Math of Social Choice Midterm Solutions (Spring 2021)

Saturday, April 3, 2021 12:25 PM

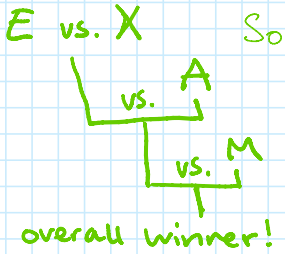
* With all the gory details *

1. Suppose you have a sequential election with the candidate sequence (E,X,A,M). Suppose E wins the election without tying. Is there a Condorcet candidate?

(A) Yes
(B) No
(C) Not enough information

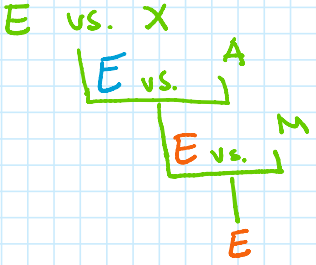
2. Explain. (optional)

Since we are asked to use the sequential election method, we use the given sequence (E,X,A,M) to run the election. The structure looks like this:



So the winner of E vs. X faces A then the winner of that H2H faces M. the winner of that H2H is the overall winner

We are told that the overall winner of our election is Candidate E (no ties.). So in order for them to be the overall winner they must have won the last head to head vs. M. we can fill in our diagram (shown in orange)



working backwards; in order for E to face M, candidate E must have won the head to head vs. A. add to diagram (shown in blue)

And in order to face A, E must have first won against X to advance to that round.

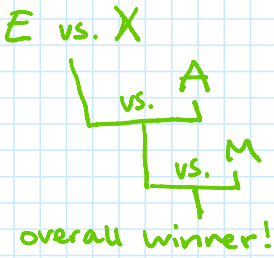
This means that for E to be the overall winner in this race, they must have won the head to head against M, A, and X. This means E must have won head to head against every other candidate (note that this is because E is first in the order.). Recall that a condorcet candidate is one that beats every other candidate head to head. Therefore E must be a condorcet candidate! □.

3. Suppose you have a sequential election with the candidate sequence (E,X,A,M). Suppose A wins the election without tying. Which of the following are definitely true? *Mark all that apply.*

- (A) A is a Condorcet candidate
- (B) A is unanimously preferred to E and X
- (C) A beats X head-to-head
- (D) A beats M head-to-head
- (E) A is a Dictator
- (F) A has at least two pairwise comparison points

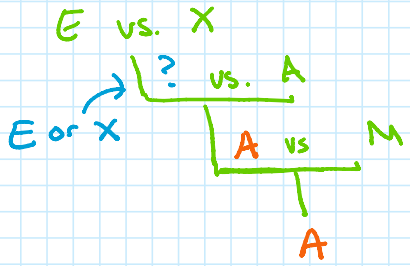
4. Explain. (optional)

We will assess whether or not each option is definitely true, but let us first observe the structure of the election. As with the previous question we have the sequential election:



This time we are given that A wins overall without tying.

So we can draw a new diagram taking into account the winner:



Similar to the previous problem we can work out that A must have faced M (and won) if they are the overall winner. and in order to have faced M they must have won the previous head to head.

However, we do not know who A faced. It could have been E or X, but we don't know which are. All we know is that A won that head to head and advanced.

So now let's look at each answer option:
(Remember we are looking for statements that are definitely true)

a) We do not know for sure if A is a condorcet candidate. In order for A to be a condorcet candidate, A would have to beat every other candidate head to head. We know for sure that A beat either X OR E. For us to be sure A is condorcet, we would have to know A beat M, X and E. We do not

we know A is either X or E.

For us to be sure A is condorcet, we would have to know A beat M, X and E. We do not know this for sure so

This answer choice is NOT definitely true. □

b) Since we do not have a preference schedule we do not know the margins of victory, we only know info who won some of the head to head, but we don't know by how much

For example, we know A beat M, but the preference schedule could have looked like this:

100x
A
M

↳ A unanimously preferred to M

or

51x	49x
A	M
M	A

↳ A barely beats M head to head.

or infinitely many other schedules!

Similarly, even if we knew A beat X and E (which we don't) there is no way to know if it was unanimous.

This answer choice is NOT definitely true. □

c) As said before, we know A beat either X or E H2H, so we do not know A beat X H2H for sure.

