

LOCAL SINGULARITY RECONSTRUCTION FROM INTEGRALS OVER CURVES IN \mathbb{R}^3

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(Communicated by Margaret Cheney)

ABSTRACT. We define a general curvilinear Radon transform in \mathbb{R}^3 , and we develop its microlocal properties. Singularities can be added (or masked) in any backprojection reconstruction method for this transform. We use the microlocal properties of the transform to develop a local backprojection reconstruction algorithm that decreases the effect of the added singularities and reconstructs the shape of the object. This work was motivated by new models in electron microscope tomography in which the electrons travel over curves such as helices or spirals, and we provide reconstructions for a specific transform motivated by this electron microscope tomography problem.

1. Introduction. We will present the microlocal analysis to understand a tomographic transform that integrates over curves in space. Using these results, we will develop and test an algorithm to reconstruct singularities such as boundaries and cracks in objects from this data.

This work is motivated by new research in electron microscopy. In *electron microscopy tomography (ET)* an object is placed in an electron microscope and data are taken. Then, the object is rotated through a series of positions and a tomographic algorithm is used to reconstruct the shape of the individual molecules. The standard model of electron microscopy [24, 43, 7] assumes that electrons travel along straight lines through the object and the data are essentially integrals of the electrostatic potential along those lines. The model is the classical Radon line transform, and standard ET algorithms use this model. However, new practical results in ET [33, 36, 37] show that in larger fields of view, e.g., one micron square (in which one looks for proteins around 7nm in size and viruses around 30nm), electron paths farther from the center beam can be helical or spiral rather than linear. Therefore, a more accurate transform for large object ET must use integrals over

2000 *Mathematics Subject Classification.* Primary: 44A12, 92C55, 35S30; Secondary: 58J40.

Key words and phrases. Electron Microscopy, Curvilinear Radon Transform, Fourier Integral Operator, Microlocal Analysis.

The first author was supported by the U.S. National Science Foundation under grants DMS-0456858 and DMS-0908015 and the Swedish Wenner Gren Foundation. Some of this work was done while the first author was at MSRI. Tufts University FRAC provided travel and research support for this work. This work was begun while the second author was at Tufts University.

curves. The algorithm in [33, 36, 37] finds the curves as part of the reconstruction method and so a description of the curves could be available in practice.

There are no inversion formulas for general curvilinear transforms, and even for line transforms, inversion is impossible for the limited data in electron microscope tomography. Therefore, we will develop a singularity detection algorithm that will show the shape of the objects, not the values of their electrostatic potentials, from data over known curves. Our algorithm is of derivative-filtered backprojection type; that is, if $\mathcal{P}_{\mathbf{p}}$ is the forward operator (transform that collects the data—that integrates a function over electron paths) our reconstruction of an object f is

$$\mathcal{L}f = \mathcal{P}_{\mathbf{p}}^* D \mathcal{P}_{\mathbf{p}} f \quad (1)$$

where $\mathcal{P}_{\mathbf{p}}^*$ is an adjoint or backprojection for $\mathcal{P}_{\mathbf{p}}$ (8) and D is a differential operator acting on the data $\mathcal{P}_{\mathbf{p}} f$. The problem is that singularities can be added to the reconstruction (or masked), and in Section 3.3, we will use microlocal analysis to choose an operator D_g that will de-emphasize some of those singularities.

The standard case of linear electron paths will guide us. The original derivative-backprojection type algorithm was Lambda Tomography [9, 8, 46]. This algorithm gives excellent reconstructions from data over lines in the plane. It does not reconstruct the density of the object, f , but $\sqrt{-\Delta}f$. It shows singularities such as region boundaries very clearly, and with a contour factor added, reconstructions look like those from the standard algorithm, filtered back projection. It is a region of interest algorithm since, to reconstruct at a point in the plane, one needs data only over lines near that point.

For lines in \mathbb{R}^3 , the picture changes. The set of lines in \mathbb{R}^3 is four dimensional, but in many tomographic problems, including ET, one has data only over a three-dimensional set of lines. The best studied case is that of an *admissible Radon transform* [15]. We will discuss one such transform, the model for ET if the electron paths are linear, in Example 2.1. The cone beam X-ray transform is another admissible transform [19].

It is well-known that for admissible transforms on lines, $\mathcal{P}_{\mathbf{p}}^* \mathcal{P}_{\mathbf{p}}$ spreads singularities in well-defined ways. This was proven in the seminal work of Greenleaf and Uhlmann (e.g., [19, 21]) and then in subsequent work for cone beam tomography such as [12, 30]. This creates one of our challenges: to choose the differential operator D to decrease the effect of these added singularities.

Several authors have developed derivative-backprojection type algorithms for admissible Radon transforms on lines in \mathbb{R}^3 . The first such algorithm for cone beam tomography was developed in [34]. It provides very good images from X-ray CT data. As already noted, derivative backprojection algorithms can spread singularities, and by using a well-chosen differential operator D_g in the reconstruction, $\mathcal{P}^* D_g \mathcal{P} f$ can show the shape of the object clearly and decrease the magnitude of added singularities. Differential operators have been found that decrease all added singularities for the cone beam X-ray transform over lines (e.g., [29, 1, 30]) and for the transform over lines in electron tomography and slant-hole SPECT, Example 2.1, [40, 11, 41]. In [12, pp. 215-216], the authors suggest a pseudodifferential operator of order one that can be used to decrease singularities. Their operator is often but not always the square root of the differential operator used in [40, 41].

There are inversion methods and uniqueness results for classes of curves in the plane (e.g., [17, 38]), in \mathbb{R}^n (e.g., [44, 3, 4, 32, 15, 16]), and in manifolds (e.g.,

[45, 13]). Often the curves are assumed to be close to lines or to have some intrinsic symmetry.

Microlocal analysis has been applied in many integral geometric problems, starting with the seminal material in [22, 23]. It has been applied to the hyperplane transform (e.g., [39, 2]), to radar (e.g., [35, 10, 31]) sonar and seismic imaging (e.g., [6, 42]), admissible transforms (e.g., [19, 21, 30]) and other transforms (e.g., translations of curves in \mathbb{R}^3 [21]).

Important recent work, including uniqueness theorems and microlocal results, has been done for the Radon transform over magnetic geodesics [5, 26]. Their model is very close to the physical model for electron microscopy, but it does not apply to our setup. They assume there is a curve through every point in every direction and that the curves come from an ODE [26, first par. on p. 112] (although local data are considered in [26]), but this is not necessarily the case for our model. More importantly, since we rotate the object in a fixed electron beam to get our data, the field is rotated in relation to the object. In [5, 26], the field is fixed in relation to the object.

In Section 2 we give the required notation. In Section 3 we develop the microlocal analysis of our general transform and provide examples. In Section 3.1 we develop the precise microlocal theorems. In Section 3.2 we describe in a more geometric way how the operator can add or mask singularities. In Section 3.3, we use the analysis we have developed to choose a “good” differential operator D_g in our reconstruction operator (1) in the following sense: D_g will be chosen to de-emphasize nearby singularities that can be added when applying the backprojection \mathcal{P}^* . Finally in Section 4 we give reconstructions from our algorithm for helical trajectory paths.

