58, .75)$, and Republicans would win all three seats; if the original election swung five points in the Democratic direction, we'd have $(.43, .48, .65)$, and now Republicans would just win one seat out of three.

The figure on the next page shows a real example from New Mexico 2016, where the dot is the actual election outcome and the blue curve comes from swinging the votes up and down.

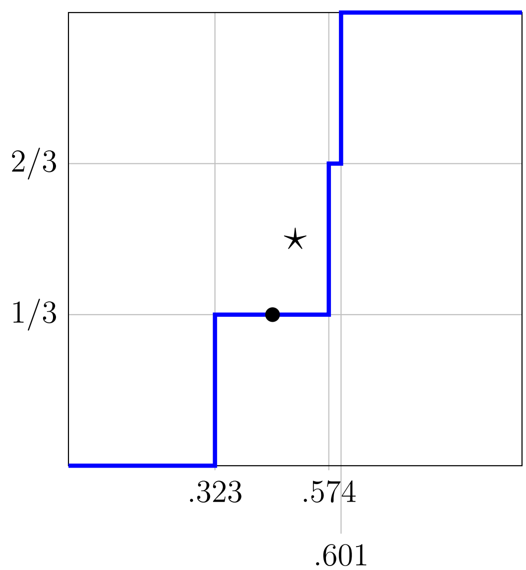
Partisan symmetry

The idea of partisan symmetry scores is very appealing: the number of seats one party got with its vote share should equal the number the other party *would* have gotten if it had the same vote share. This means if you take a seats versus votes curve and flip it over the center, it should stay the same!

Two ways of seeing how far the curve is from the center are the **mean-median score** and the **partisan bias score**. The mean-median score MM of an election is defined as the horizontal distance of the SV curve from the center point $\star = (.5, .5)$. The partisan bias score PB is the vertical distance of that curve from the center point $\star$.

Let's do an example. In 2016, New Mexico's congressional results were $(.349, .627, .376)$, from a Republican point of view. That means Rs got a vote share of $\overline{V} = (.349 + .627 + .376)/3 \approx .45$ and a seat share of one out of three seats, or $\overline{S} = 1/3$. The heavy dot in the plot marks the outcome of this election: $(\overline{V}, \overline{S}) = (.45, .33)$.

Here is the seats-votes curve generated by uniform partisan swing:



So now we can see that this isn't perfectly symmetric about the center point $(.5, .5)$ (which is marked with a $\star$), because in particular the curve doesn't go through that point!

Let's compute MM and PB. In this case, the horizontal distance goes from the star to the curve, which passes it at the value $V = .574$. So the distance from $.5$ to $.574$ gives us a mean-median score of $MM = .074$. This means that Republicans would have needed 57.4% of the vote in order to get half of the representation, which is a gap of 7.4 percentage points in favor of Democrats.

How about the vertical distance? Easy: the center point is at height $\frac{1}{2}$ and the curve passes below it at height $\frac{1}{3}$, so the difference is $PB = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \approx .167$. The interpretation is that with half of the vote, Rs would be receiving only $\frac{1}{3}$ of the representation, and PB records that gap.

Theorem 1. *Recall that* $\overline{V}$ *denotes the mean R vote share,* V_{med} *denotes the median, and* $m_{\uparrow}$ *denotes the number of seats whose R vote share is greater than average. Under uniform partisan swing, the* SV *curve goes through the points*

$$\left(\frac{1}{2}+\overline{V}-V_{\text{med}}\ ,\quad \frac{1}{2}\right) \qquad \text{and} \qquad \left(\frac{1}{2}\ ,\ \frac{m_{\uparrow}}{m}\right).$$

Proof. The middle seat for Republicans is the one with vote share V_{med}. We want to find the level at which the control of that seat flips one way or the other. So we have to add just enough to move that vote share to $1/2$. To move V_{med} to $1/2$, we must add $1/2 - V_{\text{med}}$. Since we're assuming uniform partisan swing, we add that same amount to all districts, and the statewide vote average shifts by the same number of percentage points. Since the statewide vote share was $\overline{V}$, it becomes $V = \overline{V} + \frac{1}{2} - V_{\text{med}}$ after shifting, which will cause us to arrive at split control $S = \frac{1}{2}$. That means that $\left(\frac{1}{2}+\overline{V}-V_{\text{med}}\ ,\ \frac{1}{2}\right)$ is on the curve.

