Worksheet 9

Mathematics of Social Choice

Duchin, Spring 2021

Problem 1. A classic compactness metric is the *Polsby-Popper score* of a planar region $\mathbf{R}$, defined as the ratio $\text{PoPo}(\mathbf{R}) = 4\pi A/P^2$, where P is the perimeter and A is the area of the region. The idea is that this score depends only on shape and not on size. We'll explore that here.

(a) Verify that the Polsby-Popper score of a circle of radius 10 is the same a for a circle of radius 3. Going further, verify that PoPo of a circle of radius r does not depend on r.

$$\text{PoPo (circle of radius 10)} = \frac{4\pi A}{P^2} = \frac{4\pi(\pi \cdot 10^2)}{(20\pi)^2} = \frac{400\pi^2}{400\pi^2} = \boxed{1}$$

$$\text{PoPo (circle of radius 3)} = \frac{4\pi A}{P^2} = \frac{4\pi(\pi \cdot 3^2)}{(6\pi)^2} = \frac{36\pi^2}{36\pi^2} = \boxed{1}$$

$$\text{PoPo (circle of radius r)} = \frac{4\pi A}{P^2} = \frac{4\pi \cdot (\pi r^2)}{(2\pi r)^2} = \frac{4\pi^2 r^2}{4\pi^2 r^2} = \boxed{1}$$

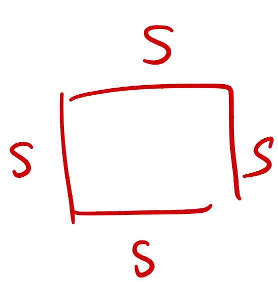
$A = \pi r^2$

$P = 2\pi r$

does not depend on r

(b) Verify that PoPo of a square with a side of length s does not depend on s. Going further, suppose that a rectangle has length ℓ and width w. Show that the PoPo score of the rectangle only depends on the ratio $x = \ell/w$. (That is, come up with a formula for PoPo of a rectangle that only depends on x but not on ℓ and w individually.)

$$\text{PoPo(square of side length s)} = \frac{4\pi A}{P^2}$$



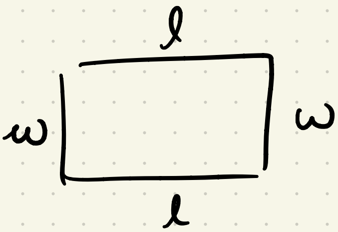
$$= \frac{4\pi s^2}{(4s)^2}$$

$$= \frac{4\pi s^2}{16 s^2}$$

$$= \frac{\pi}{4}$$

does not depend on s

$$\text{PoPo}(l \times w \text{ rectangle}) = \frac{4\pi A}{P^2}$$



$A = lw$

$P = 2l + 2w$

$ = 2(l+w)$

$$= \frac{4\pi(lw)}{(2(l+w))^2}$$

$$= \frac{\cancel{4}\pi lw}{\cancel{4}(l+w)^2}$$

$$= \frac{\pi}{\frac{(l+w)^2}{lw}}$$

$$= \frac{\pi}{\left(\frac{l+w}{l}\right)\left(\frac{l+w}{w}\right)}$$

$$= \frac{\pi}{\left(\frac{l}{l}+\frac{w}{l}\right)\left(\frac{l}{w}+\frac{w}{w}\right)}$$

$$= \frac{\pi}{\left(1+\frac{1}{x}\right)(x+1)}$$

only depends on x!

$$= \boxed{\frac{\pi}{x+2+\frac{1}{x}}}$$

Problem 2. For any region $\mathbf{R}$, let P be its perimeter and A be its area. Derive a formula for *the ratio of the area of a region* $\mathbf{R}$ *to the area of a circle with the same perimeter*. (Your answer should depend only on P and A and constants.) Compare this formula to the formula for PoPo.

What is the area of a circle with perimeter P?

Solve for the radius r first:

$$P = 2\pi r \implies r = \frac{P}{2\pi}$$

$$A = \pi r^2 = \pi \cdot \left(\frac{P}{2\pi}\right)^2 = \frac{\pi P^2}{4\pi^2} = \frac{P^2}{4\pi}$$

$$\frac{\text{area of region R}}{\text{area of circle with same perimeter}} = \frac{A}{\left(\frac{P^2}{4\pi}\right)}$$

$$= \boxed{\frac{4\pi A}{P^2}}$$

PoPo!

Problem 3. Let **H** be a regular hexagon, and let **H**$'$ be a hexagon with vertices $(1,0),(1,1),(0,1),(-1,0),$ $(-1,-1),(0,-1)$. Let **O** a regular octagon and **O**$'$ an octagon with vertices $(2,1),(1,2),(-1,2),$ $(-2,1),(-2,-1),(-1,-2),$ $(1,-2),(2,-1)$. Sketch these shapes and find their PoPo scores. (We've included some grid lines to help with some of those shapes.)