2. Notation. In this section, we define the curvilinear Radon transform and provide the examples that motivated our research. In ET, a beam of electrons goes through an object and electron counts are recorded on a detector. Then, the object is successively rotated to other angles, θ , and more data are taken on the detector plane. For this reason we consider the following parametrization of curves. Let $]a, b[$ be an open interval and let

$$Y =]a, b[\times \mathbb{R}^2.$$

For each fixed $\theta \in]a, b[$ we specify a diffeomorphism

$$\mathbf{P}_\theta : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (2)$$

and for each $(\theta, \mathbf{y}) \in Y$, the curve (electron path) associated to (θ, \mathbf{y}) is

$$\gamma_{\theta, \mathbf{y}} := \{\mathbf{P}_\theta(t, \mathbf{y}) \mid t \in \mathbb{R}\}. \quad (3)$$

This implies that, for each θ , the curves $\gamma_{\theta, \mathbf{y}}$ for different \mathbf{y} are disjoint, they fill out \mathbb{R}^3 ($\cup_{\mathbf{y} \in \mathbb{R}^2} \gamma_{\theta, \mathbf{y}} = \mathbb{R}^3$), and each $\gamma_{\theta, \mathbf{y}}$ is diffeomorphic to \mathbb{R} (by the map $t \mapsto \mathbf{P}_\theta(t, \mathbf{y})$). We will provide more assumptions on \mathbf{P}_θ in Hypothesis 1. In ET, θ represents an angle between the electron beam and object (a tilt) for one data acquisition, and $\mathbf{y} \in \mathbb{R}^2$ represents a point on the detector plane for that tilt (see Example 2.1).

The map \mathbf{P}_θ naturally defines a projection map $\mathbf{p}_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\mathbf{p}_\theta(\mathbf{x}) := \text{proj}_{\mathbf{y}}(\mathbf{P}_\theta^{-1}(\mathbf{x})) \quad (4)$$

where $\text{proj}_{\mathbf{y}}$ is projection onto the \mathbf{y} coordinate. Note that $\gamma_{\theta, \mathbf{y}} = \mathbf{p}_\theta^{-1}(\{\mathbf{y}\})$

Using this notation, we define *the curvilinear Radon transform* for $f \in L^2(B)$ where B is a fixed ball

$$\mathcal{P}_{\mathbf{p}}f(\theta, \mathbf{y}) := \int_{\mathbf{x} \in \gamma_{\theta, \mathbf{y}}} f(\mathbf{x}) ds \quad (5)$$

where ds is the arc length measure on the curve $\gamma_{\theta, \mathbf{y}}$. This integral can also be written $\mathcal{P}_{\mathbf{p}}f(\theta, \mathbf{y}) := \int_{t \in \mathbb{R}} f(\mathbf{P}_{\theta}(t, \mathbf{y})) dt$ where dt is chosen to give arc length measure.

The *backprojection operator* for $\mathcal{P}_{\mathbf{p}}$ is defined as follows. First, we define, for $\mathbf{x} \in \mathbb{R}^3$, the set parametrizing all curves passing through \mathbf{x} :

$$S_{\mathbf{x}} := \{(\theta, \mathbf{p}_{\theta}(\mathbf{x})) \mid \theta \in]a, b[\} \quad (6)$$

The backprojection is an integral over $S_{\mathbf{x}}$. However, if $S_{\mathbf{x}}$ is not compact, as in our general setup, one needs a cutoff in θ so that the backprojection can be defined on all distributions on Y . With this in mind, we choose numbers in $]a, b[$: $c' < c < d < d'$, and let

$$\phi \in C_c^\infty(]a, b[), \quad 0 \leq \phi \leq 1, \quad \phi(\theta) = 1 \text{ for } \theta \in]c, d[, \quad \text{supp } \phi \subset (c', d') \quad (7)$$

Define for $g \in \mathcal{D}'(Y)$

$$\mathcal{P}_{\mathbf{p}}^*g(\mathbf{x}) := \int_{\theta \in]a, b[} \phi(\theta)g(\theta, \mathbf{p}_{\theta}(\mathbf{x}))d\theta. \quad (8)$$

Since we have not chosen measures on \mathbb{R}^3 and on Y , $\mathcal{P}_{\mathbf{p}}^*$ is not necessarily the operator dual to $\mathcal{P}_{\mathbf{p}}$.

Remark 1. If the map $(\theta, t, \mathbf{y}) \mapsto \mathbf{P}_{\theta}(t, \mathbf{y})$ (and hence $(\theta, \mathbf{x}) \mapsto \mathbf{p}_{\theta}(\mathbf{x})$) can be smoothly extended to a periodic function of period $b - a$ for $\theta \in \mathbb{R}$, then the cutoff function ϕ is not needed and the integral for $\mathcal{P}_{\mathbf{p}}^*$ will be over $[a, b]$ and the sets $S_{\mathbf{x}}$ will be smooth compact manifolds. This is the case in slant-hole SPECT, Example 2.1, when the curve is smooth and closed.

Obviously some conditions on \mathbf{p}_{θ} (equiv. \mathbf{P}_{θ}) are needed. For example, the transform would not be invertible if \mathbf{p}_{θ} did not depend on θ . To describe these conditions, we introduce notation and a hypothesis that we will use throughout the article.

Definition 2.1. We let $\partial_{\mathbf{x}}$ be the derivative (Jacobian) matrix of a function in \mathbf{x} , and $\partial_{\mathbf{y}}$ is defined similarly. Then ∂_{θ} , and ∂_t are first-derivatives in the one-dimensional variables θ and t respectively.

Let $\xi = (\xi_1, \xi_2, \xi_3)$ be a row vector in \mathbb{R}^3 and let $\eta = (\eta_1, \eta_2)$ be a row vector \mathbb{R}^2 . We define

$$\xi d\mathbf{x} := \xi_1 dx_1 + \xi_2 dx_2 + \xi_3 dx_3, \quad \eta d\mathbf{y} := \eta_1 dy_1 + \eta_2 dy_2$$

If M is a 2×3 matrix then ηM will be the row vector η times the matrix M . If M is an arbitrary matrix, when we write $\text{span}(M)$ we mean the span of the rows of M .

Hypothesis 1. We assume $(\theta, t, \mathbf{y}) \mapsto \mathbf{P}_{\theta}(t, \mathbf{y})$ is a C^∞ map from $]a, b[\times \mathbb{R} \times \mathbb{R}^2$ to \mathbb{R}^3 that satisfies the following conditions.

- (a) The map $Y \ni (\theta, \mathbf{y}) \mapsto \gamma_{\theta, \mathbf{y}}$ is one-to-one.
- (b) For each $\theta \in]a, b[$, $(t, \mathbf{y}) \mapsto \mathbf{P}_{\theta}(t, \mathbf{y})$ is a diffeomorphism from $\mathbb{R} \times \mathbb{R}^2$ to \mathbb{R}^3 .
- (c) For each $(\theta, \mathbf{y}) \in Y$ and all \mathbf{x}_0 and \mathbf{x}_1 in $\gamma_{\theta, \mathbf{y}}$, if $\mathbf{x}_1 \neq \mathbf{x}_0$, then

$$\partial_{\theta}\mathbf{p}_{\theta}(\mathbf{x}_0) - \partial_{\theta}\mathbf{p}_{\theta}(\mathbf{x}_1) \neq \mathbf{0} \quad \text{where } \mathbf{p}_{\theta}(\mathbf{x}) = \text{proj}_{\mathbf{y}}(\mathbf{P}_{\theta}^{-1}(\mathbf{x})).$$

(d) The 4×3 matrix $\begin{pmatrix} \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) \\ \partial_{\mathbf{x}} \partial_\theta \mathbf{p}_\theta(\mathbf{x}) \end{pmatrix}$ has maximal rank, three.

Remark 2. Each assumption above can be understood geometrically.

Hyp. 1(a) implies no two curves are the same.

