GEOMETRY OF STATIONARY SETS FOR THE WAVE EQUATION IN $\mathbb{R}^n$. THE CASE OF FINITELY SUPPORTED INITIAL DATA

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Abstract. We consider the Cauchy problem for the wave equation in the whole space $\mathbb{R}^n$, with initial data which are distributions supported on finite sets. The main result is a precise description of the geometry of the sets of stationary points of the solutions to the wave equation.

§0. Introduction.

The goal of this article is to clarify the structure of stationary (nodal) sets of solutions to the wave equation (1.1) when the initial data f is a distribution supported on a finite set in $\mathbb{R}^n$. Stationary sets are sets of points $x \in \mathbb{R}^n$ for which the solution to the wave equation is always zero. The problem has been solved in the plane [AQ1, AQ2] when f is an arbitrary distribution of compact support,[1] but not much is known in general. For distributions in the plane, the stationary sets have very restrictive structure; they must be the union of a finite set and a Coxeter system of lines (lines through one point generated by a finite rotation group). This Coxeter set is contained in a translate of the zero set of a homogeneous harmonic polynomial, and it is conical about the point of intersection. Loosely speaking, we will prove that a similar pattern occurs for finitely supported distributions in $\mathbb{R}^n$.

Characterizing stationary sets in $\mathbb{R}^n$ for $n > 2$ is more difficult and only partial results are known. It is known for compactly supported initial data in $\mathbb{R}^n$ that stationary sets are contained in the union of the zero sets of harmonic polynomials and algebraic varieties of lower dimension, and it is conjectured that the harmonic polynomial can be assumed to be a translate of a homogeneous polynomial [AQ1, AQ2]. It is shown in [ABK] for f sufficiently integrable at infinity that stationary

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[1] The theorems in [AQ1, AQ2] are stated for functions of compact support, but they are valid for $\mathcal{E}'(\mathbb{R}^2)$ and the proofs are the same.

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sets cannot have bounded closed components. In [A] and [AVZ] more precise analyses are given in $\mathbb{R}^n$ for stationary sets of lower dimension and conical stationary sets for f with arbitrary growth.

§1. Formulation of the Problem and Main Results.

We will use the following notation:
$\mathcal{D}(\mathbb{R}^n)$ – the space of all C^∞-functions with compact support;
$\mathcal{D}'(\mathbb{R}^n)$ – the space of distributions;
$\mathcal{E}'(\mathbb{R}^n)$ – the space of compactly supported distributions;
$C^\infty_{\text{rad}}(\mathbb{R}^n)$ and $\mathcal{D}_{\text{rad}}(\mathbb{R}^n)$ – the subspaces of corresponding spaces, consisting of radial functions f, *i.e.*, $f(x) = f(|x|)$;
$\mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ – the space of all distributions in $\mathcal{E}'(\mathbb{R}^n)$ supported on finite sets;
$\mathbb{R}_+ = (0, \infty)$.

Let us consider the Cauchy problem for the wave equation in $\mathbb{R}^n$, $\quad n \geq 2$:

$$u_{tt} = \Delta u,\ u = u(x,t),\ x \in \mathbb{R}^n,\ t > 0, \qquad (1.1)$$
$$u\big|_{t=+0} = 0, \quad u_t\big|_{t=+0} = f,$$

with the initial data $f \in \mathcal{D}'(\mathbb{R}^n)$.

After extending u to the half-space $t < 0$ by zero, the Cauchy problem (1.1) is equivalent to the equation

$$u_{tt} = \Delta u + \delta(t) f$$

having a unique solution $u \in \mathcal{D}'(\mathbb{R}^{n+1})$, which is a C^∞-function in the t-variable.

Definition 1. Let $f \in C^\infty(\mathbb{R}^n)$ and u be the (classical) solution for (1.1). Define the stationary set $S(f)$ as the set of time-invariant zeros of the solution u :

$$S(f) = \{x \in \mathbb{R}^n : u(x,t) = 0,\ t > 0\}. \qquad (1.2)$$

To extend this definition to distributional solutions, we use regularization. Namely, if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$, the convolution $f * \varphi$ is smooth and $u * \varphi$ (convolution with respect to x) is in C^∞ and solves (1.1) for the data $f * \varphi$.

Definition 2. For $f \in \mathcal{D}'(\mathbb{R}^n)$ define

$$S(f) = \bigcap_{\varphi \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)} S(f * \varphi), \qquad (1.3)$$

where $S(f * \varphi)$ is defined by (1.2).

For regular f, Definition 2 is consistent with Definition 1, as shown in Lemma 2.2. In fact, the set of all $\varphi \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$ in (1.3) can be replaced by an δ-sequence φ_n, $\varphi_n \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$.

The main question under consideration is the following.

Problem. *Which sets $S \subset \mathbb{R}^n$ are stationary sets, $S = S(f)$, for some $f \in \mathcal{D}'(\mathbb{R}^n)$ ($f \in \mathcal{E}'(\mathbb{R}^n)$)?*

Stationary sets characterize, to a certain extent, wave propagation. An important property is that any domain $\Omega \subset \mathbb{R}^n$, bounded by a stationary set, preserves energy, i.e., $E(t) = \int_\Omega (u_t^2 + |\nabla u|^2)dx = \text{const}$.

In [AQ1, AQ2] this problem was solved in full for $n = 2$ (the membrane equation) and $f \in \mathcal{E}'(\mathbb{R}^2)$. There it was discovered that the stationary sets for this case are unions of finite sets and equiangular families of straight lines through one point (Coxeter system of lines).

In this article we describe stationary sets for the case of initial data with finite support, for arbitrary dimension n. We prove that, up to a low-dimensional component, the stationary sets are affine algebraic cones with a special geometry.

A *cone* is understood to be a union of straight lines with a common point which is the vertex of the cone. We will call a cone $K \subset \mathbb{R}^n$ k*−flat* with edge L, where L is a k-dimensional plane in $\mathbb{R}^n$, if K is a union of $(k+1)$-planes containing L.

A union $\Sigma = H_1 \cup \dots \cup H_q$ of hyperplanes $H_i \subset \mathbb{R}^n$ is called a *Coxeter system* of hyperplanes if Σ is invariant with respect to any reflection σ_i around the hyperplane H_i, $i = 1, \dots, q$. The *Coxeter group* generated by the reflection $\sigma_1, \dots, \sigma_q$ will be denoted by $W(\Sigma)$.

We will call a polynomial P in $\mathbb{R}^n$, with real coefficients *a harmonic divisor* if P divides a nonzero harmonic polynomial. Zero sets of homogeneous harmonic divisors will be called *harmonic cones.*

For any set $F \subset \mathbb{R}^n$, the affine subspace spanned by F will be denoted by span F.

Our main result is the following.

Theorem 1. *Let f be a distribution supported on a finite set, $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, $f \neq 0$. If $S(f) \neq \emptyset$, then*

(a) *$S(f)$ is an algebraic variety in $\mathbb{R}^n$, contained in the zero set of a nonzero harmonic polynomial.*

(b) *After a suitable translation, the set $S(f)$ can be represented in the form*

$$S(f) = S_0 \cup V,$$

where V is an algebraic variety of $\operatorname{codim} V > 1$ and S_0, assuming it is nonempty, is a harmonic cone, which is a $(n-1)$−dimensional real algebraic variety.

In addition, the following is true:

(c) *The conical component S_0, in general, has the two components*

$$S_0 = \Sigma \cup K,$$

where K contains $\operatorname{supp} f$ *but Σ does not, $\Sigma \cap K \neq \emptyset$ provided both Σ and K are nonempty, Σ is a Coxeter system and K is a k-flat harmonic cone with the edge $L = \operatorname{span}\ (\operatorname{supp} f)$, $k = \dim L \leq n$. If* $\operatorname{supp} f$ *is a generic set, i.e., $k = n$, then $K = \emptyset$. If $k = n-1$, then K is a hyperplane and $\Sigma \cup K$ is a Coxeter system.*

(d) *If $\tilde{\Sigma}$ is the union of all hyperplanes contained in S_0, then $\tilde{\Sigma}$ is again a Coxeter system; the distribution f is odd with respect to any reflection $\sigma \in W(\tilde{\Sigma})$, i.e. $f \circ \sigma = -f$; the sets S_0, V, $S(f)$ and* $\operatorname{supp} f$ *are $W(\tilde{\Sigma})$-invariant.*

Theorem 1 says that finite sets of point sources generate stationary sets which are necessarily algebraic varieties and which are either small (empty or low-dimensional) or up to a low-dimensional component, are $(n-1)$-dimensional cones which suitably translated are determined by zeros of spatial harmonics.

The geometry of the essential, conical part is as follows. If the set of points in $\operatorname{supp} f$ is generic then the cone is a Coxeter system of hyperplanes. These stationary sets may appear only as a result of a Coxeter skew-symmetry of the initial data. If $\operatorname{supp} f$ lies in a proper affine subspace in $\mathbb{R}^n$, then another component may appear which is a cone containing $\operatorname{supp} f$. In the plane, for any compactly supported f, $S(f)$ is, up to a finite set, a Coxeter system of lines [AQ1]. However, in the plane the set K in Theorem 1 would be a collection of lines and therefore a Coxeter system by (d).