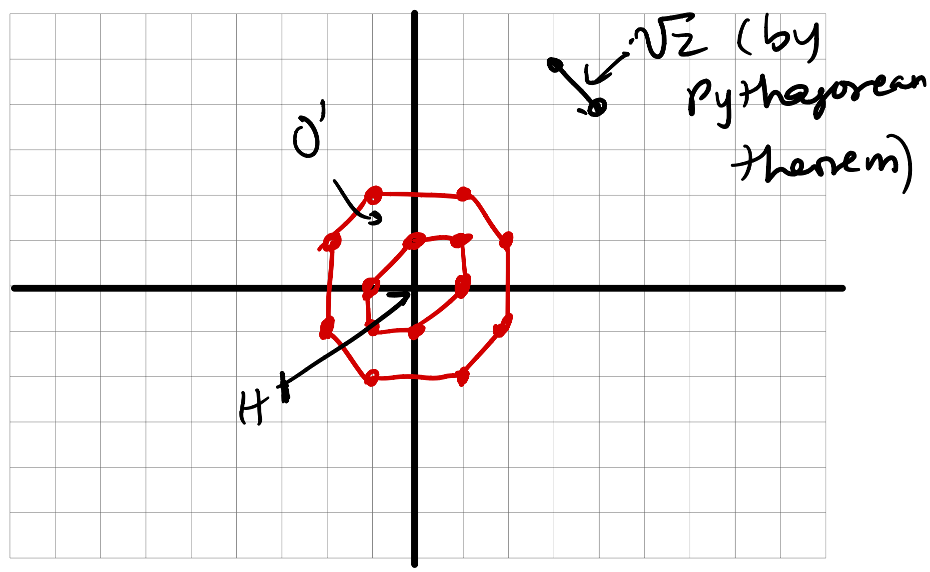
At the bottom of the page, place all the regions we've considered on the PoPo scale. Make some observations and conjectures about which polygons are the most "compact."

$\text{Area}(H') = 3$

$\text{Perimeter}(H') = 4 + 2\sqrt{2}$

$\text{Area}(O') = 14$

$\text{Perimeter}(O') = 8 + 4\sqrt{2}$



$$\text{PoPo}(H') = \frac{4\pi A}{P^2} = \frac{4\pi \cdot 3}{(4+2\sqrt{2})^2} \approx 0.81$$

$$\text{PoPo}(O') = \frac{4\pi A}{P^2} = \frac{4\pi \cdot 14}{(8+4\sqrt{2})^2} \approx 0.94$$

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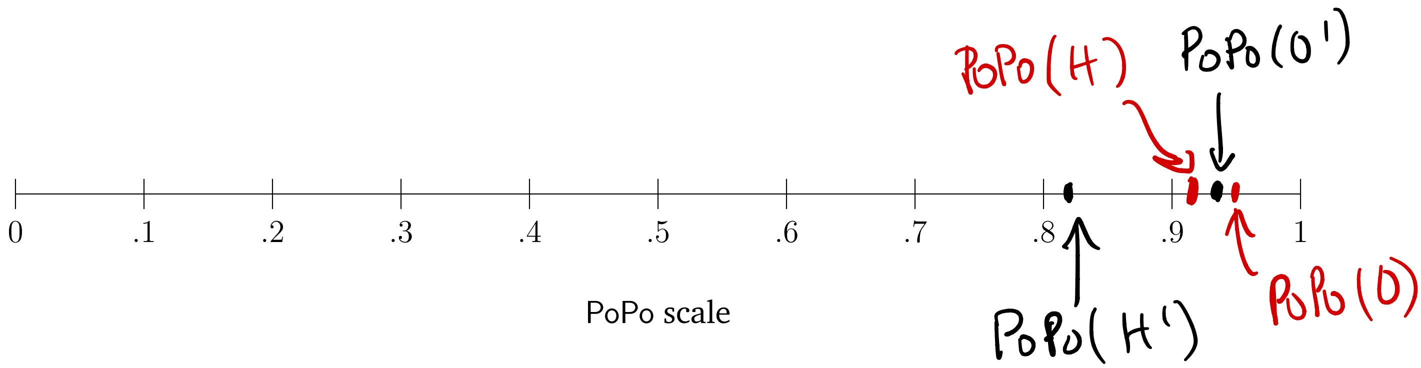
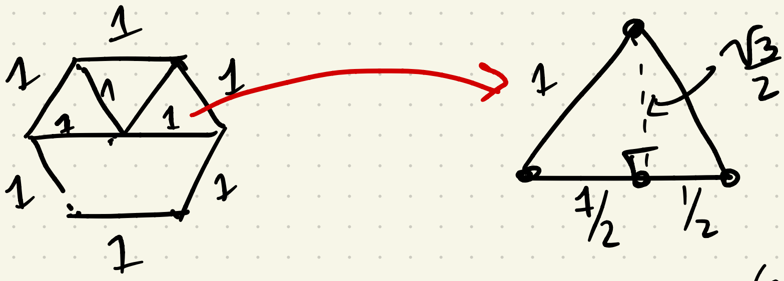




since $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

(Pythagorean theorem)