This answer choice is NOT definitely true □

d) Look at the diagram we made. We know A had to face M H2H and win (This is because A came before M in the sequence but is still the overall winner).

This answer IS definitely true. □

e) We are looking at a sequential election method. (which does not use a dictator to decide winner) and further, we don't even know who or how many voters there are. Note that a dictator is a voter not a candidate, just a voter with a lot of power, and in real life applications, the dictator voter's first choice candidate is often themselves. But here we classify a dictator as a voter not a candidate.

This answer is NOT definitely true. □

f) Remember a PWC point is given to a candidate for each other candidate they beat H2H.

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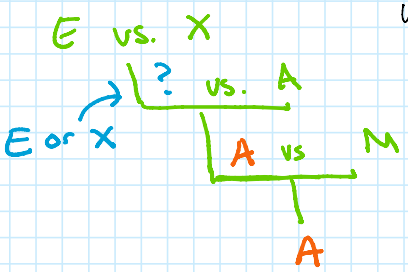
Looking back to our diagram:



We know that
A beats M → A gains 1 pt.
A beats X or A. (or both)
↳ Let's examine this further:

So no matter who wins the E vs. X H2H, A must face the winner of that race. Let's just say E won. (You could make the same argument for X.)

So then E would face A and we know A must win (see diagram). So A gains a point there.

Now we know A has at least two points so we can answer this question at this point, but let's keep looking:

So in our example when we said E beat X, we considered A vs. E as part of our sequential race. What about A vs. X? Well we only know A beat E from the seq. comp info, but we can still look at the H2H between A and X.

Either one could win. We don't know, so A could get a point from this H2H or it could not, so A can have either 2 or 3 points total (which is at least 2 points total).

this answer IS definitely true. D

5. For this election, find the primitive dominating set D_B.

(A) ∅
(B) {B}
(C) {B,D}
(D) {A,B,D}
(E) {A}
(F) {A,B,C,D}

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

Primitive dominating set D_B: smallest domset that contains candidate B.

dominating set: set where every candidate inside the set beats every candidate outside the set (all arrows point out).

First thing is to create PWC diagram (so I can see arrows)

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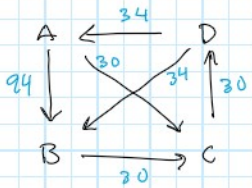
I found winner for each head to head by consolidating. I also found margins. (don't need margins for this problem but will be useful later)

- A vs. B → 94 vs. 0, margin = 94 - 0 = 0
- A vs. C → 62 vs. 32, margin = 62 - 32 = 30
- A vs. D → 30 vs. 64 margin = 64 - 30 = 34
- B vs. C → 62 vs. 32, margin = 62 - 32 = 30
- B vs. D → 30 vs. 64 margin = 64 - 30 = 34
- C vs. D → 62 vs. 32 margin = 62 - 32 = 30

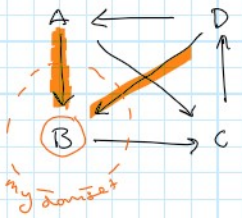
(winners highlighted)

margins = by how many votes do they win?

Then I created the resulting diagram:

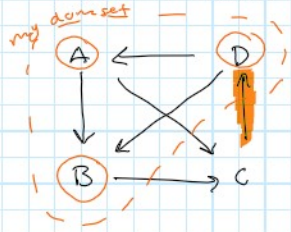


To create D_B it is smallest set including B so start by including B. I will circle every candidate I add to D_B:



So a domset means every arrow points out. So I need to add every candidate that B loses to into my domset. This means find arrows that point towards B and trace them back to candidates.

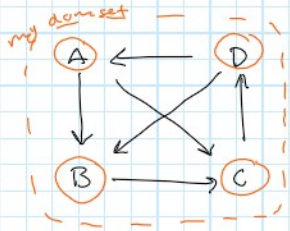
I add candidates A and D since B loses to them:



Now I ask who do my newly added candidates lose to?

- A only loses to D who is already in our set so we are good.
- D loses to C so we must add C to our domset.

I add candidate C to our domset



Now C doesn't lose to anybody outside our domset so we are done!

$D_B = \{A, B, C, D\}$

□.