An important problem in studying of the wave equation is characterizing *nodal* sets (see [CH],I, Ch.5, S.5), that is, zero sets of eigenfunctions of the Laplace operator, or, equivalently, zero sets of time-harmonic solutions of the wave equation. This problem has been studied by many authors. Results on this subject mainly say that nodal sets are hypermanifolds with singularities and the eigenfunctions cannot vanish to high order on the nodal sets (see,*e.g.,* [DF1],[DF2],[Ch],[Ba], [B1], [B2] and others).

The problem under consideration is directly related to describing nodal sets. Indeed, extending the solution $u(x,t)$ of (1.1) for $t < 0$ by $u(x,t) = -u(x,-t)$ and applying Fourier transform in t to the both sides of (1.1) yields

$$-\lambda^2 v(x,\lambda) = \Delta v(x,\lambda),$$

where $v(x,\lambda)$ is the Fourier transform evaluated at arbitrary $\lambda \in \mathbb{R}$. Thus the stationary set $S(f)$ (1.2) is just the intersection of nodal sets of all the eigenfunctions $v(\cdot,\lambda)$ which are, since the initial data f has compact support, nonzero for an infinite number of values of λ. Therefore one deals with intersections of one-parameter families of nodal sets and one can expect explicit geometric descriptions of such sets.

Indeed, let us compare our results with some known results on nodal sets. In dimension 2 (vibrating membrane) the structure of nodal sets is well understood. They consist of smooth arcs, called nodal lines, and isolated singular points where these arcs meet. Moreover, the arcs meeting at singular points form an equiangular configuration (see [B1] for the proof). The above mentioned results of [AQ1],[AQ2] show that in the case of an infinite membrane and finitely supported initial velocity one has the equiangular configuration with the nodal lines which are just straight lines.

Now, it is known that the nodal set of an eigenfunction of the Laplace-Beltrami operator on an $n-$dimensional Riemannian manifold consists of a smooth hypersurface and a singular part of dimension $\leq n-2$, ([Ba],[Ch],[BM],[HS]). Theorem 1 of the present article specifies that, in the case of $\mathbb{R}^n$ and finitely supported initial data, the hypersurface part of the nodal set is an affine cone defined by zeros of a harmonic divisor.

The plan of the present article is as follows. Sections 2 to 6 are devoted to the proof of Theorem 1. First, in Section 2, we reduce the characterization of stationary sets to the investigation of injectivity sets for the spherical transform, properly defined on distributions. In Section 3 we study the microlocal properties of the spherical Radon transform and prove a support theorem for this transform. This is the key to establishing geometric symmetry conditions on $S(f)$ relative to $\operatorname{supp} f$.

In Sections 4 and 5 we study the algebraic and geometric structure of the stationary set, and key points here are the properties of the zero sets of harmonic polynomials. In Section 6 we combine the information we have obtained and prove Theorem 1. In Section 7 we give sufficient conditions for a set in $\mathbb{R}^n$ to be a stationary set $S(f)$ for some $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$. Finally, in Section 8, we understand the geometry of $S(f)$ when $\operatorname{supp} f$ is a finite union of balls.

This article continues a series of works [AQ1], [AQ2], [AQ3], [ABK], [A], [AVZ], [AR], started by [AQ1] and devoted to the description of injectivity sets for the spherical transform, stationary sets for the wave and heat equations, and related problems. Our initial interest in the problem was motivated by a problem in approximation theory posed in [LP] (cf. [AQ1], [AQ2]).

§2. $O(n)$-averaging and the Spherical Transform.

Let $f \in \mathcal{D}'(\mathbb{R}^n)$. For any element k of the orthogonal group $O(n)$ the composition $f \circ k$ is well-defined. Denote the $O(n)$-average of f by

$$f^{\#} = \int_{k \in O(n)} (f \circ k) dk,$$

where dk is the Haar measure on the group $O(n)$. Clearly, for any test function, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ the equation $\langle f^{\#}, \varphi \rangle = \langle f, \varphi^{\#} \rangle$ holds, and this can be taken as the definition of $f^{\#}$.

Denote by $\tau_x,\ x \in \mathbb{R}^n$, the translation $\tau_x f(y) = f(x+y)$, and let

$$f_x^{\#} = (\tau_x f)^{\#}.$$

The operation $f_x^\#$ is just the radialization of f with respect to the rotation group isotropic about x.

Let $f \in C(\mathbb{R}^n)$. Define the spherical transform $C(\mathbb{R}^n) \ni f \to \hat{f} \in C(\mathbb{R}^n \times \mathbb{R}_+)$ by

$$\hat{f}(x,r) = Rf(x,r) = \int_{|\theta|=1} f(x+r\theta)dA(\theta), \tag{2.1}$$

where dA is the normalized surface measure on the unit sphere. Because the spherical transform is a Fourier integral operator, it can be defined on distributions. It is easy to see that $f_x^\#$ and $\hat{f}$ are related by

$$f_x^\#(y) = \hat{f}(x,|y|). \tag{2.2}$$

Lemma 2.1. *Let $f \in C(\mathbb{R}^n)$ and let $S(f)$ be defined by* (1.2). *Then*

$$S(f) = \{x \in \mathbb{R}^n : \hat{f}(x,r) = 0 \quad \textit{for all} \quad r > 0\}$$

and also

$$S(f) = \{x \in \mathbb{R}^n : (f * u)(x) = 0 \quad \textit{for all} \quad u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)\}.$$

Proof. The right hand sides in both formulas are the same. Indeed, we have in polar coordinates:

$$\begin{aligned}(f * u)(x) &= \int_{\mathbb{R}^n} f(x-y)u(y)dy = \int_0^\infty \int_{|\theta|=1} f(x+r\theta)u(r\theta)r^{n-1}drd\theta \\ &= \int_0^\infty \hat{f}(x,r)u(r)r^{n-1}dr.\end{aligned}$$

Therefore, $\hat{f}(x,\cdot) \equiv 0$ implies $(f*u)(x) = 0$. The opposite implication can be obtained by choosing a sequence $u_m(r)$ tending to $\delta(r-r_0)$, $r_0 > 0$ is an arbitrary fixed number.

Thus, it suffices to prove the first identity. The Poisson-Kirchhoff formula (see, *e.g.*, [He], Ch. I, S.5.2, 7) for the solution $u(x,t)$ of the Cauchy problem (1.1) yields

$$u(x,t) = \mathrm{const}(\partial_t)^{n-2}F(x,t),$$

where

$$F(x,t) = \int_0^t (t^2-r^2)^{(n-3)/2} r\hat{f}(x,r)dr\,. \tag{2.3}$$

Therefore $\hat{f}(x, \cdot) \equiv 0$ implies $u(x,t) = 0,\ t > 0$, i.e., $x \in S(f)$.

Let us prove that the converse is also true. First, we show $F(x,y) = 0$ for all $t > 0$. For $n = 2$ we have $u(x,t) = \int_0^t (t^2 - r^2)^{-1/2} r \hat{f}(x,r) dr$ and hence $u(x,t) = 0$, $t > 0$, implies $\hat{f}(x,r) = 0$, $r \in \mathbb{R}_+$, due to invertibility of the Abel transform. If $n > 2$ then $u(x,t) = 0$, $t \in \mathbb{R}_+$, implies that $F(x,t)$ is a polynomial (in the t-variable) of degree less than $n-2$.

The change of variables $r = ts$ yields

$$F(x,t) = t^{n-1} \int_0^1 (1 - s^2)^{(n-3)/2} s \hat{f}(x, ts) ds,$$

and hence $F(x,t) = O(t^{n-1})$, $t \to 0$. As $\deg F < n-2$, this is possible only if $F(x,t) = 0$ for all t. Since (2.3) can be reduced to an invertible Abel or Volterra equation of the first kind by successive differentiations, $\hat{f} = 0$ (see, *e.g.*, [Q2] Theorems A and B, Lemma 3.3, and p. 516 for this reduction). □

Lemma 2.2. *For $f \in C(\mathbb{R}^n)$, Definitions 1 and 2 coincide, that is,*

$$S(f) = \bigcap_{\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)} S(f * \varphi).$$

Proof. If $x \in S(f)$ according to Definition 1, then $(f * u)(x) = 0$ for any $u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$, due to Lemma 2.1. Let $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$ be arbitrary. Then $(f * \varphi) * u(x) = (f * (\varphi * u))(x) = 0$ because $\varphi * u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$ as it is the convolution of two radial functions from $\mathcal{D}(\mathbb{R}^n)$. Again, by Lemma 2.1 we have $x \in S(f * \varphi)$. So, $x \in S(f)$ in the sense of Definition 2. Conversely, if $x \in S(f * \varphi)$ for any $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$ then, by Lemma 2.1, $(f * \varphi * u)(x) = 0$ for all $u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$. Taking a δ-sequence φ_n we obtain $(f * u)(x) = 0$ and this means $x \in S(f)$ due to Lemma 2.1 and Definition 1. □

Now we turn to the characterization of the set $S(f)$ in terms of $O(n)$-averaging, for the case $f \in \mathcal{D}'(\mathbb{R}^n)$.

