Final Exam Practice Problems
Mathematics of Social Choice
Duchin, Spring 2021

The final will have the same format as quizzes and midterm, mostly multiple choice. Here are some problems that go over material from both halves of the semester.

1. An election is held with candidates A, B, and C. Candidate B has over 50% of the vote. (first place) Name two winner selection methods that guarantee that B is the sole winner, and two winner selection methods that do not. Show an example of a preference schedule that fits this description, and give a winner selection method for which $B \notin \mathcal{W}$.

C is a majority candidate and therefore also a Condorcet candidate

b/c they have the most 1st place votes in any consolidation so beat all others H2H.

Examples

majority-fair
plurality
runoff
elimination
PWC
beatpath
sequential

not maj-fair
secondality
Borda → example
dictatorship

x51	x49
B	A
A	C
C	B

$\mathcal{W}_{secondality} = \{A\}$, $\mathcal{W}_{Borda} = \{A\}$

$\mathcal{W}_{dict\ of\ any\ column\ 2\ votes} = \{A\}$

2. Suppose there are $m = 15$ seats on a governing council, and the zones in the town have populations $M_H = 86$, $M_R = 241$, $M_T = 332$, and $M_B = 706$. Use the Huntington Hill method to apportion seats on the council to these zones. Because not every zone will get exactly its quota, some will be moderately over-represented and others moderately under-represented. Which is the most over-represented?

$\text{tot pop} = 86 + 241 + 332 + 706 = 1365$

ideal district size $= \frac{1365}{15} = 91$

$Q_H = \frac{86}{91} \approx .945$ | $Q_R = \frac{241}{91} \approx 2.648$ | $Q_T = \frac{332}{91} \approx 3.65$ | $Q_B = \frac{706}{91} \approx 7.76$

geom. mean/boundary

$\downarrow$ 1 | $\downarrow$ 3 | $\downarrow$ 4 | $\downarrow$ 8

that's 16 seats — too many!

So I'll raise denom. from 91 people per seat to, say, 95.

$Q_H' \approx .905$ | $Q_R' \approx 2.53$ | $Q_T' \approx 3.4947$ $(\sqrt{12} \approx 3.46)$ | $Q_B' \approx 7.43$ $(\sqrt{56} \approx 7.48)$

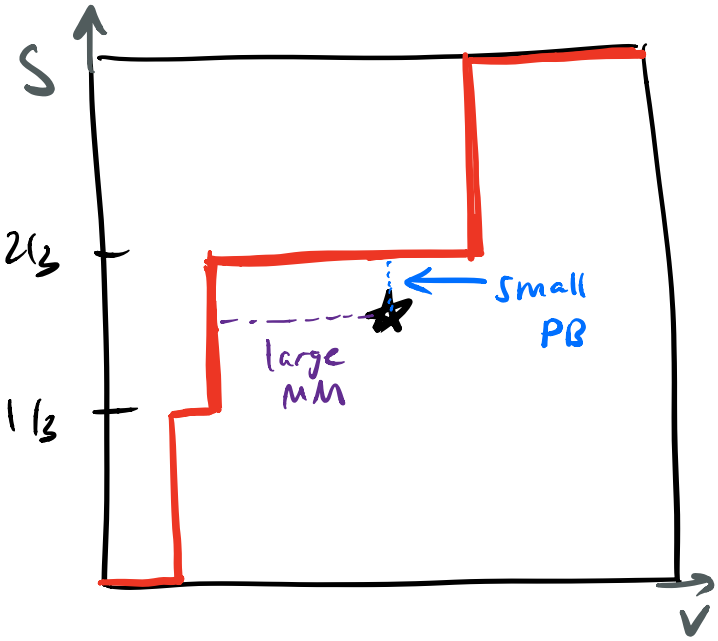
$\downarrow$ 1 | $\downarrow$ 3 | $\downarrow$ 4 | $\downarrow$ 7

86 people per district | $\frac{241}{3} = 80.3$ people per district | $\frac{332}{4} = 83$ people per district | $\frac{706}{7} \approx 100.9$ people per district

Now I've got my apportionment!

Zone R is relatively over-represented: a representative for every 80 people!

3. Draw a realistic SV curve where partisan bias shows a small Republican advantage but mean-median shows a large Republican advantage. Also, explain why it's impossible for PB and MM to disagree about which party is favored.

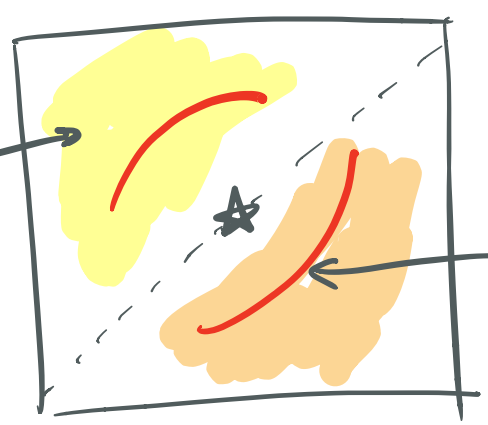


This example has $k=3$ districts.

(Large MM: Republicans can get way less than half the votes while surpassing half the seats.

(Small PB: if they had half the votes, they'd have a bit over half the seats.

Since the seat share can only go up as votes rise, every curve must be one of these types.