6. Who wins by plurality?

(A) {A}

(B) {C,D}

(C) {C}

(D) {D}

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

Plurality winner: candidate w/ most first place votes.
Instructions say no tiebreaker unless specified.
No tiebreaker specified here.
We can read straight from schedule:
A: 30 first place votes
B: 0 first place votes
C: 32 first place votes
D: 32 first place votes
C and D tie for most first place votes:
W = {C, D}

7. Who wins by Borda? (you should be able to answer this without having to compute all the Borda points.)

(A) {A}

(B) {C,D}

(C) {D}

(D) {B,C,D}

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

8. For the Borda election: whether or not you calculated Borda points, explain how you can find the winner (or check that your answer is reasonable) without crunching the numbers. (optional)

Just by observation: Borda count gives advantage to candidates that are consistently ranked higher.
Note how A dominates the rest of the candidates by placement: the number of votes in each column is roughly equal:
A's placements are 1, 2, 3 and closest is D with placements 1, 2, 4.
To check, can do calculations:
4(30) + 3(32) + 2(32) = 4(30) + 5(32) ✓

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A: $4(30) + 3(32) + 2(32) = 4(30) + 5(32)$ ✓

B: $4(0) + 3(30) + 2(32) + 1(32) = 3(30) + 3(32)$ ×

C: $4(32) + 3(0) + 2(30) + 1(32) = 2(30) + 5(32)$ ×

D: $4(32) + 3(32) + 2(30) + 1(30) = 1(30) + 7(32) = 4 + 3(30) + 5(32)$ ×

9. Who wins by elimination?

(A) {A}

(B) {B,D}

(C) {C,D}

(D) {C}

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

Elimination method: remove cardidate with smallest # of first place votes Then consolidate and repeat.

Read from schedule:

A: 30 first place B: 0 first place C: 32 firstplace, D: 32 firstplace.

B has least so remove and consolidate schedule:

×30	×32	×32
A	C	D
C	D	A
D	A	C

A: 30 first place

C: 32 first place

D: 32 first place

A has least so remove and consolidate schedule:

×30	×32	×32
C	C	D
D	D	C

$=$

×62	×32
C	D
D	C

C: 62 first place

D: 32 first place

D has least so we remove D, left with just C

$W = \{C\}$ □.

10. Who wins by runoff?

(A) {A}

(B) {C}

(C) {A,C}

(D) {D}

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

Runoffs method: Take the two candidates with the most first place votes, consolidate schedule to just them and then choose candidate with most first place votes as winner.

From schedule:

A: 30 first place, B: 0 first place, C: 32 first place, D: 32 first place.

C and D are the two candidates with most first place votes, so we consolidate to just them:

×30	×32	×32
C	C	D
D	D	C

=

×62	×32
C	D
D	C

C has the most first place now

$W = \{C\}$ □

11. Who wins by Coombs?

(A) {A}

(B) {C}

(C) {D}

(D) {B,C}

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

Coombs Method: eliminate candidate with most last place votes then consolidate and repeat.

From schedule:

A: 0 last place, B: 32 last place, C: 32 last place, D: 30 last place

Instructions for midterm say:

One math reminder.

- If there is no tiebreaker specified, you shouldn't need to assume one on this test. It's OK to have multiple winners, or multiple people eliminated at the same time.

So since B and C tie for most last place votes, we can eliminate them both at once. Then we are left with just A and D so we consolidate to those two:

×30	×32	×32
A	D	D
D	A	A

=

×30	×64
A	D
D	A

A: 64 last place

D: 30 last place

since A has most last place we eliminate A, left with just D.

$W = \{D\}$ □.

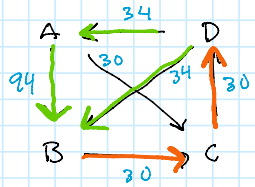
12. In the same election, is it true that D▷B? (where the symbol denotes beatpath elimination)

(T) True

(F) False

×30	×32	×32
A	C	D
B	D	A
C	A	B
D	B	C

Beatpath is based on the PWC diagram looking at margins of victory. So let's take the diagram we made in Problem #5 (since this is the same election):



The Question asks us if D ▷ B, in other words;

does D have a beatpath to B that is unmatched by any beatpaths from B to D?

Let's look:

Remember a beatpath is a path following the direction b/w two candidates (can go through other candidates to get there) and the strength of that path is the smallest margin on the path.