Hyp. 1(b) implies the curves $\gamma_{\theta, \mathbf{y}}$ are foliated and are diffeomorphic to lines and that $\mathbf{x} \mapsto \mathbf{p}_\theta(\mathbf{x})$ is a submersion

Hyp. 1(c) has several implications. First, let \mathbf{x}_0 and \mathbf{x}_1 be in the curve $\gamma_{\theta, \mathbf{y}}$. If Hyp. 1(c) fails at \mathbf{x}_0 and \mathbf{x}_1 , then basically, as θ changes, $\gamma_{\theta, \mathbf{y}}$ moves the same direction and magnitude at these two points. This means that as θ moves infinitesimally, the motion of $\gamma_{\theta, \mathbf{y}}$ is the same at both points.

Second, Hyp. 1(c) implies the condition $\mathbb{R}^3 \ni \mathbf{x} \mapsto S_{\mathbf{x}}$ is one-to-one. The reason is that, if this map were not one-to-one, then there would be two points \mathbf{x}_0 and \mathbf{x}_1 such that all curves through \mathbf{x}_0 would go through \mathbf{x}_1 and vice versa. In this case, the derivatives in θ would be equal, $\partial_\theta \mathbf{p}_\theta(\mathbf{x}_0) = \partial_\theta \mathbf{p}_\theta(\mathbf{x}_1)$. If there were two such points, then the two points could not be distinguished by curves in Y , and so in some sense the value of functions at \mathbf{x}_0 could not be distinguished from the value at \mathbf{x}_1 by integrals over curves in Y .

Finally, in Remark 5 we will show that if Hyp. 1(d) fails at $(\theta, \mathbf{y}, \mathbf{x})$, then the normal plane to $\gamma_{\theta, \mathbf{y}}$ at \mathbf{x} is stationary as θ moves infinitesimally. These last observations suggest that if the hypotheses hold, then the curves $\gamma_{\theta, \mathbf{y}}$ move enough so that they can pick up singularities at \mathbf{x} .

Remark 3. Our setup can be generalized in several ways. First, the map \mathbf{P}_θ does not have to be defined in all of $]a, b[\times \mathbb{R} \times \mathbb{R}^2$ but can be defined on $]a, b[\times]r, s[\times \Omega$ where Ω is an open set in \mathbb{R}^2 and the data are defined on $]a, b[\times \Omega$ and the support of f is in the interior of the intersection $\cap_{\theta \in]a, b[} \mathbf{P}_\theta(]r, s[\times \Omega)$.

Second, our microlocal analysis is valid for any nonzero smoothly varying measure dt on the curves in the definition of $\mathcal{P}_{\mathbf{p}}$ because the resulting operator is an elliptic Fourier integral operator (FIO) independent of the smooth measure.

We now give two examples. The first is a general model for ET on lines [41] and the second is a helical transform motivated by large-field electron microscopy.

Example 2.1. The simplest model for electron microscope tomography assumes data are over lines parallel to a curve on the sphere S^2 , and we now put this transform into our framework [7]. Let $\mathbf{c} :]a, b[\rightarrow S^2$ be a smooth regular curve such that $\|\mathbf{c}'(\theta)\| \equiv 1$ and $\mathbf{c}(\theta) \cdot \mathbf{c}''(\theta) \neq 0$ for all $\theta \in]a, b[$. Now, for each $\theta \in]a, b[$ define $\alpha(\theta) = \mathbf{c}'(\theta)$ and $\beta(\theta) = \mathbf{c}(\theta) \times \alpha(\theta)$. For each $\theta \in]a, b[$ define $\mathbf{p}_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$\mathbf{p}_\theta(\mathbf{x}) = \begin{pmatrix} \alpha(\theta) \cdot \mathbf{x} \\ \beta(\theta) \cdot \mathbf{x} \end{pmatrix}. \quad (9)$$

Then, for $(\theta, \mathbf{y}) \in Y$, where $\mathbf{y} = (y_1, y_2)$, $\gamma_{\theta, \mathbf{y}}$ is the line through $y_1 \alpha(\theta) + y_2 \beta(\theta)$ and parallel to $\mathbf{c}(\theta)$. If \mathbf{c} parameterizes a smooth closed curve, then one does not need the cutoff function ϕ in the definition of $\mathcal{P}_{\mathbf{p}}^*$ (8) because the parametrization of \mathbf{c} (and hence the curves) is smooth even at the endpoints (see Remark 1). It is a straightforward exercise to show that this example satisfies Hypothesis 1.

If the curve \mathbf{c} is a non-equatorial latitude circle of the sphere, then the resulting transform is the transform in conical tilt ET or (with the addition of a smooth weight) slant hole SPECT [40]. If the curve \mathbf{c} is part of a great circle, then the transform is the one used in single axis electron microscope tomography [41].

In [40], the authors describe a derivative backprojection algorithm of the form (1) using a well-chosen differential operator that decreases the strength all added singularities. As we discuss in Sect. 3.4, this line complex is admissible and so that theory can be used to justify why this occurs, as was done in [11]. In contrast, the transform $\mathcal{P}_{\mathbf{p}}$ is, in general, not admissible, and so our reconstruction algorithm can only decrease the magnitude of nearby singularities.

Example 2.2. The motivation for this article was stimulating research by Albert Lawrence *et al.* [37] that showed that in large field electron microscopy, the electrons travel over helix-like curves. We now put this model into our framework.

For $\theta \in [0, 2\pi]$ let A_θ be the matrix

$$A_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \quad (10)$$

which defines a rotation around the x_1 axis through the angle θ . Define

$$p(\mathbf{x}) = \begin{pmatrix} \cos(\omega x_3) & -\sin(\omega x_3) \\ \sin(\omega x_3) & \cos(\omega x_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (11)$$

where ω is a constant. Finally, let

$$\mathbf{p}_\theta(\mathbf{x}) = p(A_\theta \mathbf{x}). \quad (12)$$

Then, for $\theta = 0$, the curve $\gamma_{0,\mathbf{y}}$ is a helix with pitch $2\pi\omega$ centered on the x_3 axis and going through the point $(\mathbf{y}, 0)$ on the x_1x_2 -plane. For other θ , these helices are rotated θ radians about the x_1 axis.

It is straightforward to check that Hypothesis 1 holds for this example. In this case, the dual transform does not need the cut off function $\phi(\theta)$ in the definition of $\mathcal{P}_{\mathbf{p}}^*$ if we allow all $\theta \in [0, 2\pi]$; this is a special case of Remark 1.

In section 4, we will provide reconstructions of simulations using our algorithm and this set of the curves.

It should be pointed out that several researchers have studied the microlocal mapping properties of Radon transforms over translations of a fixed curve (e.g., [18]). This transform can be put into our setup in cases in which the curves do not have too much symmetry.

3. Microlocal Analysis of the Curve Transform. The main theoretical results are in this section. First, we put our transform in the context of double fibration. Then, we show that $\mathcal{P}_{\mathbf{p}}$ is a FIO (Thm. 3.1) and we give its canonical relation. We define sets $\mathcal{A}(\mathbf{x}, \xi)$ and $\mathcal{B}_\theta(\mathbf{x})$ that will help us understand what our transforms do to singularities. We use this to describe the microlocal regularity properties of $\mathcal{P}_{\mathbf{p}}$ and our reconstruction operator \mathcal{L} (Thm. 3.3) and to show how \mathcal{L} can add singularities (Cor. 1). In Sect. 3.2 we give geometric characterization of the added singularities, and in Sect. 3.3 we describe a way to choose the differential operator D in our reconstruction operator \mathcal{L} .