Lemma 2.3. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Then*

$$S(f) = \{x \in \mathbb{R}^n : f_x^{\#} = 0\}$$

and

$$S(f) = \{x \in \mathbb{R}^n : (f * u)(x) = 0, \quad u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)\}.$$

Proof. The second formula can be easily derived from the first one by verifying the identity on test functions.

Let us prove the first formula. The following identity can be easily verified by acting on test functions.

$$(f * u)_x^{\#} = f_x^{\#} * u, \quad u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n). \tag{2.4}$$

Now, if $x \in S(f)$, then by Definition 2, $x \in S(f * \varphi)$ for any $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$. Lemma 2.1 is applicable to the smooth function $f * \varphi$ and therefore $(f * \varphi)_x^{\#} = \widehat{(f * \varphi)}(x, \cdot) = 0$. Choosing the sequence $\varphi = \varphi_k$ converging to the δ-function as $k \to \infty$, we obtain $f_x^{\#} = 0$.

Conversely, if $f_x^{\#} = 0$ then we have from (2.4) and the fact the convolution is associative for distributions and functions of compact support [Ru, 6.3.5]:

$$((f * \varphi) * u)_x^{\#} = f_x^{\#} * (\varphi * u) \quad \text{for all} \quad \varphi, u \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n).$$

Lemma 2.1 asserts that $x \in S(f * \varphi)$. Because of Definition 2 and the arbitrariness of $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$, this means that $x \in S(f)$. $\square$

Remark. It is clear from the proof of Lemma 2.3 that in Definition 2 one can take instead of all $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$ just any sequence $\varphi_k \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$ tending to the δ-function, as $k \to \infty$, i.e.,

$$S(f) = \bigcap_{\varphi_k \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)} S(f * \varphi_k), \quad \varphi_k \to \delta.$$

§3. Microlocal Analysis and the Support theorem.

Microlocal analysis has been used to understand properties of Radon transforms beginning with Guillemin's seminal work [GS, pp. 336-337, 364-365], and others (*e.g.,* [Q3]) have used microlocal techniques to prove support theorems for generalized Radon transforms. We now give some of the basic conventions we will use in the rest of the article.

Definition 3.1. Let $r \in \mathbb{R}_+$, $S \subset \mathbb{R}^n$, and $x \in S$. The point x is called a *regular point* of S if and only if there is a connected real-analytic hypersurface, A, (an $(n-1)-$dimensional submanifold of $\mathbb{R}^n$) such that $x \in A \subset S$. We let $\operatorname{reg} S$ denote the regular points in S. Let x be a regular point of S, and let A be such an associated hypersurface ($x \in A \subset S$). Then, we let T_x denote the hyperplane tangent to A at x. The points y and y' in $\mathbb{R}^n$ are said to be T_x-*mirror* if and only if they are reflections about T_x. If $y \in T_x$, then y is its own mirror point, and we say y is *self-mirror.*

Note that the definition of regular point includes the case when S itself is not a manifold at x. For example, using our definition, $(0,0)$ is a regular point of $S = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$, and both the $x-$axis and the $y-$axis are "hypersurfaces" associated with $(0,0)$. Also, note that if $y \in S(x,r)$ then its T_x-mirror is also in $S(x,r)$.

Let X be a manifold and $f \in \mathcal{D}'(X)$. Let $H \subset X$ be a hypersurface and let $y \in H \cap \operatorname{supp} f$. We say f is zero locally near y on one side of H if and only if there is a connected neighborhood U of y such that H divides U into two disjoint open sets and f is zero on one of those sets.

Theorem 3.2. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $f \neq 0$. We let $x_0 \in S = S(f)$ be a regular point. Let $A \subset S$ be a connected real-analytic hypersurface containing x_0, and let T_{x_0} be the hyperplane tangent to A at x_0. Let $y_0 \in \operatorname{supp} f \setminus x_0$ and assume f is zero on one side of $S(x_0, |y_0 - x_0|)$ locally near y_0. Then, the $T_{x_0}-$mirror point to y_0 must also be in* $\operatorname{supp} f$.

This theorem has no conclusion if y_0 is self-mirror. The theorem (and some geometric arguments) can be used to prove a much stronger result if $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ (see §5 and Theorem 6.1). This theorem is, at least morally, a microlocal version of a *reflection principle* of Courant and Hilbert. Their theorem says if f has zero integrals over all spheres with centers on the hyperplane T_{x_0}, then f is an odd function about T_{x_0}. Theorem 3.2 provides microlocal information about wavefront at mirror points in T_{x_0}.

In Section 3.1, we will prove the needed microlocal results for the spherical transform and in Section 3.2 we will prove the support theorem.

3.1 The Microlocal Analysis. The proof of the regularity theorem, 3.3, is similar to proofs in [AQ2] and [GrQ] and so it will be sketched. In particular, we will outline what must be done to show that R is an elliptic Fourier integral operator with the given microlocal properties. We use the basic notation from [Hö1, Hö2, Tr]. If $Y \subset X$ is a C^2 manifold then the conormal bundle of Y, $N^*Y \subset T^*X$, is the set of covectors $\{(y,\xi) : y \in Y,\ \xi \in T_y^*X \text{ and } T_yY \subset \ker \xi\}$.

The analytic wavefront set of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is a conical subset, $\mathrm{WF_A}(f)$, of the cotangent bundle $T^*\mathbb{R}^n$ consisting of "directions" in which f is not real-analytic. This is defined either in terms of the very rapid decrease of localized Fourier transforms of f [Tr, Definition 1.1, p. 243] or in terms of exponential decrease of the Fourier-Bros-Iagolnitzer transform [Hö2, Theorem 9.6.3]. For example, if f is the characteristic function of an open set, D, with real-analytic boundary, then $\mathrm{WF_A}(f)$ is the conormal bundle of the boundary of D, $N^*(\partial D)$.

Theorem 3.3, a regularity theorem for the Radon transform, R, is one of the keys to the proof of the support theorem. The hypotheses include an assumption on the vanishing of f at certain points.

Theorem 3.3. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let x_0 be a regular point of S. Let A be a connected real-analytic hypersurface such that $x_0 \in A \subset S$. Assume Rf is zero in an open neighborhood of $(x_0, r_0) \in S \times \mathbb{R}_+$. Let $(y;\xi) \in N^*S(x_0,r_0)\backslash 0$, and assume that f is zero in a neighborhood of the $T_{x_0}-$mirror point to y. Then $(y;\xi) \notin \mathrm{WF_A}(f)$.*

In general, Radon transforms will not detect singularities, $\mathrm{WF}_{\mathrm{A}}(f)$, unless they are conormal to the surface being integrated over. This lemma implies that any singularity at $(y;\xi) \in N^*S(x_0, r_0)$ will be detected by data Rf as long as f is zero (or just real-analytic) near the mirror point to y on $S(x_0, r_0)$.

If y is self-mirror ($y \in T_{x_0}$), then Theorem 3.3 gives no conclusion about y. In other cases, if f is zero in a neighborhood of the $T_{x_0}-$mirror point to y, then the theorem gives specific directions above y that are not in $\mathrm{WF}_{\mathrm{A}}(f)$.

Proof. The proof is the $n-$dimensional generalization of the proof of the microlocal regularity theorem Lemma 4.3 in [AQ2].

Locally above A the incidence relation for R is defined to be $Z = \{(y, x, r) \in \mathbb{R}^n \times A \times \mathbb{R}_+ : y \in S(x, r)\}$ [He]. The appropriate microlocal diagram [GS, pp. 364-365] (see also [Q1]) is:

$$\begin{array}{ccc} \Gamma = N^*Z \setminus 0 & \xrightarrow{\pi_2} & T^*(A \times \mathbb{R}_+) \setminus 0 \\ \downarrow \pi_1 & & \\ T^*(\mathbb{R}^n) \setminus 0 & & \end{array} \tag{3.1}$$

where the maps π_1 and π_2 are projections from $\Gamma \subset T^*(\mathbb{R}^n \times (A \times \mathbb{R}_+))$ onto the indicated factors.