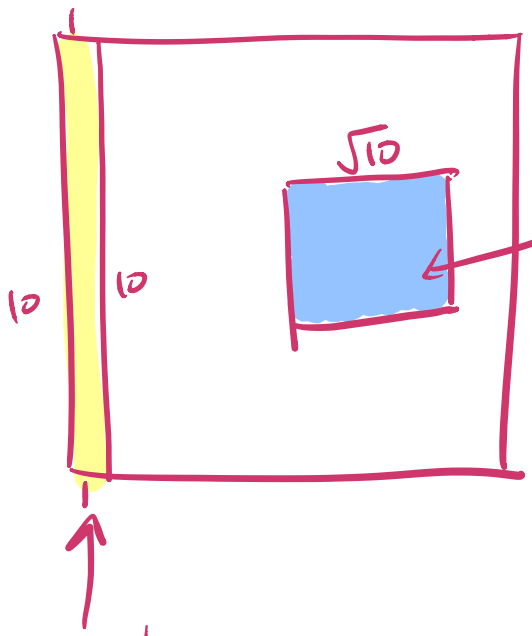


on this side, it's Republican advantage ($S>V$)

on this side, it's Republican disadvantage ($S<V$)

total area 100 area 10 each

4. Suppose a 10×10 square is to be divided into ten districts of equal area, and District 1 is required to be a rectangle. What's the best and worst PoPo score it can possibly have?



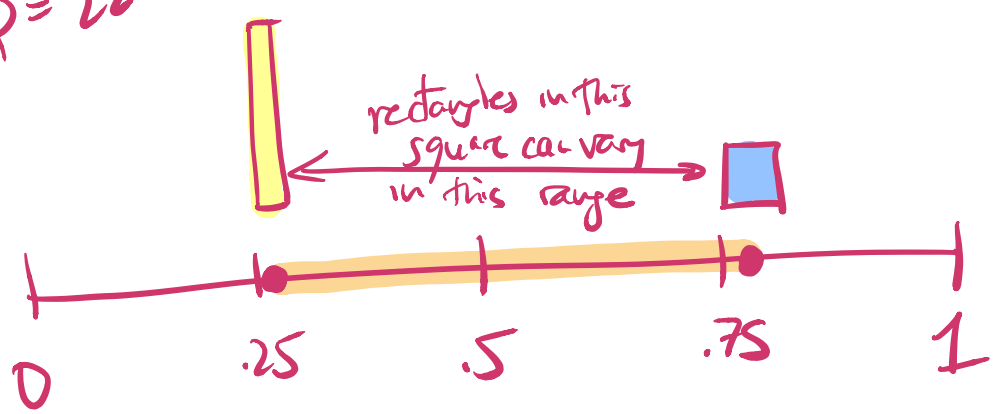
best!
$A = 10$
$P = 4\sqrt{10}$

$PoPo(\square) = \frac{4 \cdot \pi \cdot 10}{160}$

$\approx .785$
(high)

worst!
$A = 10$
$P = 22$

$\Rightarrow PoPo(\square) = \frac{4 \cdot \pi \cdot 10}{22^2} \approx .2596$
(low)



5. Recall that EG is the difference in wasted votes achieved by two parties, divided by the total number of votes. And EG' is a closely related formula that just uses seats and votes. For the following election, find EG and EG'.

District	Party X votes	Party Y votes	total	needed to win	w^X	w^Y
1	1372	1168	2540	1270	102	1168
2	1622	1030	2652	1326	296	1030
3	565	1139	1704	852	565	287
4	1012	1205	2217	1109	1012	96
			9913		1975	2581

Explain why they are not exactly equal. Do they agree on which party was given an advantage by the districting plan?

$$EG = \frac{w^X - w^Y}{\text{tot votes}} = \frac{1975-2581}{9913} = -.0611\ldots$$

Since Y wasted more votes, this is deemed to be a map that favored Party X.

these would be same under equal turnout but here, district 2 had way higher turnout than district 3, eg.

$$EG' = 2\bar{V} - \bar{S} - \tfrac{1}{2} = 2(.4875) - \tfrac{1}{2} - \tfrac{1}{2} = -.025$$

what are vote shares?

district 1	district 2	district 3	district 4
$\frac{1372}{2540}$	$\frac{1622}{2652}$	$\frac{565}{1704}$	$\frac{1012}{2217}$
$\approx .54$	$\approx .61$	$\approx .34$	$\approx .46$

average is $\bar{V} \approx .4875$

while $\bar{S} = \underbrace{2/4}_{\text{2 seats out of 4 won by X}} = .5$

6. (Bonus) "Recall" the simplified formulas for MM and PB from the notes: if you take the vote share by district, MM is the median minus the mean, and PB is the share of seats where the party did better than average, minus one-half. For instance, if there are four districts and the vote shares are $(.3, .4, .6, .8)$, then $MM = .5 - .525 = -.025$ and $PB = .5 - .5 = 0$. (That's because the vote shares have a median of .5, a mean of $\bar{V} = .525$, and the point-of-view party won $1/2$ of the districts.)

Consider a state where Republicans have $\bar{V} = 0.6$. If there are 5 districts, come up with vote shares for each District that get you a MM score of 0. What is the PB for your scenario? What are all possible PB scores holding $MM = 0$?

D/R	D/R	R won	R	R
lowest	$\leq .6$	$\boxed{.6}$	$\geq .6$	highest

median must equal mean to have $MM=0$

NOTE: I'll count exactly average as being above average (break tie in favor of R)

One example:

$(.48, .54, .6, .66, .72)$

$MM=0$

$PB = 3/5 - \frac{1}{2} = .1$

Generally, # R seats is 3, 4, or 5 out of 5

and # seats at/above average is also 3, 4, or 5

all examples average to $\bar{V}=.6$

$(.6, .6, .6, .6, .6) \leftarrow m_\uparrow = 5$

$(.4, .6, .6, .65, .75) \leftarrow m_\uparrow = 4$

$(.48, .54, .6, .66, .72) \leftarrow m_\uparrow = 3$