The microlocal properties $\mathcal{P}_{\mathbf{p}}$ can be understood using the *double fibration* of Gelfand [14] and Helgason [25]. The double fibration for $\mathcal{P}_{\mathbf{p}}$ starts with the *incidence relation*

$$Z := \{(\theta, \mathbf{y}; \mathbf{x}) \in Y \times \mathbb{R}^3 \mid \mathbf{x} \in \gamma_{\theta, \mathbf{y}}\}, \quad (13)$$

and Z is the set of all $(\theta, \mathbf{y}; \mathbf{x})$ such that \mathbf{x} is incident to the curve $\gamma_{\theta, \mathbf{y}}$. The condition can be written $\mathbf{y} = \mathbf{p}_\theta(\mathbf{x})$. This gives the double fibration

$$\begin{array}{ccc}
& Z & \\
\pi_L \swarrow & & \searrow \pi_R \\
Y & & \mathbb{R}^3
\end{array} \tag{14}$$

where π_R and π_L are the projections onto the respective factors. Note that the curve $\gamma_{\theta, \mathbf{y}}$ is $\pi_R(\pi_L^{-1}(\{(\theta, \mathbf{y})\}))$ and $S_{\mathbf{x}} = \pi_L(\pi_R^{-1}(\{\mathbf{x}\}))$.

We should point out that in the standard definition of double fibration, π_R and π_L are fiber maps and π_R is proper. Under the assumptions in Hyp. 1, π_R and π_L are both smooth fiber maps. However π_R might not be proper¹, and so the operator dual cannot be composed with $\mathcal{P}_{\mathbf{p}}$ in general. This is why we include a cutoff (7) in the definition of $\mathcal{P}_{\mathbf{p}}^*$.

Guillemin first made the connection between the double fibration and the Radon transform as a Fourier integral operator (FIO) [22, 23], and this is where we start. In a general setting, the Radon transform with incidence relation Z is a FIO with Lagrangian manifold $\Gamma := N^*(Z) \setminus \mathbf{0}$ where $N^*(Z)$ is the conormal bundle of Z ([22, 23] see also [39]) and $\mathcal{C} = \Gamma'$ is the associated canonical relation (see Theorem 3.1). The relevant microlocal diagram is the diagram on the cotangent level corresponding to this double fibration:

$$\begin{array}{ccc}
& \mathcal{C} & \\
\Pi_L \swarrow & & \searrow \Pi_R \\
T^*(Y) \setminus \mathbf{0} & & T^*(\mathbb{R}^3) \setminus \mathbf{0}
\end{array} \tag{15}$$

where Π_R and Π_L are the natural projections. If Π_L is an injective immersion, then the Radon transform satisfies *the Bolker Assumption* and when, in addition, $\pi_R : Z \rightarrow \mathbb{R}^3$ is proper, then the adjoint composed with the transform is an elliptic pseudodifferential operator. However, as we show in Theorem 3.1, $\mathcal{P}_{\mathbf{p}}$ does not satisfy the Bolker Assumption globally. That will make the composition with the adjoint a singular operator that can add singularities to the reconstruction, as we now describe.

Radon transforms detect singularities conormal to the sets of integration. This follows from the fact that the Lagrangian manifold of a Radon transform is $N^*(Z) \setminus \mathbf{0}$. Applying these ideas to $\mathcal{P}_{\mathbf{p}}$, we can conclude that, in general, we should expect to see singularities of f conormal to curves $\gamma_{\theta, \mathbf{y}}$ in the data set. However, the Bolker assumption does not hold for our transform so singularities can be added or masked in reconstruction. We begin the analysis by showing how far $\mathcal{P}_{\mathbf{p}}$ is from satisfying the Bolker assumption.

Theorem 3.1. *Let $\mathcal{P}_{\mathbf{p}}$ be a curvilinear Radon transform (5) that satisfies Hypotheses 1. Then $\mathcal{P}_{\mathbf{p}}$ is an elliptic FIO associated to the Lagrangian manifold $\Gamma = N^*(Z) \setminus \mathbf{0}$ and canonical relation $\mathcal{C} = \Gamma'$ which is given by*

$$\begin{aligned}
\mathcal{C} = \{ & (\theta, \mathbf{p}_{\theta}(\mathbf{x}), -\eta \partial_{\theta} \mathbf{p}_{\theta}(\mathbf{x}) d\theta + \eta d\mathbf{y}; \mathbf{x}, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}) d\mathbf{x}) \\
& \mid \mathbf{x} \in \mathbb{R}^3, \theta \in]a, b[, \eta \in \mathbb{R}^2 \setminus \mathbf{0} \}
\end{aligned} \tag{16}$$

The following map gives global coordinates on \mathcal{C} :

$$\begin{aligned}
c : &]a, b[\times \mathbb{R}^3 \times (\mathbb{R}^2 \setminus \mathbf{0}) \rightarrow \mathcal{C} \\
c(\theta, \mathbf{x}, \eta) = & (\theta, \mathbf{p}_{\theta}(\mathbf{x}), -\eta \partial_{\theta} \mathbf{p}_{\theta}(\mathbf{x}) d\theta + \eta d\mathbf{y}; \mathbf{x}, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}) d\mathbf{x}).
\end{aligned} \tag{17}$$

The map Π_R maps from \mathcal{C} to $T^*(\mathbb{R}^3) \setminus \mathbf{0}$ and $\Pi_L : \mathcal{C} \rightarrow T^*(Y) \setminus \mathbf{0}$.

¹If \mathbf{p}_{θ} can be extended periodically, as in Remark 1, then Π_R is proper. If not, π_R is not proper.

The map Π_L is not injective. Let $(\theta, \mathbf{y}) \in Y$ and $\eta \in \mathbb{R}^2 \setminus \mathbf{0}$. Covectors in \mathcal{C} map to the same point under Π_L if and only if they are of the form

$$(\theta, \mathbf{p}_\theta(\mathbf{x}_j), -\eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}_j) d\theta + \eta d\mathbf{y}; \mathbf{x}_j, \eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_j) d\mathbf{x}) \text{ for } j = 0, 1 \text{ where}$$

$$\mathbf{p}_\theta(\mathbf{x}_0) = \mathbf{p}_\theta(\mathbf{x}_1) \quad (18)$$

$$\eta(\partial_\theta \mathbf{p}_\theta(\mathbf{x}_1) - \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0)) = 0. \quad (19)$$

The map Π_L is not an immersion. This map drops rank by 1 on the set $\Sigma_{\mathcal{C}} \subset \mathcal{C}$ defined by

$$\lambda = (\theta, \mathbf{p}_\theta(\mathbf{x}), -\eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}) d\theta + \eta d\mathbf{y}; \mathbf{x}, \eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) d\mathbf{x}) \in \Sigma_{\mathcal{C}}$$

if and only if

$$\begin{pmatrix} \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) \\ \eta \partial_\theta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) \end{pmatrix} \text{ is not of maximal rank, three.} \quad (20)$$

Equivalently, $\lambda \in \Sigma_{\mathcal{C}}$ if and only if

$$\eta \partial_\theta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) \in \text{span}(\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x})). \quad (21)$$

The fiber of $\Sigma_{\mathcal{C}}$ above each $(\theta, \mathbf{p}_\theta(\mathbf{x}), \mathbf{x})$ is one dimensional (i.e., for each $(\theta, \mathbf{x}) \in]a, b[\times \mathbb{R}^3$ there is a one-dimensional subspace of \mathbb{R}^2 such that if η is in that subspace and $\eta \neq \mathbf{0}$, then $c(\theta, \mathbf{x}, \eta) \in \Sigma_{\mathcal{C}}$).