So let's first look at all the paths from D to B:

D → B strength = 34 (smallest, and only, # on path)

D → A → B strength = 34 (smallest # on path)

So we would choose the strongest of these to represent the strongest possible path from D to B. They are the same so we can choose either one.

Now let's look at paths from B to D:

B → C → D strength: 30 (smallest # on path)

Since there is only one path, this is the strongest path from B to D.

Since D → B strength 34 is unmatched by the strongest path from B to D:

B → D strength 30, that means

D ▷ B

□.

13. A Condorcet candidate can never be a spoiler in a Condorcet-fair system.

(T) True

(F) False

14. Explain. (optional)

- Condorcet-fair system: a voting system is condorcet fair if for any preference schedule, if a condorcet candidate exists, then that candidate must be the sole winner.
- Spoiler: is a candidate who, after being disqualified, changes the results of the election in a weird way.

We are testing to see if a condorcet candidate can be a spoiler: Since a condorcet candidate must be the sole winner in a condorcet fair election, they can't be a spoiler, because when they are removed it is unsurprising that someone new would win.

15. Consider the following election method, called "Dictatorship with Veto" or **DWV**: *The winner is the first choice of voter #1, except if that candidate is also the last choice of voter #3. In that case, go to the next choice of voter #1.*

This system is Pareto-efficient and single-winner. Which one of the following is true?

(A) Voter #1 can vote strategically.

(B) Voter #2 can vote strategically.

(C) Voter #3 can vote strategically. ← voter who vetos

(D) The system is strategy-proof, by one of the Impossibility Theorems.

16. Explain. (optional)

Let's first note that DWV is not the dictatorship that is referred to by the Impossibility Theorems. Those theorems specifically apply to "dictatorship of the k^{th} voter".

So the answer cannot be D, because since we know the system is Pareto efficient and single winner, if it was also

So the answer cannot be D, because since we know the system is Pareto efficient and single winner, if it was also strategy proof, then our system would have to be dictatorship of the k^{th} voter, which it is not.

We also know that the answer cannot be B because voter 2 has no effect on the election results, so they can't be strategic.

Now we are left with either voter 1 or voter 3 being able to be strategic.

Let's look at voter 1. The best chance they have of getting their favorite candidate is to put them first. Then if that candidate gets vetoed it moves to their second choice. Notice that voter 1 will still get one of their first two choices, so there is no way to give an advantage to their first choice over their second besides putting them in first place.

Now let's look at voter 3. Here is a strategy. Consider this preference schedule:

voter 1:	voter 3:
A	B
B	A
C	C
D	D

strategic voter 3:

B
C
D
A

if the candidate that voter 1 has in second is voter 3's first place candidate, voter 3 can move the candidate that voter 1 has to the bottom of their ballot to get their first choice elected, even if they still quite like the candidate they put on the bottom.

Before strategy: $W = \{A\}$

after strategy: $W' = \{B\}$

voter 3 likes B better than A so they succeed at strategy vote. □

17. Which of these systems are unanimity-fair? (Don't rely on memory—you can work these out in real time!) *Mark all that apply.*

- (A) Plurality
- (B) Secondality
- (C) Borda
- (D) Dictatorship

18. Explain. (optional)

Unanimity-fair: a system is unanimity-fair if X is higher than Y on every ballot, then Y can't win.

Let's check each one!

→ **Plurality**: if X has to be higher than Y on each ballot then the highest Y can be is second (need to leave room for X above). So if X > Y on each ballot, Y can't have any first place votes and can't win

So plurality is unanimity fair □

→ **Secondality**: We can have all ballots have x in first place and Y in second place. All X > Y but since all ballots have Y in second place, Y must have the most second place votes and wins! We found an example where X > Y on all ballots but Y wins, so

Secondality is NOT unanimity fair □

→ **Borda**: if X > Y on all ballots then for each ballot X will have at least one or point more than Y. Since X gains more points than Y from each voter, Y can't have ≥ points than X. So if X > Y on all ballots, then Y can't win. so

Borda is unanimity fair □

→ **Dictatorship** If X > Y on all ballots then this includes the dictators ballot. Similar to plurality argument, if X > Y the highest spot Y can be is second since we need to leave room for x above. So Y can't be at top of dictators ballot so Y can't win. So if X > Y on all ballots, Y can't win

Dictatorship is unanimity-fair □