Worksheet 10
Mathematics of Social Choice
Duchin, Spring 2021

Problem 1. In this problem, you'll be drawing a *seats-votes curve* (or "SV curve") for an election.

Go to the wikipedia page United States House of Representatives elections in Nevada, 2020 and get the results from the four congressional races.

(a) Find the Republican vote shares V_1, V_2, V_3, V_4.

	District 1	District 2	District 3	District 4
R votes	74,490	216,078	190,975	152,284
D votes	137,868	155,780	205,421	168,459
R share	0.351	0.581	0.484	0.475
	V_1	V_2	V_3	V_4

To get the vote share V_i for district i, we compute

$$\frac{\#\text{R votes}}{\#\text{R votes} + \#\text{D votes}}.$$

For instance, for district 1,

$$\text{R share} = \frac{74,490}{74,490 + 137,868} \approx 0.351.$$

Note: For simplicity, we are only using Democratic & Republican votes to compute vote shares in this problem.

(b) Find the mean Republican vote share $\overline{V}$, which is the average of these district shares V_1, V_2, V_3, V_4. Check that it is a bit different from the statewide share of Republican votes.

$$\overline{V} = \frac{V_1 + V_2 + V_3 + V_4}{4} = \frac{0.351 + 0.581 + 0.484 + 0.475}{4}$$

$$= \boxed{0.473}$$

$$\text{Statewide share} = \frac{\text{total R votes}}{\text{total votes}}$$

$$= \frac{74490 + 216078 + 190975 + 152284}{74490 + 216078 + 190975 + 152284 + 137868 + 155780 + 203421 + 168457}$$

$$= \frac{633827}{1299353}$$

$$\approx \boxed{0.488}$$

Observe that the statewide vote share is slightly higher than the average of the district vote shares.

(c) Find the Republican seat share $\overline{S}$, which is the share of seats that were won by Republican candidates in this race.

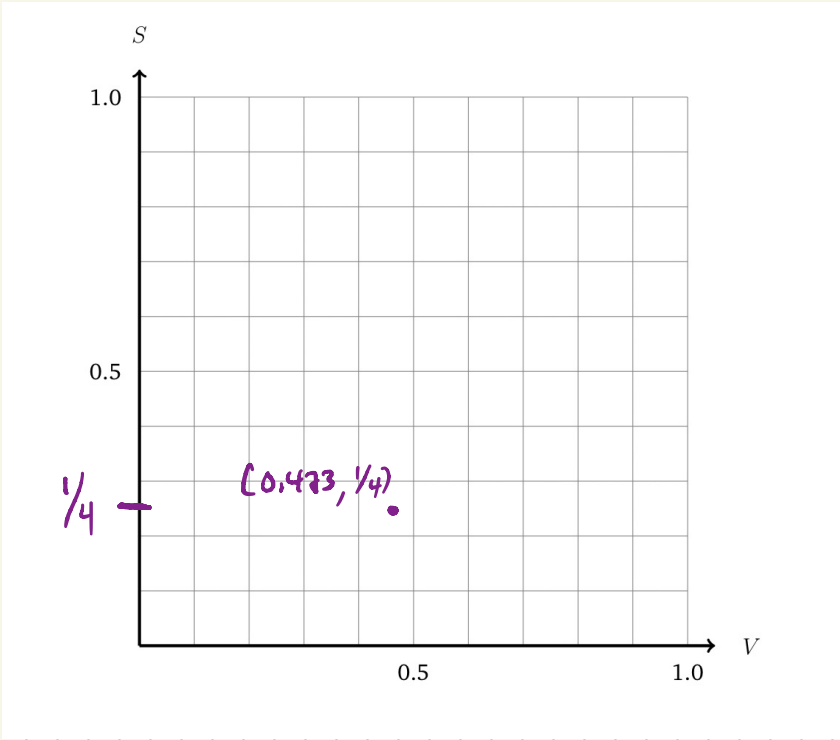
District i is won by a Republican candidate if over 50% of the votes are R votes, ie., if $V_i > 0.5$, so:

$$\overline{S} = \frac{\#R \text{ seats won}}{\#\text{seats (districts)}} = \boxed{\frac{1}{4}}$$

← district 2 satisfies $V_2 > 0.5$

(d) Now we can start drawing the SV curve. The Republican vote share will be on the x-axis, and the share of seats won by Republicans will be on the y-axis. (Note that both of these range from 0 to 1.) Using the previous two parts of this question, plot the point $(\overline{V}, \overline{S})$ on your graph.

$(\overline{V}, \overline{S}) = (0.473, 1/4)$



(e) To find the rest of the curve, we will be using the assumption of *uniform partisan swing*–that means that we consider what would happen if we add or subtract the same amount from the vote share in each district. By how much do you have to increase the Republican vote share in each district to increase the number of Republican seats by 1? Add this to your plot as a step.