Remark 4. The results in Theorem 3.1 can be understood geometrically. Condition (18) means that \mathbf{x}_0 and \mathbf{x}_1 both lie on the same curve, $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$. Condition (19) means that η is perpendicular to the vector $\partial_\theta \mathbf{p}_\theta(\mathbf{x}_1) - \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0)$. In all cases, for all \mathbf{x}_0 and \mathbf{x}_1 in $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ there are η for which this condition holds. By Hyp. 1(c), for $\mathbf{x}_1 \neq \mathbf{x}_0$, there is only a one-dimensional set of such η . In Remark 5, we will explain the geometric implications of conditions (20)-(21).

Proof of Theorem 3.1. The incidence relation Z (13) is a smooth manifold because Z is the image of the smooth coordinate map

$$]a, b[\times \mathbb{R}^3 \ni (\theta, \mathbf{x}) \mapsto (\theta, \mathbf{p}_\theta(\mathbf{x}); \mathbf{x}) \in Z.$$

Therefore, Guillemin's theory applies to show the Schwartz kernel, \mathcal{S} , of $\mathcal{P}_{\mathbf{p}}$ (and of $\mathcal{P}_{\mathbf{p}}^*$) is integration on Z . \mathcal{S} is easily shown to be a Fourier integral distribution with Lagrangian manifold $\Gamma = N^*(Z) \setminus \mathbf{0}$ (e.g., [22, 23] [39, Proposition 1.1]). To calculate Γ we note that Z is defined (13) by equations

$$(\mathbf{p}_\theta)_j(\mathbf{x}) - y_j = 0, \quad j = 1, 2, \quad (22)$$

where $(\mathbf{p}_\theta)_j$ is the j^{th} coordinate function of \mathbf{p}_θ and y_j is the j^{th} coordinate of \mathbf{y} . So above each $(\theta, \mathbf{p}_\theta(\mathbf{x}); \mathbf{x}) \in Z$, $N_{(\theta, \mathbf{p}_\theta(\mathbf{x}); \mathbf{x})}^*(Z)$ has basis given by the differentials of the two coordinate functions in (22):

$$\partial_{\mathbf{x}}(\mathbf{p}_\theta)_j(\mathbf{x}) d\mathbf{x} + \partial_\theta(\mathbf{p}_\theta)_j d\theta - dy_j, \quad j = 1, 2,$$

and therefore, taking linear combinations, we calculate Γ and see that $\mathcal{C} = \Gamma'$ is as given in (16).

To show $\mathcal{P}_{\mathbf{p}}$ is a FIO (rather than just a Fourier Integral Distribution), we use the coordinates (16). The projection Π_R maps \mathcal{C} to $T^*(\mathbb{R}^3) \setminus \mathbf{0}$ because \mathbf{p}_θ is a submersion so $\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x})$ has maximum rank; therefore, $\eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x})$ is never $\mathbf{0}$ since $\eta \neq \mathbf{0}$. Now, Π_L maps \mathcal{C} to $T^*(Y) \setminus \mathbf{0}$ because the $d\mathbf{y}$ coordinate of the projection is η , and $\eta \neq \mathbf{0}$. Finally, since the measure for $\mathcal{P}_{\mathbf{p}}$ is nowhere zero, $\mathcal{P}_{\mathbf{p}}$ is an elliptic FIO.

It is a straightforward exercise using the expression for \mathcal{C} , (16), to show that (18) and (19) characterize the points on \mathcal{C} mapping to the same point under Π_L . Note that the $d\mathbf{y}$ coordinate of the projection shows that η is the same for all preimages of the same covector in $T^*(Y)$.

To show where Π_L is an immersion, we use coordinates (17) on \mathcal{C} . In these coordinates, the Jacobian matrix of Π_L is

$$\begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \partial_\theta \mathbf{p}_\theta(\mathbf{x}) & \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) & \mathbf{0} \\ -\eta \partial_\theta^2 \mathbf{p}_\theta(\mathbf{x}) & -\eta \partial_\theta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) & -(\partial_\theta \mathbf{p}_\theta(\mathbf{x}))^T \\ \mathbf{0} & \mathbf{0} & I_2 \end{pmatrix},$$

where $(\partial_\theta \mathbf{p}_\theta(\mathbf{x}))^T$ denotes the transpose. This Jacobian matrix is invertible precisely when the matrix in (20) is of maximal rank, three. This proves the double implication involving (20).

Hyp. 1(b) and equation (4) imply that \mathbf{p}_θ is a submersion. This in turn implies

$$\text{span}(\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}))$$

has dimension two, so condition (20) is equivalent to Π_L dropping rank by one.

Finally, note for each \mathbf{x} and θ , that there is a one-dimensional subspace of η for which (21) holds, that is for which Π_L drops rank when η is in this subspace and $\eta \neq \mathbf{0}$. This holds because $\text{span}(\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}))$ and $\text{span}(\partial_\theta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}))$ are not the same planes by Hyp. 1(d). This finishes the proof. \square

3.1. Microlocal Regularity Theorem. In this section, we will use the geometry of \mathcal{C} and the mapping properties of the projections Π_R and Π_L to describe what $\mathcal{P}_{\mathbf{p}}$ and our reconstruction operator

$$\mathcal{L} = \mathcal{P}_{\mathbf{p}}^* D \mathcal{P}_{\mathbf{p}}$$

do to singularities where D is a differential or pseudodifferential operator.

To state these theorems, we will need to introduce some more notation. If X and Y are manifolds and $\Lambda \subset T^*(Y) \times T^*(X)$ and $A \subset T^*(X)$, then

$$\begin{aligned} \Lambda' &:= \{(y, \eta; x, -\xi) \mid (y, \eta; x, \xi) \in \Lambda\} \\ \Lambda^t &:= \{(x, \xi; y, \eta) \mid (y, \eta; x, \xi) \in \Lambda\} \\ \Lambda \circ A &:= \{(y, \eta) \mid \exists (x, \xi) \in A \text{ with } (y, \eta; x, \xi) \in \Lambda\} \end{aligned} \quad (23)$$

If Π_L and Π_R are the projections onto $T^*(Y)$ and $T^*(X)$ respectively, then from the definition one sees

$$\Lambda \circ A = \Pi_L(\Pi_R^{-1}(A)). \quad (24)$$

Definition 3.2. Let $\mathbf{x}_0 \in \mathbb{R}^3$ let $\xi \in \mathbb{R}^3 \setminus \mathbf{0}$. Let $\mathcal{P}_{\mathbf{p}}$ be a curvilinear Radon transform (5) that satisfies Hyp. 1.

Define

$$\Theta(\mathbf{x}_0, \xi) := \{\theta \in]a, b[\mid \xi \in \text{span}(\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0))\}. \quad (25)$$

Define

$$\mathcal{A}(\mathbf{x}_0, \xi) = \mathcal{C}^t \circ (\mathcal{C} \circ (\{(x_0, \tau \xi d\mathbf{x}) \mid \tau > 0\})) \quad (26)$$

Define

$$\mathcal{B}_\theta(\mathbf{x}_0) := \{(x_0, \eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0) d\mathbf{x}) \mid \eta \neq \mathbf{0}, \eta \partial_\theta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0) \in \text{span}(\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0))\} \quad (27)$$

Our next proposition relates \mathcal{B}_θ to Σ_C and explains what Θ is. We will use these sets in the proofs of Theorem 3.3 and Corollary 1 to describe how singularities can be added to $\mathcal{L} = \mathcal{P}_p^* D\mathcal{P}_p(f)$. Those theorems will show how $\mathcal{A}(\mathbf{x}, \xi)$ relates to the added singularities of our operator \mathcal{L} .