We must show the map π_2 is close enough to being an injective immersion (the Bolker Assumption, [GS, pp. 364-365, Q1]) that the calculus of Fourier integral operators can be used to prove the lemma. Specifically, the goal is to prove that π_2 in (3.1) satisfies:

(3.2) *covectors $(y, x, r; \xi, \eta) \in \Gamma$ and $(y', x, r; \xi', \eta) \in \Gamma$ have the same image under π_2 only if y and y' are T_x-mirror. π_2 is a local diffeomorphism except above points (y, x, r) where $y \in S(x, r)$ is self-mirror.*

To this end, we first calculate N^*Z in good coordinates. Points $(y, x, r) \in Z$ are determined by the equation $|y - x|^2 - r^2 = 0$, and the differential of this equation gives a basis of the fibers of N^*Z. Coordinates for $\Gamma = N^*Z \setminus 0$ are:

$$\begin{gathered} S^{n-1} \times A \times \mathbb{R}_+ \times (\mathbb{R} \setminus 0) \to \Gamma \\ (\theta, x, r, \alpha) \to \big(y, x, r; \alpha([r\theta]\mathbf{dy} - [P_x(r\theta)]\mathbf{dx} - r\mathbf{dr})\big) \\ \text{where } y = r\theta + x. \end{gathered} \tag{3.3}$$

Here, $(w_1, \ldots, w_n)\mathbf{dy} = w_1\mathbf{dy}_1 + \cdots + w_n\mathbf{dy}_n$ is the covector in $T^*\mathbb{R}^n$ corresponding to $(w_1, \ldots, w_n) \in \mathbb{R}^n$ and the map $P_x : \mathbb{R}^n \to (T_x - x)$ is the orthogonal projection onto the hyperplane through the origin parallel to T_x. This hyperplane is, of course, T_xA, viewed as a subspace of $\mathbb{R}^n$. Finally, $P_x(r\theta)\mathbf{dx}$ is the covector corresponding to the vector $P_x(r\theta)$.

Eq. (3.3) shows that π_1, and π_2 do not map to the zero sections, so R is a Fourier integral operator associated to the Lagrangian manifold, Γ [Tr, Theorem

2.1, p. 316]. This is one reason why R can be evaluated on distributions. R is real-analytic elliptic since the measure of integration for R, dA, is real-analytic and nowhere zero.

The map π_2 is equivalent to the corresponding map in coordinates (3.3):

$$(\theta, x, r, \alpha) \xrightarrow{\tilde{\pi}} \big(x, r; -\alpha([P_x(r\theta)]\mathbf{dx} - r\mathbf{dr})\big). \tag{3.4}$$

Since x and r are known from the image of $\tilde{\pi}$, $\tilde{\pi}$ determines $P_x(\theta)$ so $y = x + r\theta$ is known up to its T_x−mirror from π_2. The calculation that $\tilde{\pi}$ is a local diffeomorphism except at self-mirror points is left to the reader (one can do it in local coordinates on A if one likes). Since (3.2) is essentially a local statement, this proves (3.2).

Now, assume f is as in the hypotheses of Theorem 3.3. R has been shown to be an analytic elliptic Fourier integral operator associated with Γ. The calculus of such operators implies the conclusion of Theorem 3.3 in the same way as in [AQ1]. The basic idea is as follows. Let $(y;\xi) \in N^*(S(x_0, r_0)) \setminus 0$ and assume f is zero near the T_{x_0}−mirror point to y. Then by (3.2), no singularities above the T_{x_0}−mirror point, y' can cancel singularities above y. Here's why. By the calculus of these operators the only singularities of f that are detectable from Radon data are those on $N^*(S(x_0, r_0))$ $(\mathrm{WF_A}(Rf) \subset \{((x_0, r), \eta) : (y, (x_0, r); \xi, -\eta) \in \Gamma$ for some $(y;\xi) \in \mathrm{WF_A}(f)\}$, and for fixed (x_0, r_0), the only wavefront of f that affects this are covectors in $N^*S(x_0, r_0))$. Furthermore, since R is elliptic, and π_2 maps only covectors above y and y' to the same point, the only singularities of f on $N^*S(x_0, r_0)$ that can cancel singularities above y are those above y'. So, since there are no singularities above y', and since Rf is real-analytic near (x_0, r_0), then $(y;\xi) \notin \mathrm{WF_A}(f)$. The details of this argument are given in [AQ1] and [GrQ] (see also [SKK, Ka]). □

3.2 Proof of Support Theorem. The following theorem of Hörmander, Kawai, and Kashiwara [SKK, Hö2 Theorem 8.5.6] is one key to the proof.

Lemma 3.4. *Let X be a real-analytic manifold and let $h \in \mathcal{D}'(X)$ and $y \in \operatorname{supp} h$. Let H be a C^2 hypersurface with $y \in H$. Assume f is zero locally near y on one side of H. If $(y;\xi) \in N^*(S) \setminus 0$, then $(y;\xi) \in \mathrm{WF_A}(h)$.*

Under the assumptions of this lemma, h cannot be real-analytic near y because y is a boundary point of supp h. Lemma 3.4 is a strengthening of this simple observation because it provides specific wave front directions above y that must be in $\mathrm{WF_A}(h)$.

Proof of Theorem 3.2. Assume the mirror point to y_0 in T_{x_0} is not in $\operatorname{supp} f$. Let $r_0 = |y_0 - x_0|$. Then, y_0 is a boundary point of $\operatorname{supp} f$ in $S(x_0, r_0)$. Furthermore, locally near y_0, f is zero on one side of the sphere $S(x_0, r_0)$. Let $(y_0;\xi_0) \in N^*S(x_0, r_0) \setminus 0$. By Lemma 3.4, $(y_0;\xi_0) \in \mathrm{WF_A}(f)$. However, by the regularity theorem, 3.3, $(y_0;\xi_0) \notin \mathrm{WF_A}(f)$ since Rf is zero near (x_0, r_0), A is a real-analytic hypersurface, and the T_{x_0}−mirror point to y_0 is not in $\operatorname{supp} f$. This contradiction proves the theorem. □

§4. Algebraic Structure of $S(f)$ for $f \in \mathcal{E}'(\mathbb{R}^n)$.

We will need some properties of harmonic polynomials in $\mathbb{R}^n$. Let us introduce some notation. For a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ we denote $P^{\mathbb{C}} \in \mathbb{C}[z_1, ..., z_n]$ its natural extension to $\mathbb{C}^n$. Denote $N(P) = P^{-1}(0)$ and $N^{\mathbb{C}}(P)$ the complexification of the variety $N(P)$, $N^{\mathbb{C}}(P) = (P^{\mathbb{C}})^{-1}(0)$. We have $N(P) = N^{\mathbb{C}}(P) \cap \mathbb{R}^n$.

Lemma 4.1. *Let $Q \neq 0$ be a harmonic divisor in $\mathbb{R}^n$ with real coefficients and*

$$Q^{\mathbb{C}} = A_1^{\mathbb{C}} \ldots A_q^{\mathbb{C}}$$

be the decomposition into a product of irreducible factors (over $\mathbb{C}$). Then the following is true:

(1) *All polynomials $A_j^{\mathbb{C}}$ are distinct and have real coefficients, so that decomposition above gives also the irreducible decomposition in $\mathbb{R}[x_1, \cdots, x_n]$.*
(2) *Any irreducible component $N_j = N(A_j)$ of the variety $N(Q)$ is an $(n-1)$-dimensional real algebraic variety in $\mathbb{R}^n$. The complement $\mathbb{R}^n \setminus N_j$ is disconnected.*
(3) *If $G \in \mathbb{R}[x_1, \cdots, x_n]$ vanishes on an open subset $U \subset N_j$ of some irreducible component N_j then A_j divides G. In particular, Q divides any polynomial with real coefficients, vanishing on $N(Q)$.*

Proof. To prove (1), let $(Q^{\mathbb{C}})^*(z) = \overline{Q^{\mathbb{C}}}(\overline{z})$ be the polynomial with complex conjugate coefficients. By assumption $(Q^{\mathbb{C}})^* = Q^{\mathbb{C}}$ and therefore if $(A_i^{\mathbb{C}})^* \neq A_i^{\mathbb{C}}$ for some i, then both $(A_i^{\mathbb{C}})^*$ and $A_i^{\mathbb{C}}$ are presented in the decomposition of the polynomial $Q^{\mathbb{C}}$. Then $Q(x)$, $x \in \mathbb{R}^n$, is divisible by $A_i(x)A_i^*(x) = |A_i(x)|^2$. This is impossible since Q is a harmonic divisor and the Choquet-Brelot theorem [BC] states that no nonnegative polynomial in $\mathbb{R}^n$ can divide a nonzero harmonic polynomial. Thus $A_i^{\mathbb{C}} = (A_i^{\mathbb{C}})^*$, $i = 1, \ldots, q$, which means that all $A_i^{\mathbb{C}}$ and therefore A_i have real coefficients. In addition, this implies that all the polynomials A_i are distinct as otherwise Q would have a nonnegative divisor A_i^2.

We now prove (2). If some irreducible component $N_i = N(A_i)$ has lower dimension, then its complement $\mathbb{R}^n \setminus N_i$ would be connected. The polynomial A_i preserves sign in $\mathbb{R}^n \setminus N_i$ and vanishes on N_i. Since A_i is a harmonic divisor, this is again impossible due to the Choquet-Brelot theorem.