For the other part, we ask ourselves how both parties would fare if each one got half of the vote. Since Republicans initially got $\overline{V}$ share of the vote, we need to shift that by adding $\frac{1}{2}-\overline{V}$ in all districts. This produces vote levels of

$$(V_1 + \frac{1}{2} - \overline{V}\ ,\quad \ldots\quad ,\ V_m + \frac{1}{2} - \overline{V}).$$

How many of those numbers are greater than $\frac{1}{2}$? The answer to that tells us how many R seats will be produced when $V = \frac{1}{2}$. But note that $V_i + \frac{1}{2} - \overline{V} \geq \frac{1}{2}$ exactly if $V_i \geq \overline{V}$. So we simply count how many districts have vote share greater than $\overline{V}$, and that is $m_{\uparrow}$, by definition. This produces a seat share of $S = m_{\uparrow}/m$, as desired. We've now shown that $\left(\frac{1}{2}\ ,\ \frac{m_{\uparrow}}{m}\right)$ is on the curve. □

Corollary 1. *Unless the* SV *curve has a step at* $S = 1/2$ *or* $V = 1/2$*, we have*

$$|MM| = \overline{V} - V_{\text{med}} \qquad \text{and} \qquad |PB| = \frac{m_{\uparrow}}{m} - \frac{1}{2}.$$

Efficiency gap

Let's say that a *wasted vote* is any winning vote in excess of half the votes in a district, or any losing vote at all. Then let's let W^R be the total votes wasted by Republicans statewide, and W^D be the votes wasted by Democrats. Then the efficiency gap is defined as $EG = \frac{W^R - W^D}{Tot}$, which is the difference in wasted votes, divided by the total turnout of the election.

Example: let's do the New Mexico 2016 election from above, with R vote shares $(.349, .627, .376)$. For instance, suppose that each district had $T = 1000$ voters. Then we'd get the following outcomes.

i	R votes	D votes	W_i^R	W_i^D
1	349	651	349	151
2	627	373	127	373
3	376	624	376	124
total	1352	1648	852	648

so we get $EG = \dfrac{852 - 648}{3000} = .068.$

The authors of efficiency gap, Nick Stephanopoulos and Eric McGhee, proposed that any election with $|EG| > .08$ is probably gerrymandered! So this election is skewed but not officially gerrymandered (according to them). This EG value is positive, meaning that Republicans wasted more votes, so according to EG it's tilted in favor of Democrats.

Theorem 2. *Under the assumption that every district has equal turnout, simplification gives us*

$$EG = 2\overline{V} - \overline{S} - \frac{1}{2}.$$

Proof. Let's suppose every district has turnout T so that the total turnout in the election is $Tot = Tm$. Suppose Rs won k out of m districts, so that $\overline{S} = k/m$. That means that Ds won the other $m - k$ districts. In each district won by Ds, the Rs waste all their votes, which is V_iT votes, while the Ds waste their excess votes, which is $((1 - V_i) - 1/2)T$ votes. In each district won by Rs, the Ds waste all their votes, which is $(1 - V_i)T$ votes, while the Rs waste their excess votes, which is $(V_i - 1/2)T$ votes. Adding these up we get total R wastage equal to $W^R = \overline{V}Tm - \frac{k}{2}T$, and D wastage of $-\overline{V}Tm + T\frac{k}{2} + T\frac{m}{2}$.

Note that if we add these together we get $W^R + W^D = \frac{1}{2}Tm$, which is correct—that's half the total number of votes!

But the efficiency gap is computed by subtracting:

$$EG = \frac{W^R - W^D}{Tot} = \frac{2\overline{V}Tm - kT - T\frac{m}{2}}{Tm} = 2\overline{V} - \frac{k}{m} - \frac{1}{2} = 2\overline{V} - \overline{S} - \frac{1}{2}. \quad \square$$

This is kind of surprising! It means that in the end, the efficiency gap doesn't really depend on what's happening in each district; it only depends on the statewide vote share vs the statewide seat share.