Consider the quadruple of vote shares

$(V_1, V_2, V_3, V_4) = (0.351, 0.581, 0.484, 0.475)$

District 3 is closest to the 0.5 threshold for winning a seat; it is $0.5 - 0.484 = 0.016$ percentage points away. Add this margin to each of the four vote shares in the quadruple above: ⟶

$(0.351 + 0.016,\ 0.581 + 0.016,\ 0.484 + 0.016,\ 0.475 + 0.016)$

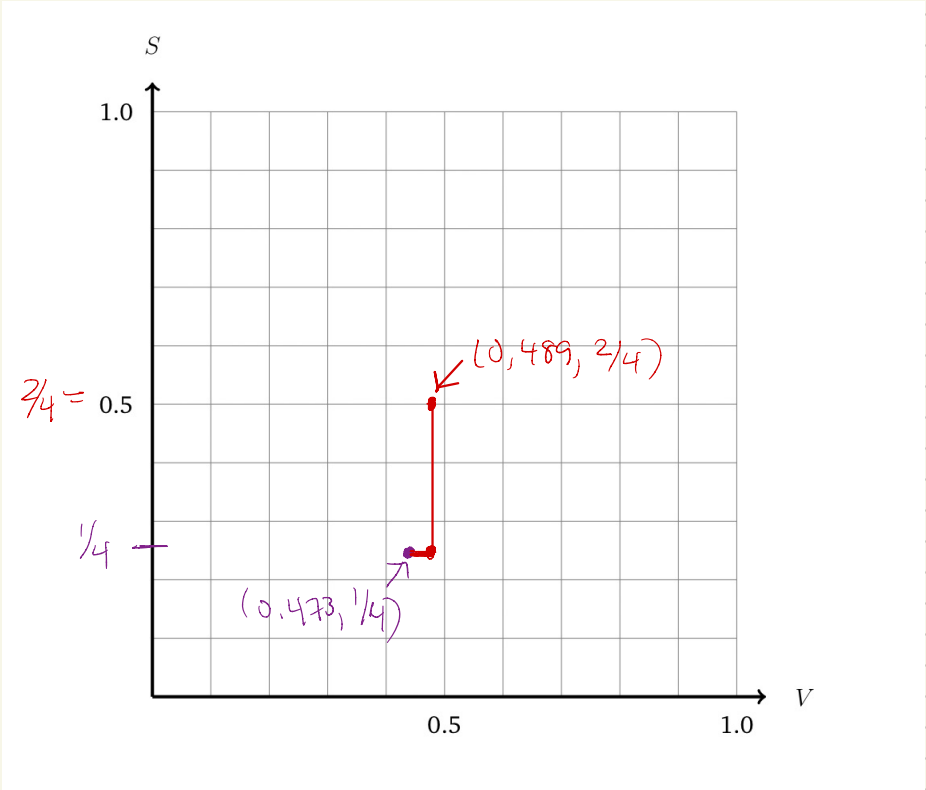
$= (0.367,\ 0.597,\ 0.5,\ 0.491)$

Consider this new quadruple. We have identified a tipping point for Republicans to move from a seat share of 1/4 to 2/4 via an added seat in district 3. $(0.489, 2/4)$

Calculate the average of the four new vote shares in this new quadruple:

$$\frac{0.367 + 0.594 + 0.5 + 0.491}{4} \approx 0.489$$

So we now plot the point $(0.489,\ 2/4)$:



Next we will take the new triple

$$(0.367, 0.597, 0.5, 0.491)$$

and identify the district closest to the 0.5 threshold. This is district 4, which is $0.5 - 0.491 = 0.009$ percentage points away. Add this margin to each of the four vote shares in the quadruple above:

$$(0.367 + 0.009,\ 0.597 + 0.009,\ 0.5 + 0.009,\ 0.491 + 0.009)$$

$$= (0.376,\ 0.606,\ 0.509,\ 0.5)$$

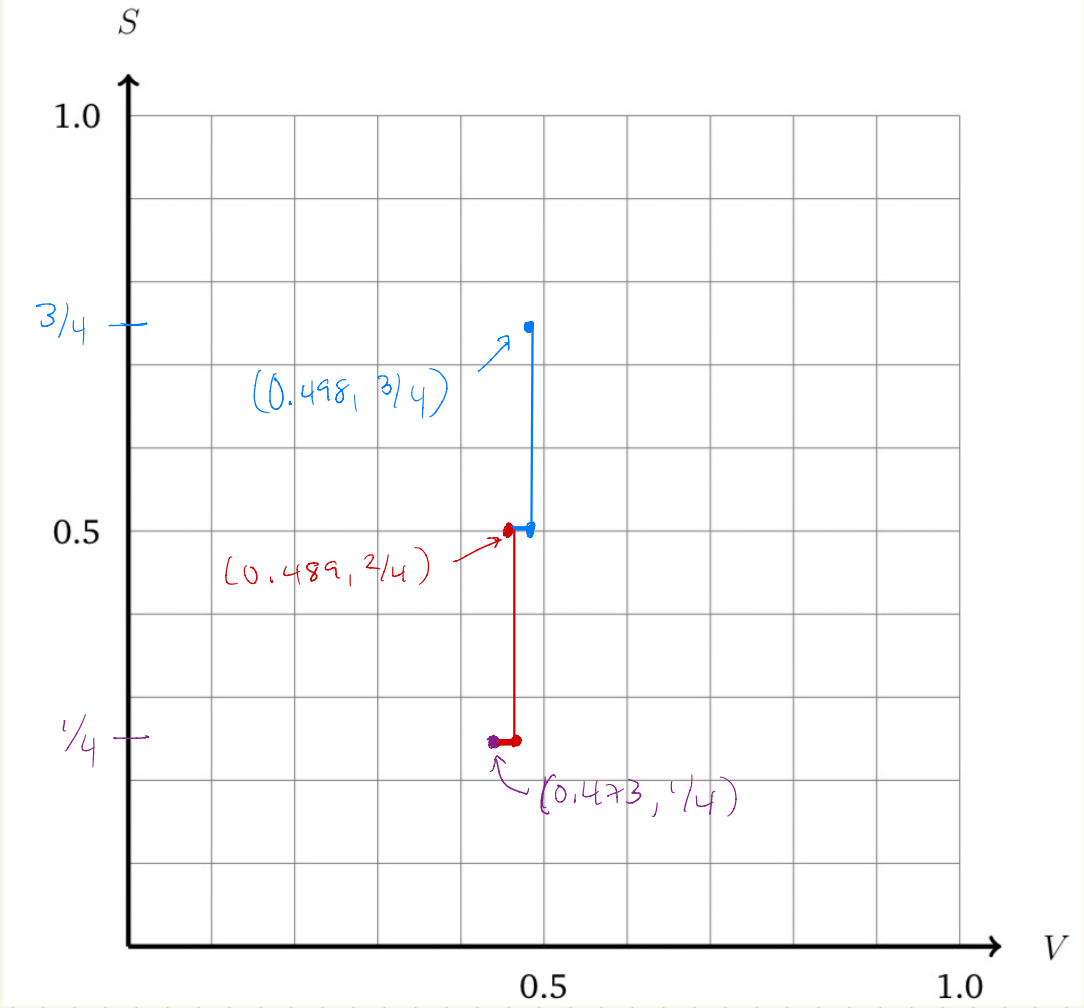
We have identified a tipping point for Republicans to move from a seat share of 2/4 to 3/4.

Calculate the average of the four new vote shares in this new quadruple:

$$\frac{0.376 + 0.6006 + 0.509 + 0.5}{4} \approx 0.498$$

So now we plot the point $(0.498, 3/4)$:

→



We take the most recent quadruple

$$(0.376, 0.606, 0.509, 0.5).$$

The first distinct is the only remaining one not to have reached the 0.5 threshold. It requires $0.5 - 0.376 = 0.124$ more percentage points. We add this amount to every coordinate in the above quadruple: $\longrightarrow$

$= (0.376 + 0.124,\ 0.606 + 0.124,\ 0.509 + 0.124,\ 0.5 + 0.124)$

$= (0.5,\ 0.730,\ 0.633,\ 0.624)$

We have identified a tipping point for Republicans to move from a seat share of $3/4$ to $4/4$.

Calculate the average of the four new vote shares in this new quadruple:

$$\frac{0.5 + 0.730 + 0.633 + 0.624}{4} \approx 0.622.$$

So now we plot the point $(0.622, 4/4)$.

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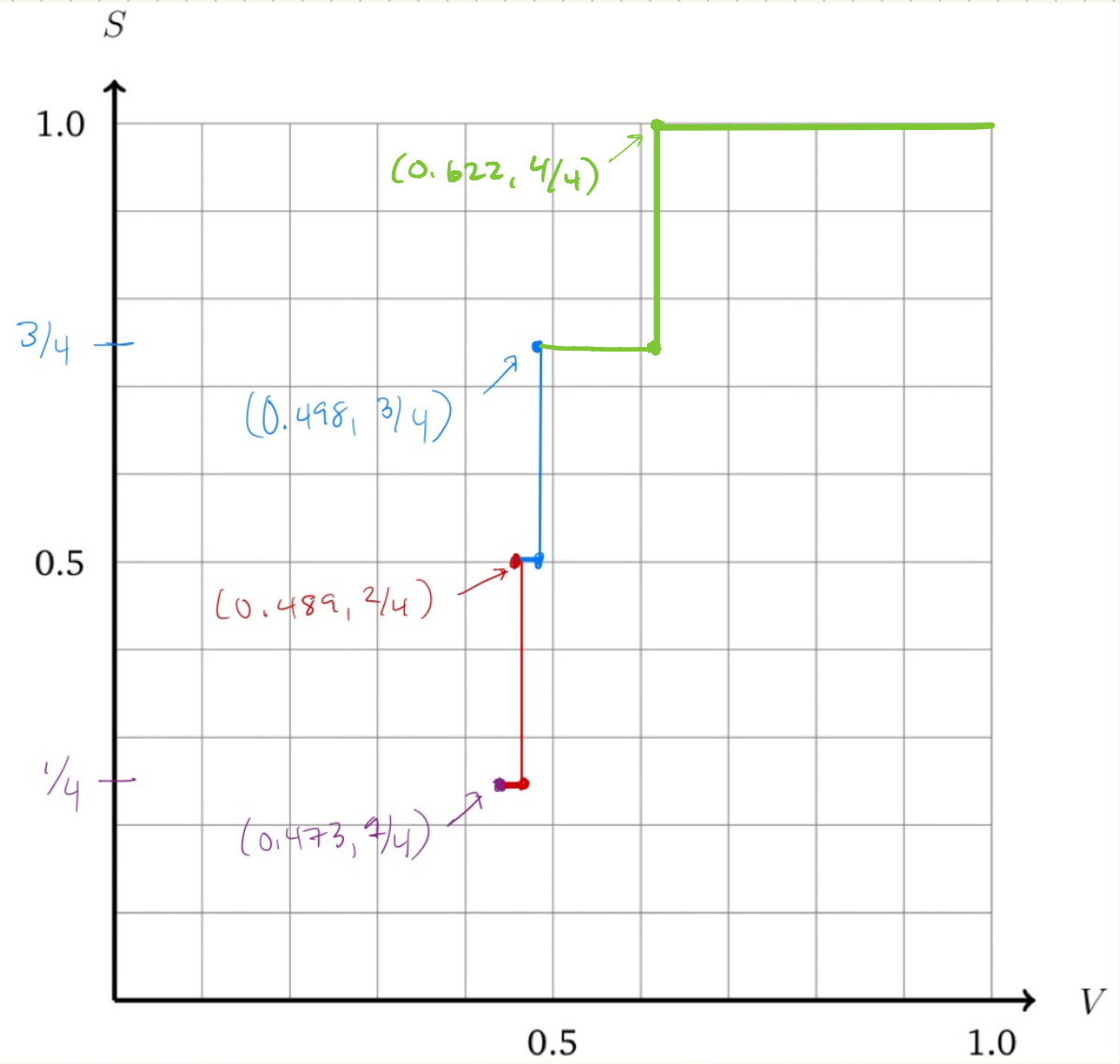