Area of unit equilateral triangle $= \left(\frac{1}{2}\right)(\text{base})(\text{height})$

$$= \left(\frac{1}{2}\right)(1)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{4}$$

$\Rightarrow$ Area of unit hexagon $= 6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$

Perimeter of unit hexagon $= 6$

PoPo (regular hexagon) $= \frac{4\pi A}{P^2} = \frac{4\pi \cdot \left(\frac{3\sqrt{3}}{2}\right)}{36}$

$$= \frac{6\pi\sqrt{3}}{36} = \frac{\pi\sqrt{3}}{6} \approx \boxed{0.91}$$

Area of unit octagon $= 2(1+\sqrt{2})$

Perimeter of unit octagon $= 8$

$$\text{PoPo(regular octagon)} = \frac{4\pi A}{P^2} = \frac{4\pi \cdot 2(1+\sqrt{2})}{64}$$

$$= \frac{\pi(1+\sqrt{2})}{8}$$

$$\approx \boxed{0.95}$$

Observations:

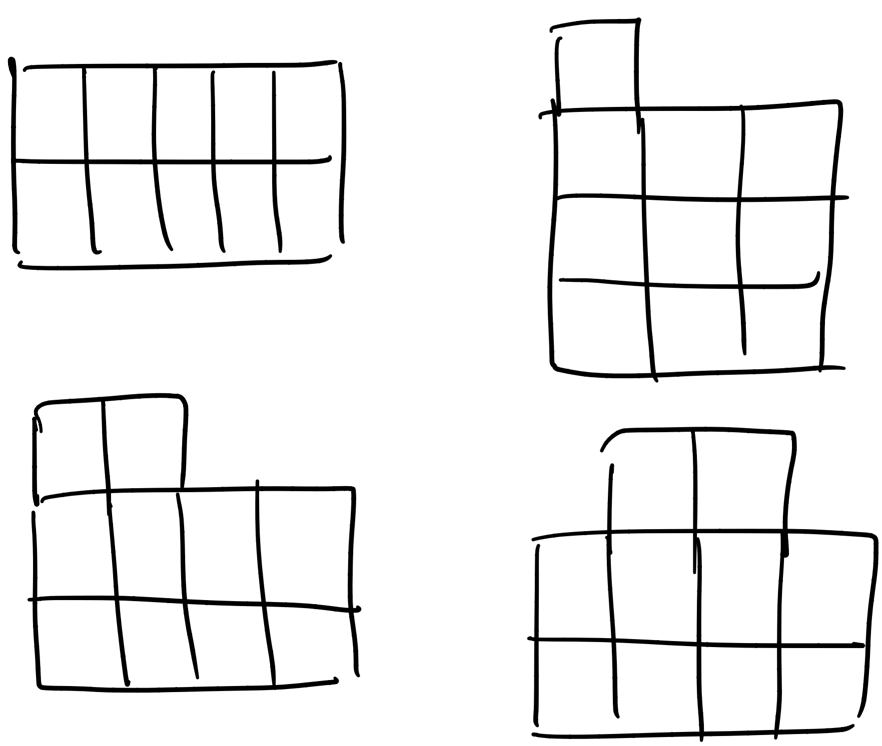
- Regular polygon more compact than its non-regular counterpart.
- Regular polygons with more sides are more compact. This is because they more closely approximate circles (which have the maximum PoPo score of 1).

Problem 4. You have a 10×10 grid with 40 orange squares (lighter gray on printout) and 60 pink (darker gray). You want to divide it into 10 districts.

Here are some redistricting agendas you might adopt:
(1) proportional representation (4 orange seats), as compact as possible
(2) max orange representation (6 orange seats), as compact as possible
(3) competitiveness (seek districts that are 6-4 or 5-5), as compact as possible
(4) safe seats (seek 8-2, 9-1, 10-0), as compact as possible
(5) simply as compact as possible

Your assignment: try to advance each agenda as much as possible while keeping good compactness scores. For each of the five agendas, score your plan on two compactness metrics.
(A) PoPo: $4\pi A/P^2$ for each district, averaged over the districts in the plan.
(B) *cut edges*: the number of pairs of neighboring tiles that lie in *different* districts.
We'll compile the most extreme results from the whole class to investigate the efficacy of compactness metrics at detecting gerrymandering.

To maximize $\frac{4\pi A}{P^2}$ when A is fixed at 10, we want to minimize P. The smallest P can be is 14, as in the districts below. These are the most compact districts in terms of PoPo.



Note:
higher PoPo
$\Downarrow$
more compact

more cut edges
$\Downarrow$
less compact