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Generalized Radon transforms and applications in tomography

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The Radon transform and its generalizations play a significant role in the development of many imaging techniques [25]. For example, the ordinary Radon transform is the foundation of the mathematical model of conventional X-ray computerized tomography (CT) [20], and weighted Radon transforms are used in emission tomography, such as in Single Photon Emission Computed Tomography (SPECT) [20]. The circular and spherical Radon transforms have been used in mono-static setups of ultrasound reflection tomography (URT) [22, 23], SONAR [18] and RADAR [7], as well as in thermo- and photo-acoustic tomography [17]. Bi-static setups of URT, SONAR, RADAR and certain seismic imaging techniques have been based on the elliptical Radon transform [3, 4, 10]. Recent developments in imaging technology based on particle scattering, such as Compton cameras, single-scattering optical tomography, and other modalities, have led to the study of the broken-ray (or V-line) transform, conical Radon transform and their various generalizations [1]. The cone beam transform of vector fields has been used in Doppler tomography [16] and Lorentz transmission electron tomography [24]. Certain 3D models of emission tomography [5, 27] and X-ray transmission tomography [26, 29], accounting for scattered particles, have used the so-called lemon transform and apple transform, integrating over interior and exterior surfaces of spindle tori.

This special issue features articles on recent advances in theoretical and practical aspects of image reconstruction using generalized Radon transforms. Below we give a brief description of each of these works.

Paper [6] deals with electrical capacitance tomography, where external capacitance measurements of an object are used to recover the dielectric permittivity distribution in its interior. The authors first develop a mathematical model for data acquisition by such an imaging device. Then they show that the measured data in that model corresponds to a weighted Radon transform of the electrical permittivity of the object near its surface. Using approximate inversion techniques, the authors obtain image reconstructions from experimental data with good surface resolution and short depth imaging. The high quality of these reconstructions points to the possibility of developing an excellent imaging technique, if more accurate inversion formulas are discovered. Consequently, the paper raises several interesting and unexplored theoretical questions on imaging in this setup.

The author of [8] considers the problem of inverting the cone beam transform of vector fields in $\mathbb{R}^3$. This transform maps a vector field to its integrals along rays emanating from points on a given curve (source trajectory). It is proved that the solenoidal part of the field can be reconstructed uniquely, if the source trajectory satisfies the Kirillov–Tuy condition of order 2. The latter implies that every plane, intersecting the support of the field, intersects the trajectory twice. The paper presents an exact inversion formula and a related reconstruction algorithm. The efficiency of the method is demonstrated through numerical experiments.

Paper [9] deals with the problem of image reconstruction in photo- and thermo-acoustic tomography, under the assumption of constant speed of sound. Standard mathematical models for this setup use the circular Radon transform in 2D and spherical Radon transform in 3D. There exist multiple exact inversion formulas for these transforms when the centers of integration circles (spheres) are located either on closed curves (surfaces) surrounding the image, or on unbounded ones. This paper presents an alternative approach to the problem, which yields a theoretically exact reconstruction of the standard Radon projections of the image function from the data measured on a finite open surface. In addition to that, the proposed method reduces the time interval for which the data should be known. The efficiency of the method is demonstrated through numerical experiments.

In [12], the authors consider the inverse source problem for photoacoustic tomography with directionally dependent data. They assume that the data are a fixed linear combination of the pressure and its normal derivative at the boundary, rather than just the pressure itself. The authors provide a series inversion method using spherical harmonics and orthogonal polynomials when the detectors are on a sphere surrounding the object and for an arbitrary linear combination of pressure and normal derivative. For odd dimensions, their method is exact if data are taken for sufficiently large finite time because of the strict Huygens' property, and it is exact in even dimensions only if data are taken for all time. The authors provide very good reconstructions of a phantom for pure pressure data, pure normal derivative data, and for a linear combination of pressure and derivative. The reconstruction methods work well even when they are not exact.

In [13], the authors study microlocal properties of imaging (generalized backprojection) operators for elliptical Radon transforms (ERT) and related Fourier integral operators. These transforms send a function to its integrals over ellipses in 2D or ellipsoids of revolution in 3D. In the case considered in this paper, the ellipses/ellipsoids have foci fixed distance apart from each other. The resulting transform corresponds to data acquisition in the linear model of seismic imaging with common offset scanning geometry, where sources and receivers are offset by a constant vector. The generalized backprojection of ERT leads to a pseudo-differential operator. The computation and analysis of its symbol allow the authors to explain how the proposed imaging operators map and allow them to produce reconstructions that are more uniform in depth.