To prove (3), denote for simplicity $N = N_j$. We can assume that U consists only of smooth points of N and $U = B \cap N$ where $B = B(a, r)$ is a ball in $\mathbb{R}^n$. Denote $\hat{B}$ the corresponding complex ball in $\mathbb{C}^n$, with the same center and radius, so that $B = \hat{B} \cap \mathbb{R}^n$. The $\hat{B} \cap N^{\mathbb{C}}$ is an open subset of the complex algebraic variety $N^{\mathbb{C}} \subset \mathbb{C}^n$ and the complex polynomial $G^{\mathbb{C}}$ vanishes on the $(n-1)-$dimensional real smooth submanifold $B \cap N = \hat{B} \cap N^{\mathbb{C}} \cap \mathbb{R}^n$ of the complex manifold $\hat{B} \cap N^{\mathbb{C}}$. Due to the uniqueness theorem for holomorphic functions, the polynomial $G^{\mathbb{C}}$ vanishes on $\hat{B} \cap N^{\mathbb{C}}$. Then $G^{\mathbb{C}}$ vanishes on the Zariski closure of $\hat{B} \cap N^{\mathbb{C}}$ which is exactly the irreducible component $N^{\mathbb{C}}$. By Hilbert's Nullstellensatz this implies that the defining irreducible polynomial $A_j^{\mathbb{C}}$ divides $G^{\mathbb{C}}$ and, as $A_j^{\mathbb{C}}$ has real coefficients, A_j divides G. □

Lemma 4.2. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$. Then*

a) *The set $S(f)$ is an algebraic variety in $\mathbb{R}^n$ contained in the zero sets of a nonzero harmonic polynomial.*
b) *$S(f) = S_0 \cup V$, where V is an algebraic variety of $\operatorname{codim} V > 1$, $S_0 = N(Q)$ and Q is a harmonic divisor.*

Proof. According to Lemma 2.3, $S(f) = \{x \in \mathbb{R}^n : (f * u)(x) = 0,\ u \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)\}$. Since $\operatorname{supp} f$ is compact, $u \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$ can be replaced by any radial C^∞-function. Let $u(y) = u_m(y) = |y|^{2m}$, $m = 0, 1, \ldots$. The functions $\{u_m\}$ constitute a complete system in the space C^∞_{rad} on any compact set, hence

$$\begin{aligned} S(f) &= \{x \in \mathbb{R}^n : (f * u_m)(x) = 0, m \in \{0\} \cup \mathbb{N}\} \\ &= \bigcap_{m=0}^{\infty} N(P_m), \end{aligned}$$

where $P_m = f * u_m$. By the Hilbert Finiteness Theorem [VW] $S(f)$ is defined by a finite family of polynomials P :

$$S(f) = \bigcap_{m=m_0}^{M} N(P_m).$$

Here $m_0 = \min\{m : P_m \neq 0\}$.

Let P be the greatest common polynomial divisor (over $\mathbb{C}$) of all the polynomials $P_m^{\mathbb{C}}$, $\quad m = m_0, \cdots, M$. Then the variety $N(P)$ is the union of all common irreducible components of the varieties $N(P_m^{\mathbb{C}})$. Since the intersection of any two distinct irreducible components has at any smooth point complex codimension greater than 1, then the intersection of all the complex algebraic varieties $N(P_m)$ is representable as the union of the zero variety of P, $N(P)$, and a variety of codimension greater than 1:

$$\bigcap_{m=m_0}^{M} N(P_m^{\mathbb{C}}) = N(P) \cup W\,,\ \operatorname{codim}_{\mathbb{C}} W \geq 1.$$

Denote Q the restriction of P to the real space $\mathbb{R}^n$, so that $N(Q) = N(P) \cap \mathbb{R}^n$. Then intersecting with the real space $\mathbb{R}^n$ yields

$$S(f) = \bigcap_{m=m_0}^{M} N(P_m) = (\bigcap_{m=m_0}^{M} N(P_m^{\mathbb{C}})) \cap \mathbb{R}^n = N(Q) \cup V,$$

where $V = W \cap \mathbb{R}^n$ and $\operatorname{codim}_{\mathbb{R}} V \geq \operatorname{codim}_{\mathbb{C}} W \geq 1$.

It remains to prove that Q is a harmonic divisor. One proves that by the following argument.

Since $\Delta|y|^{2m} = c_m|y|^{2(m-1)}$, $c_m = 2m(2m+n-2)$, we have that the Laplace operator Δ acts on the polynomial P_m by

$$\Delta P_m = \Delta(f * u_m) = f * \Delta u_m = c_m(f * u_{m-1}) = c_m P_{m-1}.$$

Therefore, $\Delta P_{m_0} = c_m P_{m_0-1} = 0$ and we conclude that P_{m_0} is harmonic. Since Q divides P_{m_0} then Q is a harmonic divisor. $\square$

§5. Geometric Structure of $S(f)$ for $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$.

From now on the distribution f is assumed of finite support, $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, and $f \neq 0$. The distribution f is determined by a finite set of differential operators D_i, $i = 1, \dots d$, with constant coefficients, and points $a_1, \dots, a_d \in \mathbb{R}^n$, and acts on test functions $\varphi \in C^\infty(\mathbb{R}^n)$ by

$$\langle f, \varphi \rangle = \sum_{i=1}^{d} (D_i\varphi)(a_i).$$

The points $a_1, \dots, a_d$ constitute $\operatorname{supp} f$.

Definition. Let $x \in \mathbb{R}^n$ and let $a \in \operatorname{supp} f$. We say that x *simply touches* $\operatorname{supp} f$ at the point a if the sphere $S(x, |x-a|)$ intersects $\operatorname{supp} f$ only at the point a. We say that x *multiply touches* $\operatorname{supp} f$ at the point a if $S(x, |x-a|) \cap \operatorname{supp} f$ contains at least two points.

Clearly, the set of simply touching points is open in $\mathbb{R}^n$ because $\operatorname{supp} f$ is finite.

Let S_0, V and Q be as in Lemma 4.2 and $A_i, i = 1, \dots, q$, the irreducible divisors of the polynomial Q. All the polynomials A_i have real coefficients due to Lemma 4.1.

Lemma 5.1. *Let $N_j = N(A_j)$ be an irreducible component of the algebraic variety S_0. If N_j contains a point x_0 which simply touches* $\operatorname{supp} f$ *at some point $a \in \operatorname{supp} f$ then the polynomial $A_j(x+a)$ is homogeneous and, correspondingly, N_j is an affine cone with vertex a.*

Proof. Denote for simplicity $A = A_j$, $N = N_j$. There exists a neighborhood $U \subset N$ of x_0, consisting of points simply touching $\operatorname{supp} f$ at the point a. We can take a smaller subset and assume that U consists of smooth points of N.

If U and $\varepsilon > 0$ are sufficiently small, then each sphere $S(x, |x-a|+t)$, $x \in U$, $0 < |t| < \varepsilon$, contains no point from $\operatorname{supp} f$. Now we can apply Theorem 3.2 and conclude that the point a must be a self-mirror for the tangent planes $T_x(N)$, $x \in U$, that is, $a \in T_x(N)$ for each $x \in U$.

Then the vector $x-a$ is orthogonal to the normal vector $\nabla A(x)$ to $T_x(N)$, and therefore the polynomial

$$G(x) = (x-a, \nabla A(x))$$

vanishes for all $x \in U$. Lemma 4.1,(3), implies that A divides G and as $\deg G \leq \deg A$, we have $G = \lambda A$, $\lambda \in \mathbb{R}$. Thus we obtain the Euler equation

$$\sum_{i=1}^{n} (x_i - a_i) \frac{\partial A}{\partial x}(x) = \lambda A(x)$$

which implies that the polynomial $A(x+a)$ is λ-homogeneous, where automatically $\lambda = \deg A$. $\square$

Corollary 5.2. *If f is supported at one point a, then S_0 is an affine (algebraic) cone with the vertex a.*

Indeed, in this case every point x of any irreducible component N_j simply touches $\operatorname{supp} f$ at a, and therefore $S = \bigcup_j N_j$ is a cone with respect to a.

Lemma 5.3. *Let N_j be an irreducible component of S_0. If there exists an open set $U \subset N_j$ consisting of points which multiply touch* $\operatorname{supp} f$, *then N_j is a hyperplane, bisector between some two points a, $a' \in \operatorname{supp} f$.*

Proof. Denote $N = N_j = N(A_j)$. Choosing U smaller, one can assume that all points of U are smooth points of N. Pick $x \in U$. By the assumption, there exists a point $a \in \operatorname{supp} f$ such that the sphere $S(x, |x-a|)$ contains another point $a' \in \operatorname{supp} f$, $a' \neq a$. Then $|x-a| = |x-a'|$ and $(x, a-a') = \frac{1}{2}(|a|^2 - |a|^2)$. Therefore, x belongs to the hyperplane $H_{a,a'}$ which is a bisector between the points a and a'.

Thus, U is contained in the (finite) union of the hyperplanes $H_{a,a'}$ and, as U is an $(n-1)$-dimensional smooth manifold, U is contained in one hyperplane $H_{a,a'}$. Then the entire irreducible component N coincides with the hyperplane $H_{a,a'}$. $\square$

Proposition 5.4. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $f \neq 0$. If $S(f)$ contains hyperplanes $H_1, \dots, H_\ell$ then $S(f)$ contains the Coxeter system Σ generated by the reflections σ_j about H_j.*

The hyperplanes $H_1, \dots, H_\ell$ have a common point, and the Coxeter group $W(\Sigma)$ is finite. The distribution f is odd with respect to reflections $\sigma \in W(\Sigma)$, $f \circ \sigma = -f$, and the sets $S(f)$ and $\operatorname{supp} f$ *are $W(\Sigma)$-invariant.*

Proof. The definition (1.3), $S(f) = \bigcap_{\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)} S(f * \varphi)$, implies that it suffices to prove the proposition for regular f. By Lemma 2.1 the spherical transform $\hat{f}(x,r) = 0$ for all $x \in H_i$, $i = 1, \dots, s$, and $r > 0$.