S
1.0
3/4
0.5
1/4
(0.622, 4/4)
(0.498, 3/4)
(0.489, 2/4)
(0.473, 1/4)
0.5
1.0
V

There is one more thing to do: we need to complete the side of the curve to the left of our original point $(0.473, 1/4)$.

That is, we start off with our original quadruple

$$(0.351, 0.581, 0.484, 0.475)$$

and compute the threshold for losing a seat. The only seat that is won (and thus can go from winning to losing) is that in district 2, and it must lose $0.581 - 0.5 = 0.081$ percentage points. Subtract this amount from each vote share in the original quadruple:

$$(0.351 - 0.081, 0.581 - 0.081, 0.484 - 0.081, 0.475 - 0.081)$$

$$= (0.27, 0.5, 0.403, 0.394)$$

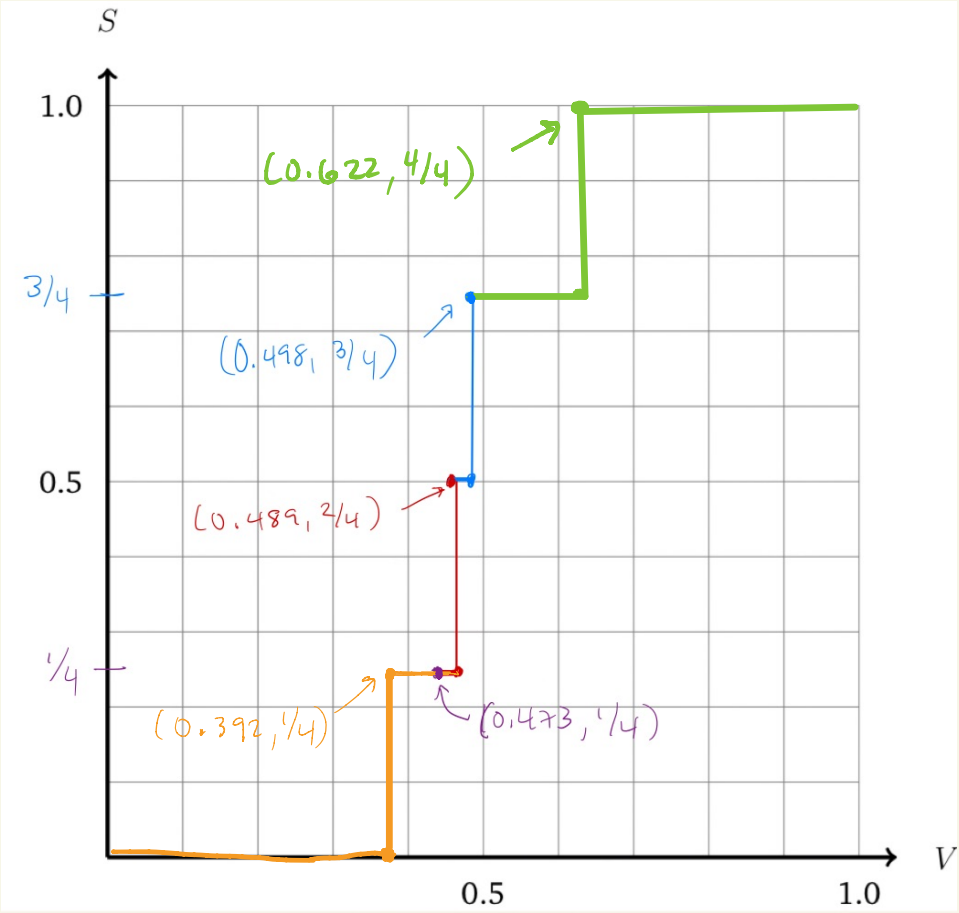
We have identified a tipping point for Republicans to move from a seat share of $1/4$ to $0/4$.