Proposition 1. *Let \mathcal{P}_p be a curvilinear Radon transform (5) that satisfies Hypothesis 1. Let $\mathbf{x}_0 \in \mathbb{R}^3$ let $\xi \in \mathbb{R}^3 \setminus \mathbf{0}$.*

1. *Let $\theta \in]a, b[$, then $\mathcal{B}_\theta(\mathbf{x}_0)$ is the set of projections to $T^*(\mathbb{R}^3)$ of covectors in Σ_C above $(\theta, \mathbf{p}_\theta(\mathbf{x}_0); \mathbf{x}_0) \in Z$, and $\mathcal{B}_\theta(\mathbf{x}_0)$ has a one-dimensional fiber.*
2. *$\Theta(\mathbf{x}_0, \xi)$ is the set of all angles in $]a, b[$ such that $(\mathbf{x}_0, \xi d\mathbf{x})$ is conormal to the curve $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$.*
3. *Now, assume $\Theta(\mathbf{x}_0, \xi) \neq \emptyset$ and let $\theta_0 \in \Theta(\mathbf{x}_0, \xi)$. There is a unique $\eta_0 \in \mathbb{R}^2 \setminus \mathbf{0}$ such that $\xi = \eta_0 \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)$. If $\lambda = c(\theta_0, \mathbf{x}_0, \eta_0)$ (see (17)) then $\Pi_R(\lambda) = (\mathbf{x}_0, \xi d\mathbf{x})$ and*

$$\gamma_Y := \Pi_L(\lambda) = (\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0), -\eta_0 \partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{x}_0) d\theta + \eta_0 d\mathbf{y}). \quad (28)$$

Proof. The claims in part 1 the proposition about \mathcal{B}_θ are just restatements in terms of \mathcal{B}_θ of the last part of Theorem 3.1, in particular equation (21).

Part 2 follows from the fact that covectors conormal to $\gamma_{\theta, \mathbf{y}}$ are all of the form

$$(\mathbf{x}, \eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x})) \text{ for } \mathbf{x} \in \gamma_{\theta, \mathbf{y}}, \eta \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

since $\mathbf{x} \in \gamma_{\theta, \mathbf{y}}$ iff $\mathbf{p}_\theta(\mathbf{x}) = \mathbf{y}$.

To prove part 3, note that if $\theta \in \Theta(\mathbf{x}_0, \xi)$, then for some

$$\eta \in \mathbb{R}^2 \setminus \mathbf{0}, \xi = \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0).$$

Now η is unique because the matrix $\partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)$ has maximum rank. Finally, the statements about Π_L and Π_R follow from the definitions of these maps. \square

We will use our next theorem to explain the added singularities in our examples in Sect.4. This microlocal regularity theorem describes what \mathcal{P}_p and our reconstruction operator \mathcal{L} can do to singularities of the function f .

Theorem 3.3. *Let \mathcal{P}_p be a curvilinear Radon transform (5) that satisfies Hypothesis 1. Let $f \in \mathcal{E}'(\mathbb{R}^3)$. Let D be a pseudodifferential operator on \mathbb{R}^2 acting on \mathbf{y} and*

$$\mathcal{L}(f) = \mathcal{P}_p^* D\mathcal{P}_p(f).$$

Then,

$$\text{WF}(\mathcal{P}_p(f)) \subset \mathcal{C} \circ \text{WF}(f) = \Pi_L(\Pi_R^{-1} \text{WF}(f)) \quad (29)$$

$$\text{WF}(\mathcal{L}(f)) \subset \mathcal{C}^t \circ (\mathcal{C} \circ \text{WF}(f)) = \Pi_R(\Pi_L^{-1}(\Pi_L(\Pi_R^{-1} \text{WF}(f)))) \quad (30)$$

Assume D is elliptic and $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(f)$. Assume

$$\begin{aligned} \Theta(\mathbf{x}_0, \xi_0) \text{ is a single point, } \theta_0, \text{ and } \theta_0 \in]c, d[, \\ (\mathbf{x}_0, \xi_0 d\mathbf{x}) \notin \mathcal{B}_{\theta_0}(\mathbf{x}_0), \text{ and} \end{aligned} \quad (31)$$

$$\mathcal{A}(\mathbf{x}_0, \xi_0) \cap \text{WF}(f) = \{(\mathbf{x}_0, \tau \xi_0 d\mathbf{x}) \mid \tau > 0\}.$$

Then,

$$(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(\mathcal{P}_p^* D\mathcal{P}_p(f)). \quad (32)$$

The proof of (29) and (30) follow from the basic calculus of conormal distributions and the rest of the proof follows from the theory of Fourier integral operators. It will be given in the appendix. We can use this theorem to relate the set \mathcal{A} to added singularities from our operators.

Corollary 1. *Let $\mathcal{P}_{\mathbf{p}}$ be a curvilinear Radon transform (5) that satisfies Hypothesis 1. Let $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x}) \in \Pi_R(\mathcal{C})$ and let $f \in \mathcal{E}'(\mathbb{R}^3)$. Let D be a pseudodifferential operator for $\mathbf{y} \in \mathbb{R}^2$.*

1. *If $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x}) \in \text{WF}(f)$ then this singularity can create a singularity of $\mathcal{L}(f)$ at any other covector in $\mathcal{A}(\mathbf{x}_0, \xi)$.*
2. *The only covectors that can add singularities to $\mathcal{L}(f)$ at $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x})$ are in $\mathcal{A}(\mathbf{x}_0, \xi)$. That is if*

$$\mathcal{A}(\mathbf{x}_0, \xi) \cap \text{WF}(f) = \emptyset, \quad (33)$$

then $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x}) \notin \text{WF}(\mathcal{L}(f))$.

Note that $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x}) \in \mathcal{A}(\mathbf{x}_0, \xi)$ and so assumption (33) means that $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x}) \notin \text{WF}(f)$ and so f does not have a singularity at $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x})$ that could appear in $\mathcal{L}(f)$. On the other hand, if assumption (31) holds, then any singularity of f at $(\mathbf{x}_0, \xi \mathbf{d}\mathbf{x})$ will be in $\mathcal{L}(f)$.

Proof of Corollary 1. The proof of Part 1 follows from (30) and the definition of \mathcal{A} , equation (26).

The proof of Part 2 follows from (30) and the fact that the *inverse image* of

$$\{(\mathbf{x}_0, \tau \xi_0 \mathbf{d}\mathbf{x}) \mid \tau > 0\}$$

under the relation

$$\mathcal{C}^t \circ \mathcal{C} = \Pi_R \Pi_L^{-1} \Pi_L \Pi_R^{-1}$$

is the same as the *image* of this set under this relation, so $(\mathbf{x}_1, \xi_1 \mathbf{d}\mathbf{x}) \in \mathcal{A}(\mathbf{x}_0, \xi_0)$ if and only if $(\mathbf{x}_0, \xi_0 \mathbf{d}\mathbf{x}) \in \mathcal{A}(\mathbf{x}_1, \xi_1)$. \square

3.2. The geometric and practical meaning of the microlocal regularity theorems. In this section, we will describe geometrically our microlocal regularity theorems and discuss their implications for singularity detection using our operator $\mathcal{L} = \mathcal{P}_{\mathbf{p}}^* D \mathcal{P}_{\mathbf{p}}$.