A *reflection principle* is proved in [CH, II, pp. 699 *ff.*] (in a slightly different formulation) that states that if a function integrates to zero on all spheres centered on a hyperplane, then the function is odd about that hyperplane. This reflection principle shows that the function f is odd around each hyperplane H_i, $f \circ \sigma_i = -f$, $i = 1, \ldots, \ell$.

Since $S(f \circ \sigma_i) = \sigma_i(S(f))$ then $S(f)$ is σ_i-invariant, $i = 1, \ldots, \ell$, and $S(f)$ contains the generated Coxeter system Σ. The skew-symmetry $f \circ \sigma = -f$ is preserved for the reflections $\sigma \in W(\Sigma)$.

The Coxeter group $W(\Sigma)$ has a finite number of mirrors because $S(f)$ is an algebraic variety. Then all $W(\Sigma)$-orbits are finite and $W(\Sigma)$ is a finite group. By Kakutani's theorem [DS, Ch.8, S.10, Th. 8.] the convex hull of any orbit contains a fixed point of the group $W(\Sigma)$. This point a belong to any hyperplane H_i in Σ, and therefore Σ is a cone with the vertex a. Finally, the group $W(\Sigma)$ is finite because it is generated by reflections and has a finite set of mirrors [GB, Prop. 4.1.3]. $\square$

§6. Proof of Theorem 1.

Fix $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, $f \neq 0$ and let $S(f)$, S_0, V, Q be as above. We assume that $S_0 \neq \emptyset$.

The statement (a) of Theorem 1 is already proved in Lemma 4.2a). Our next goal is to prove that S_0 is an affine cone.

Let $Q = A_1 \ldots A_q$ be the irreducible composition of the polynomial Q, and $S_0 = N_1 \cup \cdots \cup N_q$ be the corresponding decomposition of the algebraic variety $S_0 = N(Q)$ into irreducible components $N_i = N(A_i)$.

6.1. Structure of the Component K. Denote by K the union of all irreducible components N_i such that $\operatorname{supp} f \subset N_i$. Let $a \in \operatorname{supp} f$. Then a is a non-isolated point of N_i as $\mathbb{R}^n \setminus N_i$ is disconnected due to Lemma 4.1. Take $x \in N_i$ as close to a so that $S(x, |x-a|) \cap \operatorname{supp} f = \{a\}$. The point x simply touches $\operatorname{supp} f$ at the point a and Lemma 5.1 says that the polynomial $A_j(x+a)$ is homogeneous, and the component N_j is a cone with the vertex a.

Thus N_j is conical about any point $a \in \operatorname{supp} f$. Then N_j is a cone with the edge $L = \text{span } (\operatorname{supp} f)$. This follows from geometric arguments or it can be proved in terms of the defining polynomial A_j. Indeed, applying a rigid motion of $\mathbb{R}^n$, we can assume that $L = \mathbb{R}^k \times \{0\}$, $k = \dim L$. Then for any $a \in \operatorname{supp} f$, the Euler equation holds:

$$(x, \nabla A_j(x)) = (a, \nabla A_j(x)), \quad x \in \mathbb{R}^n.$$

By linearity, it is true for all $a \in L$.

Substituting $a = 0$, we obtain that A_j is homogeneous and also $(a, \nabla A_j(x)) = 0$, $x \in \mathbb{R}^n$, which implies $\frac{\partial A_j}{\partial x_1}(x) = \cdots = \frac{\partial A_j}{\partial x_k}(x) = 0$, due to arbitrariness of $a \in \mathbb{R}^k \times \{0\}$.

If $k = n$, then $A_j = \text{const} \neq 0$ ($A_j = 0$ would imply $Q = 0$, $S(f) = \mathbb{R}^n$ and, correspondingly, $f = 0$). Therefore, $N_j = \emptyset$, and we obtain that $K = \emptyset$ if $k = n$.

If $k = n-1$, then $A_j(x) = \tilde{A}_j(x_n)$ and $N_j = \{x_n = 0\}$.

If $k < n-1$, then $A_j(x_1, \ldots, x_n) = \tilde{A}_j(x_{k+1}, \ldots, x_n)$ and $N_j = \mathbb{R}^k \times N(\tilde{A}_j)$. Since $\tilde{A}_j$ is homogeneous, then $N(\tilde{A}_j)$ is a cone in $\mathbb{R}^{n-k}$. Then N_j is a cone with the edge L, and we have proven the assertion about K in (c).

6.2. Structure of the Component Σ. Let Σ be the union of all components N_j which do not entirely contain $\operatorname{supp} f$. In other words, Σ includes all the irreducible components of S_0 which are not in K. So $S_0 = \Sigma \cup K$. Let N_j be such a component, U is open in N_j and consists of smooth points. Let $x \in U$ and $a \in \operatorname{supp} f \backslash N_j$. Then the sphere $S(x, |x-a|)$ contains another point from $\operatorname{supp} f$ since otherwise x simply touches $\operatorname{supp} f$ at the point a and $a \in N_j$ according to Lemma 5.2. Thus U consists of points multiply touching $\operatorname{supp} f$, and N_j is a bisector hyperplane between two points $a, a' \in \operatorname{supp} f \setminus N_j$, according to Lemma 5.2.

We have proven that Σ is a union of hyperplanes $\Sigma = H_1 \cup \cdots \cup H_\ell$. Proposition 5.4 yields that Σ is a Coxeter system. Also $\Sigma \cup K$ is a Coxeter system if $\dim(\operatorname{span}\ f) = k-1$ and, correspondingly, K is a hyperplane. This proves (c) and (d).

6.3. Conical Structure of the Set S_o. Now we want to prove that S_0 has conical structure as claimed in (b). We have $S_0 = \Sigma \cup K$ where Σ is a cone with respect to any point in $H_1 \cap \cdots \cap H_\ell$, and K is a cone with respect to any point in $L = \operatorname{span}\ (\operatorname{supp} f)$.

To prove that S_0 is a cone it suffices to prove that $H_1 \cap \cdots \cap H_\ell \cap L \neq \emptyset$. Lemma 5.4 claims that $\operatorname{supp} f$ is invariant under the action of the Coxeter group $W(\Sigma)$. Then the convex hull of $\operatorname{supp} f$ possesses the same invariance. The invariant set conv hull$(\operatorname{supp} f)$ is compact and Kakutani's theorem implies that there exists a $W(\Sigma)$-fixed point $a \in$ conv hull$(\operatorname{supp} f) \subset L$. Clearly, a belongs to any hyperplane H_i, $i = 1, \ldots, \ell$.

Thus S_0 is an affine cone with the vertex $a \in$ conv hull$(\operatorname{supp} f)$. Let Q be the defining polynomial for S_0 introduced in Lemma 4.2, and Φ is a nonzero harmonic polynomial ($\Phi = P_{m_0}$ in the proof of Lemma 4.2), having Q as a divisor. Since S_0 is a cone with respect to a, then $P(x) = Q(x-a)$ is homogeneous (Lemma 5.1 says that all the irreducible factors of P are homogeneous). It remains for us to note that P divides the harmonic polynomial $\Psi(x) = \Phi(x+a)$, and $S_0 = a + N(P)$. Since P is homogeneous, then P divides any homogeneous term in the decomposition of Ψ and therefore P is a divisor of some nonzero harmonic homogeneous polynomial in $\mathbb{R}^n$. This completes the proof of Theorem 1.

6.4. Geometric Conditions on $S(f)$. We summarize the geometric essence of what we have proven in the following theorem. The proof involves the microlocal arguments we have developed using Theorem 3.2 as well as the algebraicity of the set $S(f)$, which is proved in Section 4 and enables us to analytically continue a locally conical set to a globally conical set.

Theorem 6.1 (Support Theorem). *Let $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, $f \neq 0$. Assume $S = S(f) \neq \emptyset$. Assume there are regular points in S, and let $x_0 \in S$ be a regular point.*

Let A be a connected real-analytic hypersurface in $\mathbb{R}^n$ such that $x_0 \in A \subset S$. Let T_{x_0} be the hyperplane tangent to A at x_0. There are two possibilities.

(a) *For some $a_0 \in \operatorname{supp} f$, $a_0 \notin T_{x_0}$. In this case $A \subset T_{x_0} \subset S$ and $\operatorname{supp} f$ is symmetric about T_{x_0}. Furthermore, f is odd about T_{x_0}.*

(b) *Or, $\operatorname{supp} f \subset T_{x_0}$. In this case, near x_0, S is conical about $L = \operatorname{span}\ \operatorname{supp} f$. Precisely, A generates a subset of S that is conical with edge L. In this case, $k = \dim L < n$.*

§7. Sufficient Conditions for Stationary Sets.