Calculate the average of the four new vote shares in this new quadruple:

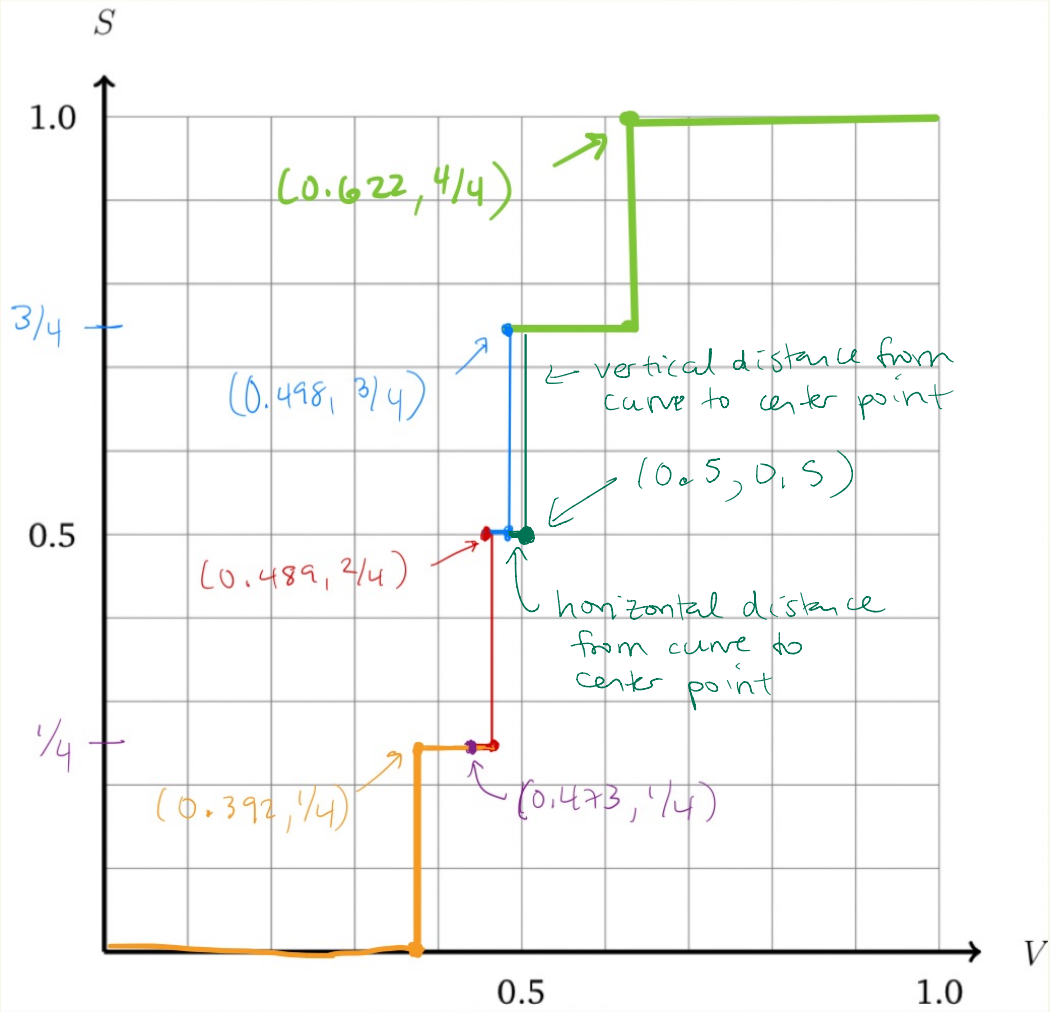
$$\frac{0.27 + 0.5 + 0.403 + 0.394}{4} \approx 0.392.$$

So we plot the point $(0.392, 0/4)$.

This completes our SV curve!



(g) Plot the point $(0.5, 0.5)$ on your graph. We define the *mean-median* score (or MM) to be the horizontal distance of the curve from the center point, and the *partisan bias* (or PB) to be the vertical distance of the curve from the center point. Find MM and PB for this election. Interpret them—from the partisan symmetry point of view, is this districting plan biased in favor of Democrats, Republicans, or neither?



MM = 0.5 − 0.498 = 0.002

PB = 3/4 − 0.5 = 0.25 = 1/4

From a partisan symmetry point of view, this is biased in favor of Republicans:

1) With between 48.9% and 49.8% of the vote share, R gets half the seats.

2) With half of the average vote share, R gets 3/4 of the seats.

Problem 2. Go to the wikipedia page United States House of Representatives elections in Oregon, 2020 and get the results from the five congressional races.

(a) Compute V_1, V_2, V_3, V_4, V_5, $\overline{V}$, and $\overline{S}$ for this election.