A singularity of f at $(\mathbf{x}, \xi \mathbf{d}\mathbf{x})$ is said to be *visible* if it can affect singularities of $\mathcal{P}_{\mathbf{p}} f$.

Theorem 3.4. *Let $\mathcal{P}_{\mathbf{p}}$ be a curvilinear Radon transform satisfying Hypothesis 1 and let f have compact support. Define the set*

$$\mathcal{V} := \Pi_R(\mathcal{C}). \quad (34)$$

\mathcal{V} is the union of all conormals to curves in Y , and the visible singularities of f are contained in \mathcal{V} . That is, the visible wavefront are codirections conormal to curves in Y , and $\mathcal{L}(f)$ is smooth off of \mathcal{V} .

Proof. According to Theorem 3.3 and, in particular (30), a singularity at $(\mathbf{x}, \xi \mathbf{d}\mathbf{x})$ must satisfy $\mathcal{C} \circ \{(\mathbf{x}, \xi \mathbf{d}\mathbf{x})\} \neq \emptyset$ to be visible. This is equivalent to

$$\Pi_L \Pi_R^{-1} \{(\mathbf{x}, \xi \mathbf{d}\mathbf{x})\} \neq \emptyset$$

which is equivalent to

$$(\mathbf{x}, \xi \mathbf{d}\mathbf{x}) \in \Pi_R(\mathcal{C}) = \mathcal{V} \quad (35)$$

By examining the projection Π_R and using Proposition 1 parts 2 and 3, one sees that these visible directions are exactly those conormal to $\gamma_{\theta, \mathbf{y}}$ for some $(\theta, \mathbf{y}) \in Y$ because the normal space to $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ is $\text{span } \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0)$ since $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ is defined by $\mathbf{p}_\theta(\mathbf{x}) = \mathbf{p}_\theta(\mathbf{x}_0)$. Thus, the visible wavefront codirections are those that are conormal to curves in Y . Directions not conormal to any curves in Y will be invisible (i.e., smoothed) by $\mathcal{P}_\mathbf{p}$ and $\mathcal{P}_\mathbf{p}^* D\mathcal{P}_\mathbf{p}$.

Note that, $\mathcal{V} = \mathcal{C}^t \circ \mathcal{C} \circ (T^*(\mathbb{R}^3) \setminus \{\mathbf{0}\}) = \Pi_R (\Pi_L^{-1} (\Pi_L (\Pi_R^{-1} (T^*(\mathbb{R}^3) \setminus \{\mathbf{0}\}))))$. By equation (30) for any f , $\text{WF}(\mathcal{L}(f)) \subset \mathcal{V}$. So for any f , $\mathcal{L}(f)$ is smooth in directions not in \mathcal{V} . \square

The singularity detection properties of \mathcal{L} will now be described geometrically. Let $\mathbf{x}_0 \in \mathbb{R}^3$ and let

$$\mathbb{S}_{\mathbf{x}_0} = \bigcup_{\theta \in]a, b[} (\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)} \setminus \{\mathbf{x}_0\}) \quad (36)$$

be the union of curves in Y that pass through \mathbf{x}_0 minus the ‘‘vertex’’ \mathbf{x}_0 . In the next theorem we show that the set $\mathbb{S}_{\mathbf{x}_0}$ is an immersed submanifold and added singularities above \mathbf{x}_0 come from singularities of f conormal to $\mathbb{S}_{\mathbf{x}_0}$.

Theorem 3.5. *Under Hypothesis 1, for each $\mathbf{x}_0 \in \mathbb{R}^3$, $\mathbb{S}_{\mathbf{x}_0}$ is an immersed submanifold of \mathbb{R}^3 .*

A singularity of $f \in \mathcal{E}'(\mathbb{R}^3)$ at $(\mathbf{z}, \xi \mathbf{d}\mathbf{x})$ can add a singularity to $\mathcal{L}f$ above \mathbf{x}_0 only if $(\mathbf{z}, \xi \mathbf{d}\mathbf{x}) \in N^(\mathbb{S}_{\mathbf{x}_0}) \setminus \mathbf{0}$.*

If $\theta_0 \in]a, b[$ and if $\mathbf{z} \in \gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$ and $(\mathbf{z}, \xi \mathbf{d}\mathbf{x})$ is conormal to $\mathbb{S}_{\mathbf{x}_0}$ along $\gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$ ² then there is a specific co-direction above \mathbf{x}_0 at which a singularity of f at $(\mathbf{z}, \xi \mathbf{d}\mathbf{x})$ could be added. In fact, if $\eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{z}) = \xi$ then a singularity could be added at $(\mathbf{x}_0, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0))$. That is, $(\mathbf{z}, \xi \mathbf{d}\mathbf{x}) \in \mathcal{A}(\mathbf{x}_0, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0))$

Thus, singularities can be added to $\mathcal{L}(f)$ at \mathbf{x}_0 only if they are at points on $\mathbb{S}_{\mathbf{x}_0}$ and in co-directions conormal to $\mathbb{S}_{\mathbf{x}_0}$. For each conormal to $\mathbb{S}_{\mathbf{x}_0}$, the theorem describes the possible added direction above \mathbf{x}_0 . This result is suggested by the observation that $\mathcal{P}_\mathbf{p}^* \mathcal{P}_\mathbf{p} f(\mathbf{x}_0)$ is integration of f over $\mathbb{S}_{\mathbf{x}_0}$ in a weight that is singular at \mathbf{x}_0 . An analogous observation for $\mathcal{P}_\mathbf{p}^* \mathcal{P}_\mathbf{p}$ in the admissible case is in [19].

Proof. Let $\mathbf{x}_0 \in \mathbb{R}^3$ and define

$$\mathcal{S}_{\mathbf{x}_0} := \{(\mathbf{x}, \theta) \in (\mathbb{R}^3 \setminus \{\mathbf{x}_0\}) \times]a, b[\mid \mathbf{p}_\theta(\mathbf{x}) - \mathbf{p}_\theta(\mathbf{x}_0) = \mathbf{0}\}. \quad (37)$$

We consider the map

$$s(\mathbf{x}, \theta) = \mathbf{p}_\theta(\mathbf{x}) - \mathbf{p}_\theta(\mathbf{x}_0) \quad (38)$$

for $\mathbf{x} \neq \mathbf{x}_0$ and note that $\mathcal{S}_{\mathbf{x}_0} = s^{-1}(\mathbf{0})$. The derivative matrix of s is

$$A(\mathbf{x}, \theta) = (\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}), \partial_\theta \mathbf{p}_\theta(\mathbf{x}) - \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0)). \quad (39)$$

Since $\mathbf{x} \mapsto \mathbf{p}_\theta(\mathbf{x})$ is a submersion, the first three columns of the matrix have maximum rank, so $\mathcal{S}_{\mathbf{x}_0}$ is an immersed two-dimensional submanifold of $\mathbb{R}^3 \times]a, b[$.