Paper [14] presents another study of microlocal properties of FBP type formulas for generalized Radon transforms. The transforms considered in this article correspond to acquisition of tomographic data corrupted by the motion of an object, e.g. in dynamic imaging applications. The authors derive a novel reconstruction method approximating the solution via a filtered backprojection-type algorithm. They demonstrate the efficiency of the proposed technique through numerical results applied to dynamic photoacoustic tomography data.

Paper [21] studies microlocal properties of a family of filtered backprojection (FBP) type formulas for two specific setups of the spherical Radon transform in $\mathbb{R}^n$. The transforms considered here map a function f to its integrals over spheres with centers restricted to a convex, closed hypersurface S. In the case when f is supported outside of S, it is shown that FBP leads to a pseudodifferential operator with a singular symbol. In the second case, when $n = 2$, S is a square and f is supported inside S, it is shown that the resulting operator generates artifacts along circles centered at the vertices of S. Hence, there can not exist an FBP type exact inversion formula for the spherical Radon transform in this setup.

We describe together the articles that are on themes related to Compton tomography, the V-line transform, and related transforms.

The authors of [2] investigate various properties of the V-line and conical Radon transforms. The V-line transform (VLT), also often called the broken ray transform (BRT), maps a 2D function to two-parameter families of its integrals along V-lines (i.e. unions of two rays with a common vertex). The conical Radon transform (CRT) considered in this paper maps a function in 3D to a three-parameter family of its integrals over surfaces of polyhedral cones. The authors provide new inversion formulas for CRT and for VLT with different constant weights on each branch. They also reprove some previously known results using geometrically intuitive ideas. Using the novel inversion techniques they derive a range description for VLT and prove several support theorems.

The BRTs are typically considered reciprocal, i.e. the value of the integral of f depends on the path (locus of points), but not the direction in which it is traversed. However, in certain imaging applications the reciprocity condition may not hold. Paper [11] deals with nonreciprocal BRTs arising in optical fluorescence imaging. Here, the photon travels along a line until it is absorbed by a fluorophore molecule and then is re-emitted in a different direction and at a different frequency. It is shown, that by reversing the path of a photon and using the non-reciprocity of the data function, one can reconstruct simultaneously and independently all optical properties of the medium (the attenuation coefficient at both frequencies, as well as the concentration of the contrast agent). The results are also applicable to inverting BRTs that arise due to single Compton scattering.

In the two papers described above, the vertices of the broken rays and cones of integration were located inside the image domain. In the case of Compton camera imaging the resulting GRTs integrate over surfaces of cones with vertices outside of the image domain, typically restricted to a hypersurface. The author of [19] derived a new inversion formula for an overdetermined (5-dimensional) weighted conical Radon transform with vertices on a plane outside of the image domain. Three methods for numerical computations of the inverse are presented. The proposed techniques take advantage of overdetermined data to compensate for low number of events measured by the Compton camera and the measurement noise. It is shown, that adding a constraint on the total variation of the final image strongly improves the results.

Paper [15] presents another study on conical Radon transforms arising in Compton camera imaging. In this work, the data are collected on a detector with spherical surface, i.e. the vertices of integration cones are limited to a sphere. Moreover, in contrast to the previous paper, the conical transform is not overdetermined, as it uses a family of cones of the same dimension as the image function, and the integrals include a general weight function that depends on the distance from the vertex of the cone. The authors prove uniqueness of inversion of that transform and use variational regularization techniques for stability of the inversion. Similar to the previous work, it is shown that TV-regularization improves the quality of reconstruction from conical Radon data.

Another generalized Radon transform arising in 2D Compton scattering tomography is studied in [28]. Here the authors consider a mapping that sends the image function to a two parameter rotationally invariant set of circular arcs in the plane. New inversion formulas are obtained for limited data interior and exterior problems, where the image function is supported correspondingly inside and outside of a disc. An interesting feature of the presented approach is the connection between the circular arcs transform and the classical Radon transform, linking the problems considered in this article with the interior and exterior problems formulated for the classical Radon transform.

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