	District 1	District 2	District 3	District 4	District 5
R votes	161,928	273,835	110,570	216,081	204,372
D votes	297,071	168,881	343,574	240,950	234,863
R share	0.353	0.619	0.243	0.473	0.465

Compute R share for each district the same way as in Problem 1:

$$V_1 = \frac{161,928}{161,928+297,071} \approx 0.353$$

$$V_2 = \frac{273,835}{168,881+273,835} \approx 0.619$$

$$V_3 = \frac{110,570}{110,570+343,574} \approx 0.243$$

$$V_4 = \frac{216,081}{216,081+240,950} \approx 0.473$$

$$V_5 = \frac{204,372}{204,372+234,863} \approx 0.465$$

$$\overline{V} = \frac{V_1+V_2+V_3+V_4+V_5}{5} \approx \boxed{0.431}$$

$\boxed{\overline{S} = \frac{1}{5}}$ (district 2 is only district i with $V_i > 0.5$)

(b) Let's say that a *wasted vote* is any winning vote in excess of half the major-party votes in a district, or any losing vote at all. Find the Democratic and Republican wasted votes in each district, and add them up to find W^D and W^R overall.

2 252 125

	District 1	District 2	District 3	District 4	District 5
R votes	161,928	273835	110,570	216,081	204,372
D votes	297,071	168,881	343,574	240,950	234,863
R share	0.353	0.619	0.243	0.473	0.465

District — # votes needed to win

1 $\frac{161,928 + 297,071}{2} \rightarrow 229,500$

2 $\frac{273,835 + 168,881}{2} \rightarrow 221,358$

3 $\frac{110,570 + 343,574}{2} \rightarrow 227,072$

4 $\frac{216,081 + 240,950}{2} \rightarrow 228,516$

5 $\frac{204,372 + 234,863}{2} \rightarrow 219,618$

Let W_i^D be wasted votes of D in district i

W_i^R be wasted votes of R in district i

District 1

R = 161,928 votes needed to win: 229,500

D = 297,071 D wins

$W_1^D = 297,071 - 229,500 = \boxed{67,571}$

votes in excess of what was needed

$W_1^R = \boxed{161,928}$

R lost ⇒ all R votes wasted

District 2

R = 273,835 votes needed to win: 221,358

D = 168,881 R wins

$W_2^D = \boxed{168,881}$ (all votes wasted because D lost)

$W_2^R = 273,835 - 221,358 = \boxed{52,477}$

District 3

R = 110,570 votes needed to win: 227,072

D = 343,574 D wins

$W_3^D = 343,574 - 227,072 = \boxed{116,502}$

$W_3^R = \boxed{110,570}$

District 4

R = 216,081 votes needed to win: 228,516

D = 240,950 D wins

$W_4^D = 240,950 - 228,516 = \boxed{12,434}$

$W_4^R = \boxed{216,081}$

District 5

R = 204,372 votes needed to win: 219,618

D = 234,863 D wins

$W_5^D = 234,863 - 219,618 = \boxed{15,245}$

$W_5^R = \boxed{204,372}$

In summary:

i	W_i^R	W_i^D
1	161,928	67,571
2	52,477	168,881
3	110,570	116,502
4	216,081	12,434
5	204,372	15,245
+		+
	745,428	380,633
	W_R	W_D

tot: total # votes
= 2,252,125

↑ add up all votes cast across all 5 districts

(c) There are two ways to define the *efficiency gap* for a districting plan. One is the difference in wasted votes as a share of the total votes: $EG = \frac{W^R - W^D}{Tot}$. The other can be computed just by comparing seat share to vote share: $EG' = 2\overline{V} - \overline{S} - \frac{1}{2}$.

Compute EG and EG' for this election. (If the districts had equal numbers of votes cast, these would be equal.)

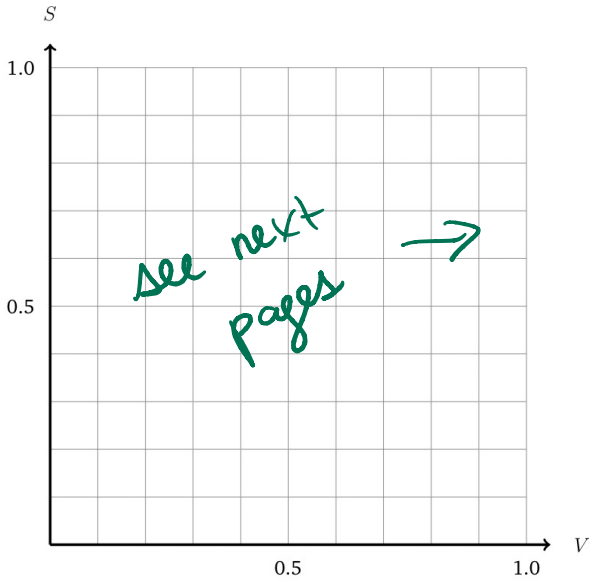
$$EG = \frac{W_R - W_D}{tot} = \frac{745,428 - 380,633}{2,252,125}$$

$$EG \approx 0.162$$

$$EG' = 2\overline{V} - \overline{S} - \frac{1}{2} = (2 \cdot 0.431) - \frac{1}{5} - \frac{1}{2}$$

$$EG' \approx 0.162$$

(d) Plot $(\overline{V}, \overline{S})$ and draw the SV curve for this election.