Theorem 1 says that the stationary sets $S(f)$ for $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ may consist of three parts: a low-dimensional variety V, a Coxeter system Σ, and a cone K having all points in span $(\operatorname{supp} f)$ as vertices. In addition, the union $S_0 = \Sigma \cup K$ must be a cone containing the zero set of some shifted harmonic homogeneous polynomial and the entire stationary set $S(f) = \Sigma \cup K \cup V$ must belong to the zero set of some nonzero harmonic (not necessarily homogeneous, if V is not a cone with the common vertex with S_0).

Now the question is whether all the possibilities are realizable. Namely, whether each of the sets Σ, K, V and any unions of sets of these three types are the stationary sets $S(f)$ for some $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ or, more generally, are contained in a stationary set?

Below we give positive answers for the sets Σ, K, V and $\Sigma \cup V$. The case of $\Sigma \cup K \cup V$, where each of the three sets is nonempty, remains unsolved.

Given a polynomial $G \in \mathbb{R}[x_1, \dots, x_n]$, denote by T_G the distribution $\langle T_G, \varphi \rangle = G(\partial)\varphi(0),\ \varphi \in \mathcal{D}(\mathbb{R}^n)$. Here $G(\partial) = G\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

Lemma 7.1. *Let Ψ be a homogeneous harmonic polynomial in $\mathbb{R}^n$, with complex coefficients. Then $\Psi(\partial)|x|^{2m} = d_{m,k}|x|^{2(m-k)}\Psi(x)$, if $\deg \Psi \le m$, and $\Psi(\partial)|x|^{2m} = 0$, if $\deg \Psi > m$. Here $d_{m,k} = \frac{2^k m!}{(m-k)!}$.*

Proof. The polynomial Ψ can be represented as a linear combination of polynomials $(c_1x_1 + \dots + c_nx_n)^k,\ k = \deg \Psi$, with $c_i \in \mathbb{C}$, $c_1^2 + \dots + c_n^2 = 0$. Thus it suffices to check the identity for such simple polynomials. This is done by straightforward computation. □

The following theorem shows that the stationary set generated by a homogeneous distribution (of finite order) supported at a single point coincides with common zeros of iterated Laplacians of the symbol of the corresponding differential operator:

Theorem 7.2. *For any homogeneous polynomial $G \in \mathbb{R}[x_1 \dots x_n]$, we have*

$$S(T_G) = \bigcap_{j \ge 0} N(\Delta^j(G)).$$

Proof. According to Lemma 2.1, $S(T_G)$ is the set of common zeros of the convolutions $G(\partial)u = T_G * u$, where $u \in \mathcal{D}_{\text{rad}}(\mathbb{R}^n)$. This set coincides with common zeros of the polynomials $G(\partial)|x|^{2m}$, $m = 0, 1, \cdots$ as the radial polynomials $|x|^{2m}$ form a complete system in C^∞_{rad} on any compact set. Thus, $S(T_G) = \{x \in \mathbb{R}^n : G(\partial)|x|^{2m} = 0,\ m = 0, 1, \dots\}$.

Represent G in the form

$$G(x) = h_k(x) + |x|^2 h_{k-2}(x) + |x|^4 h_{k-4}(x) + \dots,\ k = \deg G, \tag{7.1}$$

where h_{k-2j} is a harmonic homogeneous polynomial of degree $k - 2j$.

We have for $m \geq j$:

$$\begin{aligned}(|x|^{2j}h_{k-2j})(\partial)|x|^{2m} &= h_{k-2j}(\partial)\Delta^j|x|^{2m} \\ &= c_m c_{m-1} \dots c_{m-j+1} h_{k-2j}(\partial)|x|^{2(m-j)},\ c_i = 2i(2i+n-2).\end{aligned}$$

This expression vanishes for $m < j$. We proceed by using Lemma 7.1:

$$G(\partial)|x|^{2m} = \sum_{j=k-m}^{[k/2]} a_{mj}|x|^{2(m+j-k)}h_{k-2j}(x)\ ,$$

where $a_{mj} = d_{m,k-2j}c_m c_{m-1} \dots c_{m-j+1}$ and $2m \geq k$.

We can assume that $x \neq 0$ and as all the polynomials under consideration are homogeneous, we can take $|x| = 1$. Now take $m = k, k-1, \cdots, [(k+1)/2]$ and consider the system of $k - [(k+1)/2] + 1 = [k/2] + 1$ linear equations

$$G(\partial)|x|^{2m} = \sum_{j=k-m}^{[k/2]} a_{mj}h_{k-2j}(x) = 0, k \geq m \geq\ [(k+1)/2],$$

for $[k/2]+1$ unknown $h_{k-2j}(x)$, $j = 0, \cdots, [k/2]$. We obtain the linear system with upper triangular matrix having nonzero diagonal entries and therefore conclude that the condition $G(\partial)|x|^{2m} = 0$ for all m is equivalent to $h_k(x) = h_{k-2}(x) = \cdots = 0$.

In turn, the last equalities hold if and only if $G(x) = \Delta G(x) = \Delta^2 G(x) = \cdots = 0$. To check this, observe that $\Delta(|x|^{2j}h_{k-2j}) = c_j|x|^{2(j-1)}h_{k-2j} + 4|x|^{2(j-1)}(k-2j)h_{k-2j} = (c_j + 4(k-2j))|x|^{2(j-1)}h_{k-2j}$. We have used here the Euler equation for homogeneous polynomial h_{k-2j} and also its harmonicity.

Then, applying the iterated Laplacians to both sides of (7.1), we obtain

$$\Delta^s G(x) = \sum_{j \geq s} b_{s,j}|x|^{2(j-s)}h_{k-2j}(x),$$

where $b_{s,j} = (c_j + 4(k-2j))(c_{j-1} + r(k - 2_j + 2)) \dots (c_{j-s+1} + 4(k - 2j + 2s - 2))$. The matrix $b_{s,j}$ is again upper triangular and nondegenerate; hence $\Delta^s G(x) = 0$, $s = 0, 1, \dots$ is equivalent to $h_{k-2_j}(x) = 0$, $j = 0, 1, \dots$. This completes the proof. □

The following two corollaries prove that zero sets of harmonics are stationary sets of a homogeneous distribution supported at a single point and describe all such distributions:

Corollary 7.3. *If Ψ is a homogeneous harmonic polynomial, then $N(\Psi) = S(T_\Psi)$.*

Corollary 7.4. *Let Ψ be a homogeneous harmonic polynomial and G a polynomial in $\mathbb{R}^n$. Then $N(\Psi) \subset S(T_G)$ if and only if Ψ divides all the polynomials G, ΔG, $\Delta^2 G, \dots$.*

Proof. Since Ψ is homogeneous, then it is easy to check that $N(\Psi) \subset S(T_G)$ is equivalent to $N(\Psi) \subset S(T_{G_m})$, where G_m is any homogeneous term of G. In turn, by Theorem 7.2, this is equivalent to $G_m, \Delta G_m, \Delta^2 G_m, \dots$ vanishing on $N(\Psi)$. Vanishing on zeros of a harmonic polynomial is equivalent to divisibility (see Lemma 4.1); therefore all the homogeneous terms G_m, along with their iterated Laplacians, are divisible by Ψ. This proves the corollary. □

Finally, any low-dimensional real algebraic variety can be stationary for some solution of the wave equation with point supported initial data and, moreover, any Coxeter system can be added:

Theorem 7.5. [A] *Let V be an algebraic variety in $\mathbb{R}^n$, $\operatorname{codim} V > 1$. Let Σ be either empty or a Coxeter system of hyperplanes. Then there exists a nontrivial polynomial $G \in \mathbb{R}[x_1, \dots, x_n]$ such that $\Sigma \cup V \subset S(T_G)$.*

Remark. In order to prove that $N(\Psi) \cup V$, *where* Ψ is a homogeneous harmonic polynomial and $\operatorname{codim} V > 1$, can be realized as a stationary set, it would be sufficient to prove, according to Corollary 7.4, that the set of homogeneous polynomials G such that Ψ divides all $\Delta^s G$, $s = 0, 1, \dots$ is big enough to satisfy the additional condition on the low-dimensional part: $\Delta^s G\big|_V = 0$, $s = 0, 1, \dots$. Because of what has been proven in Theorem 7.2, this means that all the harmonic homogeneous polynomials h_{k-2j} in the decomposition (7.1) are divisible by Ψ and vanish on V. However, showing the space of harmonic homogeneous polynomials h divisible by a given harmonic Ψ is big enough turned out to be very nontrivial in $\mathbb{R}^n$ for $n > 2$ (*cf.* [A]). We even do not know whether this space is always infinite dimensional or not.

§8. The Case of Balls.

Similar arguments can be used to prove a theorem similar to Theorem 1 if $\operatorname{supp} f$ is (roughly) the disjoint union of balls. Let $\mathcal{E}'_D(\mathbb{R}^n)$ be the set of distributions whose support is contained in the union of a finite number of disjoint closed balls and whose support contains the boundaries of these balls. Distributions in $\mathcal{E}'_D(\mathbb{R}^n)$ can be arbitrary inside each closed ball, but their support must contain the entire boundary of each ball. Our next theorem is the analogue of Theorem 6.1 for $\mathcal{E}'_D(\mathbb{R}^n)$.