original vote share vector

(0.353, 0.619, 0.243, 0.473, 0.465) $(\overline{V}, \overline{S}) = (0.431, 1/5)$

↓ add one seat (margin = 0.5 − 0.473 = 0.027)

(0.380, 0.646, 0.270, 0.5, 0.492) $(\overline{V}, \overline{S}) = (0.458, 2/5)$

↓ add one seat (margin = 0.5 − 0.492 = 0.008)

(0.388, 0.654, 0.278, 0.508, 0.5) $(\overline{V}, \overline{S}) = (0.466, 3/5)$

↓ add one seat (margin = 0.5 − 0.388 = 0.112)

(0.5, 0.766, 0.390, 0.620, 0.612) $(\overline{V}, \overline{S}) = (0.578, 4/5)$

(0.5, 0.766, 0.390, 0.620, 0.612) $(\bar{v}, \bar{s}) = (0.578, 4/5)$

↓ add one more seat

(margin = 0.5 − 0.390 = 0.110)

(0.610, 0.876, 0.5, 0.730, 0.722) $(\bar{v}, \bar{s}) = (0.688, 5/5)$

We also need to start with the original vector and compute the threshold for going from 1 to 0 seats:

(0.353, 0.619, 0.243, 0.473, 0.465)

↓ subtract one seat (margin = 0.619 − 0.5 = 0.119)

(0.234, 0.5, 0.124, 0.354, 0.346)

$(\bar{v}, \bar{s}) = (0.312, 0/5)$

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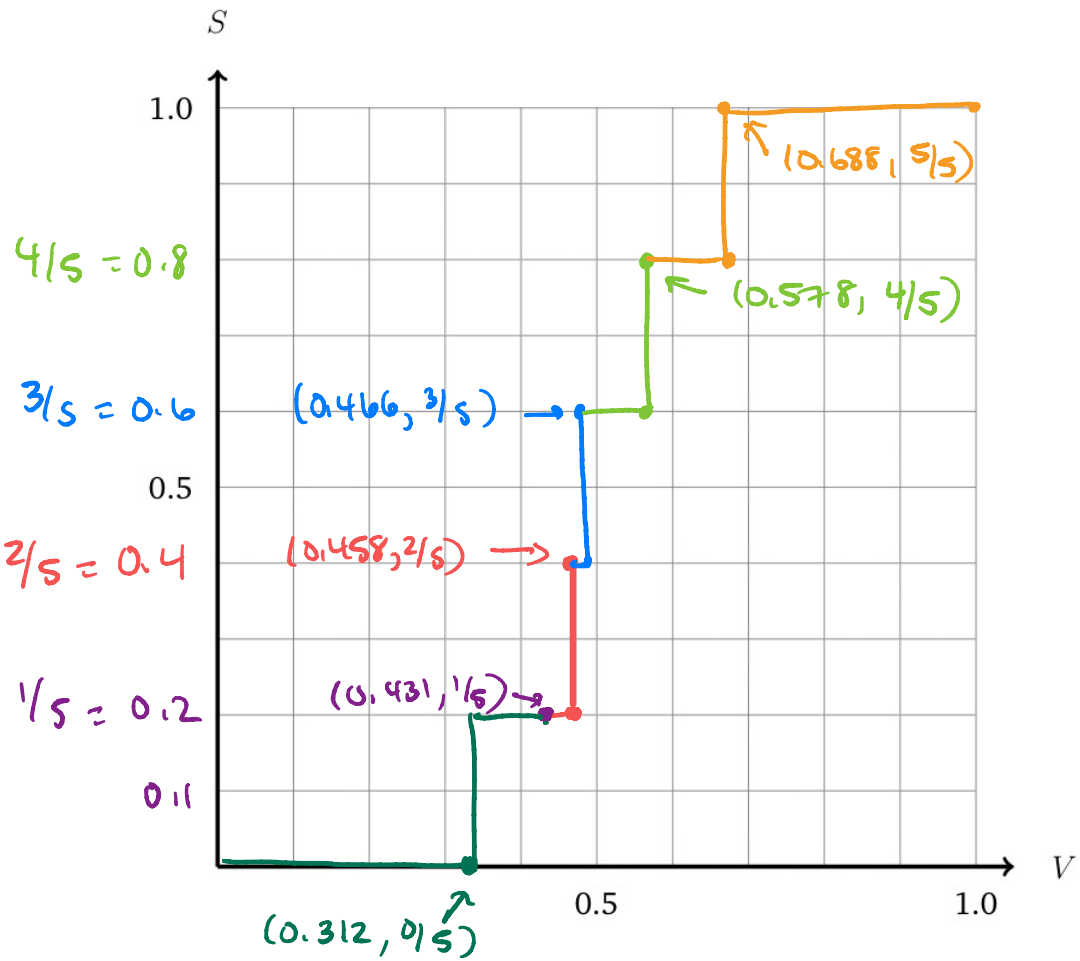
(d) Plot $(\overline{V}, \overline{S})$ and draw the SV curve for this election.



(e) Let's consider the "simplified" efficiency gap $EG' = 2\overline{V} - \overline{S} - \frac{1}{2}$. Solve for $EG' = 0$ algebraically (find $\overline{S}$ as a function of $\overline{V}$) and plot that line on the SV plane above.