Theorem 8.1 (Support Theorem). *Let $f \in \mathcal{E}'_D(\mathbb{R}^n)$, $f \neq 0$. Assume $S = S(f) \neq \emptyset$. Assume there are regular points in S, and let $x_0 \in S$ be a regular point and $x_0 \notin \operatorname{supp} f$. Let A be a connected real-analytic hypersurface in $\mathbb{R}^n$ such that $x_0 \in A \subset S$. Let T_{x_0} be the hyperplane tangent to A at x_0. Let C be the set of* *centers of the disks making up* $\operatorname{supp} f$. *There are two possibilities.*

(a) *For some* $c_0 \in C$, $c_0 \notin T_{x_0}$. *In this case* $A \subset T_{x_0} \subset S$ *and* $\operatorname{supp} f$ *is symmetric about* T_{x_0}. *Furthermore,* f *is odd about* T_{x_0}.

(b) *Or,* $C \subset T_{x_0}$. *In this case, near* x_0, S *is conical about* $L = \operatorname{span} C$. *Precisely,* A *generates a subset of* S *that is conical with edge* L. *In this case,* $k = \dim L < n$.

Proof. The proof is similar to the proof of Theorem 6.1. Consider case (a). Let r_0 be the smallest radius such that $S(x_0, r_0)$ meets $\operatorname{supp} f$ on a disk D_0 not centered on T_{x_0}. This implies that the point, y_0, of tangency of $S(x_0, r_0)$ and D_0, is not self mirror. Then, by Theorem 3.2, its mirror point, y_1, must also be in $\operatorname{supp} f$. If this mirror point was in a disk in $\operatorname{supp} f$ centered on T_{x_0}, then y_0 would lie in the same disk by symmetry; but, the disks in $\operatorname{supp} f$ are disjoint. So, by the choice of r_0, y_0 and y_1 must both be boundary points of disks D_0 and D_1 in $\operatorname{supp} f$ that are not centered on T_{x_0}. Assume the disk D_j has center c_j and radius t_j, $j = 0, 1$. We show that $t_0 = t_1$, that $T_{x_0} \subset S$, and that f is odd about T_{x_0}.

For each $x \in A \setminus D_0$ let $S(x)$ be the disk of smaller radius tangent to ∂D_0 and let r_x be its radius. We claim that there is a neighborhood of x_0, $U \subset A$, such that for every point $x \in U$, the sphere $S(x)$, that is tangent to D_0, is also tangent to D_1. We prove it by contradiction. If $x_1 \in A$ is close enough to x_0 and $S(x_1)$ is tangent to D_0 at a point $\tilde{y}_0$, then the mirror point, $\tilde{y}_1$ to $\tilde{y}_0$ must be in D_1 by Theorem 3.2 and the assumption that the disks in $\operatorname{supp} f$ are all disjoint. Again, since these disks are disjoint (and perhaps by making x_1 closer to x_0), we can find an $r < r_{x_1}$ such that $S(x_1, r)$ does not meet D_0 but is tangent to D_1 at a point near y_1. Because (x_1, r) is sufficiently close to (x_0, r_0), the mirror point to the point of tangency on D_1 is not in $\operatorname{supp} f$. This contradiction to the support theorem, Theorem 3.2, explains the claim.

By the claim above, if $x \in U$ and $r > 0$ is such that $S(x, r)$ is tangent to D_0, then it is tangent to D_1. This means x must satisfy the equation

$$|c_0 - x| - |c_1 - x| = t_0 - t_1 \tag{8.1}$$

First, assume $t_0 \neq t_1$. Equation (8.1) is the equation of a hyperboloid of two sheets and so U is an open set on a hyperboloid. We will show that the entire hyperboloid of two sheets is contained in $S(f)$. Let $G(x) = 0$ be the second order polynomial (irreducible over $\mathbb{R}$) that defines the hyperboloid and let the set $S(f) = N(Q) \cup V$ as in Lemma 4.2, where Q is a harmonic divisor and V is a low-dimensional variety. Clearly, the hypersurface A and therefore the set U belong to the $(n-1)$-dimensional part $S_o = N(Q)$.

Thus, the polynomial G vanishes on an open subset of $N(Q)$, then G vanishes on an open subset of some irreducible component of $N(Q)$. Now, by Lemma 4.1, G must vanish on the entire component and the irreducible factor defining the component is a divisor of G. As G is itself irreducible, the component is just the quadric $N(G)$ and we have the entire two-sheeted hyperboloid, S_1, in $N(Q)$. This case is eliminated by our next lemma.

Lemma 8.2. *Let S_1 be a regular real-analytic surface (possibly disconnected). Assume that S_1 contains two points, a and b, $a \neq b$, such that the segment $\overline{ab}$ is perpendicular to the tangent planes T_a and T_b to S_1 at the point a and b respectively. Then the spherical Radon transform defined by (2.1) is injective on S_1.*

This theorem is the $n-$dimensional generalization of the main support theorem of [AQ2], Theorem 4.1. The proof in $\mathbb{R}^n$ is the exact analogue of the proof on p. 397-398 of [AQ2] because the microlocal properties of the Radon transform are analogous, as shown above, and because the geometry is the same.

This implies $t_0 = t_1$ and U is an open set on a hyperplane. Now, using the fact that S is an algebraic variety (Lemma 4.2) allows us to infer $T_{x_0} \subset S$. Finally, using the reflection principle [CH, II, pp. 699 *ff.*], we see f is odd about T_{x_0}.

Case (b) is very similar to the case for finite support. Let D_0 be a disk in $\operatorname{supp} f$ with center $c_0 \in T_{x_0}$. We will show, near x_0, A generates a subset of S that is conical about c_0. For $x \in A \setminus D_0$ let $S(x)$ be the sphere of smaller radius that is tangent to ∂D_0. Let y_0 be the unique point of intersection of $S(x_0)$ and D_0. Then, by construction, y_0 is self-mirror. By smoothness of A, we can find a neighborhood $U \subset A$ of x_0 and a neighborhood $V \subset \mathbb{R}^n$ of y_0 such that for each $x \in U$ the T_x-mirror point to the single point $S(x) \cap D_0$ is also in V. By perhaps making U and V smaller, we can assume $(V \cap \operatorname{supp} f) \subset D_0$. So, for each $x \in U$, the mirror point to $S(x) \cap D_0$ must be in $\operatorname{supp} f$ (Theorem 3.2) and it must be in D_0. By convexity, this point is self-mirror. This implies that $c_0 \in T_x$ $\forall x \in U$. As with the case of finite support, we see that U generates a subset of S that is conical about c_0. Since this is true for all $x \in A \setminus \operatorname{supp} f$ and all centers in C, $A \setminus \operatorname{supp} f$ generates a subset of S that is conical about all points of L. $\square$

Theorem 8.1 can be used to prove a version of Theorem 1 at least for the part of $S(f)$ disjoint from $\operatorname{supp} f$. By analytic continuation, this gives information about the part of $S(f)$ in $\operatorname{supp} f$: the conical sets defining $S(f)$ continue into $\operatorname{supp} f$. As result we obtain Theorem 1 for the case when $\operatorname{supp} f$ is the union of finite number of disjoint balls. The geometry of the stationary set in this case is the same as is described in Theorem 1 for $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$.

§9. Concluding Remarks.

Theorem 1 asserts that for an initial distributions with finite support, the essential $(n-1)-$dimensional part of the stationary set is a cone. From Section 7, we learn this cone appears as the set of common zeros of spatial harmonics in the Fourier decomposition of the initial distribution. Correspondingly, this happens only when these harmonics simultaneously vanish on a large set (i.e., are coherent). More specifically, the cone may contain a system of Coxeter mirrors, if the initial data (sources) admit a corresponding symmetry. In this case vanishing of the solution of the wave equation on the mirrors is the result of cancelling of waves propagated by symmetric sources.

We expect that the stationary sets have a similar geometry for compactly supported initial data and, more generally, for distributions vanishing sufficiently fast at infinity.

The main difficulty in proving that is obtaining the conical structure of the essential part of stationary sets. This was done in [AQ1,AQ2] for $n = 2$, by using symmetry (mirror points) of the support of the initial data, given by the support theorem (Theorem 3.2), and the simple structure of zero sets of harmonic polynomials of two variables.

Lack of information about zero sets of harmonic polynomials of more than two variables was the main obstacle for us in extending our approach to describing stationary sets for compactly supported or rapidly decreasing initial data in $\mathbb{R}^n$. Nevertheless, we hope to succeed using a deeper analysis of the algebraic and geometric structure of stationary sets and by refining the microlocal results that go into the proof of Theorem 1 to be valid more generally, such as for rapidly decreasing functions.

Note, in conclusion, that the conical structure is related to rate of decay of the initial data at infinity and does not occur in general. For instance, radial time-harmonic solutions of the wave equation provide stationary sets which are concentric spheres with radii determined by the zeros of a Bessel function.

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