Next, solve for $EG' = -.08$ and for $EG' = .08$ and plot those lines. Using those, shade the region that corresponds to $-.08 \leq EG' \leq .08$ on your seats-votes plot.

Interpret this—from the efficiency gap point of view, is this districting plan biased in favor of Democrats, Republicans, or neither?

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Find $\overline{S}$ as a function of $\overline{V}$

$$0 = 2\overline{V} - \overline{S} - \frac{1}{2}$$

$$\Rightarrow \overline{S} = 2\overline{V} - \frac{1}{2}$$

(Plot the line $y = 2x - \frac{1}{2}$)

Solve $EG' = -0.8$

$$-0.08 = 2\overline{V} - \overline{S} - \frac{1}{2}$$

$$\Rightarrow \overline{S} = 2\overline{V} - \frac{1}{2} + 0.08 = 2\overline{V} - 0.42$$

(Plot the line $y = 2x - 0.42$)

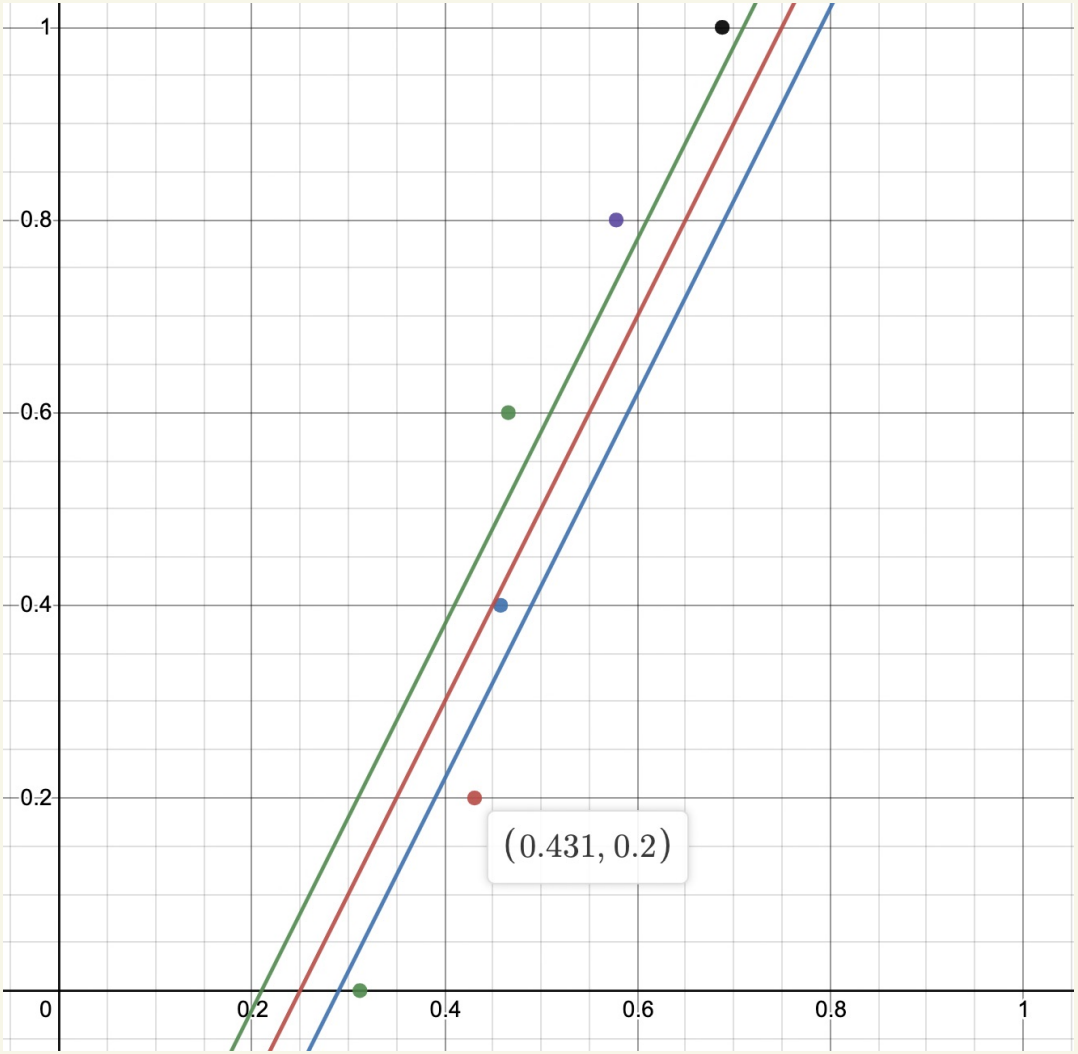
Solve $EG' = 0.8$

$$0.08 = 2\overline{V} - \overline{S} - \frac{1}{2}$$

$$\Rightarrow \overline{S} = 2\overline{V} - \frac{1}{2} - 0.08 = 2\overline{V} - 0.58$$

(Plot the line $y = 2x - 0.58$

Biased in favor of Democrats → our EG' falls outside of the $[-0.08, 0.08]$ and $W^R > W^D$


1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
(0.431, 0.2)