We now use this and the Implicit Function Theorem to show $\mathbb{S}_{\mathbf{x}_0}$ is an immersed submanifold of \mathbb{R}^3 . Let $(\mathbf{z}, \theta_0) \in \mathcal{S}_{\mathbf{x}_0}$. By Hyp. 1(c), since the last column of the matrix $A(\mathbf{z}, \theta_0)$ (which is $\partial_\theta (\mathbf{p}_\theta(\mathbf{x}) - \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0))$) is not zero and the first three columns have maximum rank, one can solve $s(\mathbf{x}, \theta) = \mathbf{0}$ for θ and one coordinate of \mathbf{x} in terms of the other two \mathbf{x} coordinates on $\mathcal{S}_{\mathbf{x}_0}$ near (\mathbf{z}, θ_0) . Without loss of generality assume

²That is, $\mathbf{z} \in \gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$ and $\xi \mathbf{d}\mathbf{x}$ is conormal at \mathbf{z} to a coordinate patch of $\mathbb{S}_{\mathbf{x}_0}$ (as given in (40)) through the curve $\gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$. This clarification is needed if $\mathbb{S}_{\mathbf{x}_0}$ intersects itself in two distinct coordinate patches at \mathbf{z} .

one can solve for the third coordinate x_3 and θ in terms of (x_1, x_2) . This implies that on some neighborhood U of (z_1, z_2) there are smooth functions $x_3(x_1, x_2)$ and $\theta(x_1, x_2)$ such that for $(x_1, x_2) \in U$, $(x_1, x_2, x_3(x_1, x_2), \theta(x_1, x_2)) \in \mathcal{S}_{\mathbf{x}_0}$. Let $\pi : \mathbb{R}^3 \times]a, b[\rightarrow \mathbb{R}^3$ be the projection onto the \mathbf{x} coordinates. Then,

$$U \ni (x_1, x_2) \mapsto \pi(x_1, x_2, x_3(x_1, x_2), \theta(x_1, x_2)) = (x_1, x_2, x_3(x_1, x_2)) \quad (40)$$

gives smooth local coordinates on $\mathbb{S}_{\mathbf{x}_0}$ (because this map is clearly injective from $(x_1, x_2) \in U$ to $\mathbb{S}_{\mathbf{x}_0}$ and the function $x_3(x_1, x_2)$ is smooth). Therefore $\mathbb{S}_{\mathbf{x}_0}$ is an immersed two-dimensional submanifold of \mathbb{R}^3 . This proves our first claim.

We now show that conormals to $\mathbb{S}_{\mathbf{x}_0}$ are exactly those that give rise to added singularities at the prescribed covectors given in the last statement of the theorem.

We denote the natural pairing between cotangent vectors and tangent vectors as $\langle \Xi, T \rangle := \Xi(T)$. Let $\mathbf{z} \in \mathbb{S}_{\mathbf{x}_0}$ and $\theta_0 \in]a, b[$ such that $\mathbf{x} \in \gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x}_0)$. Using this notation, we pull $(\mathbf{z}, \xi d\mathbf{x})$ back to $\mathcal{S}_{\mathbf{x}_0}$ and show the condition for $(\mathbf{z}, \xi d\mathbf{x})$ to be conormal to $\mathbb{S}_{\mathbf{x}_0}$ is equivalent to the condition for $\pi^*(\xi d\mathbf{x})$ to be conormal to $\mathcal{S}_{\mathbf{x}_0}$ at (\mathbf{z}, θ_0) . This follows because $\pi^*(\xi d\mathbf{x})$ is conormal to $\mathcal{S}_{\mathbf{x}_0}$ if and only if for all tangent vectors $T \in T_{(\mathbf{z}, \theta_0)}(\mathcal{S}_{\mathbf{x}_0})$, we have that $0 = \langle \pi^*(\xi d\mathbf{x}), T \rangle$. However, using the definitions of π^* and π_* we see

$$0 = \langle \pi^*(\xi d\mathbf{x}), T \rangle = \langle \xi d\mathbf{x}, \pi_*(T) \rangle.$$

This last condition implies that $\xi d\mathbf{x}$ is conormal to $\mathbb{S}_{\mathbf{x}_0} = \pi(\mathcal{S}_{\mathbf{x}_0})$ since

$$\pi_*(T_{(\mathbf{z}, \theta_0)}(\mathcal{S}_{\mathbf{x}_0})) = T_{\mathbf{z}}(\mathbb{S}_{\mathbf{x}_0}).$$

It is straightforward exercise to show that $\pi^*(\xi d\mathbf{x}) = \xi d\mathbf{x} + 0d\theta$. For $\xi d\mathbf{x} + 0d\theta$ to be conormal to $\mathcal{S}_{\mathbf{x}_0}$ at (\mathbf{z}, θ_0) , $(\xi, 0)$ must be a linear combination of the rows of the derivative matrix $A(\mathbf{z}, \theta_0)$ since $\mathcal{S}_{\mathbf{x}_0}$ is defined by $s(\mathbf{x}, \theta) = \mathbf{0}$. For this to be true, for some $\eta \in \mathbb{R}^2$, $(\xi, 0) = \eta A(\mathbf{z}, \theta_0)$ or equivalently

$$\xi = \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{z}), \quad \text{and} \quad \eta (\partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{z}) - \partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{x}_0)) = 0. \quad (41)$$

Condition (41) shows that being conormal to $\mathbb{S}_{\mathbf{x}_0}$ along the curve $\gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x}_0)$ is equivalent for $(\mathbf{z}, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{z}) d\mathbf{x})$ to be conormal to $\gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x}_0)$ and $(\mathbf{z}, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{z}) d\mathbf{x}) \in \mathcal{A}(\mathbf{x}_0, \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0))$, the set of possible added singularities at \mathbf{x}_0 in co-direction $\eta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0) d\mathbf{x}$.

The added singularities above \mathbf{x}_0 are certain singularities conormal to curves in Y through \mathbf{x}_0 , and we have shown the condition for conormality to $\mathbb{S}_{\mathbf{x}_0}$ is the same as the condition for being a (potential) added singularity in Theorem 3.3. This finishes the proof. \square

Now we give the geometric meaning of Hyp. 1(d).

Remark 5. One can understand the rank hypothesis Hyp. 1(d) geometrically. First, chose θ_0 and \mathbf{x}_0 and let $\mathbf{y}_0 = \mathbf{p}_{\theta_0}(\mathbf{x}_0)$. Note that because \mathbf{p}_{θ_0} is a submersion, $\text{span}(\partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0))$ has dimension two. Then, because $\gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x})$ is defined by the equations $\mathbf{p}_{\theta_0}(\mathbf{x}) = \mathbf{y}$, the normal plane to $\gamma_{\theta_0, \mathbf{y}_0}$ at \mathbf{x}_0 is $\text{span}(\partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0))$.

If the rank condition Hyp. 1(d) fails at θ_0 and \mathbf{x}_0 this means that

$$\text{span}(\partial_{\theta} \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)) \subset \text{span}(\partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)). \quad (42)$$

This means that, at least infinitesimally at θ_0 the normal plane to $\gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x}_0)$ at \mathbf{x}_0 is stationary in θ , so the curves near $\gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x}_0)$ are infinitesimally rigid at \mathbf{x}_0 .

However, if the rank condition Hyp. 1(d) holds, the curves are not rigid and, by Theorem 3.3 \mathcal{P}_p is able to detect wavefront conormal to $\gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$ if f does not have too many other singularities (by Theorem 3.3).

3.3. The “good” Differential Operator $D_g = D_g(\theta, \mathbf{x}_0)$ and the Reconstruction Operator \mathcal{L}_g . In this section, we will define the differential operator $D_g = D_g(\theta, \mathbf{x}_0)$ that we use in our reconstruction algorithm. This operator is “good” in the sense that it will de-emphasize added singularities, at least from points near \mathbf{x}_0 . In order to understand the properties of $D_g(\theta, \mathbf{x}_0)$, we analyze the properties of the added singularities in terms of η in the coordinate map for \mathcal{C} in (17).

Let $\mathbf{x}_0 \in \mathbb{R}^3$, then there are two types of “bad” cotangent directions in \mathcal{V} that can cause problems for the composition $\mathcal{L} = \mathcal{P}_p^* D \mathcal{P}_p$ above \mathbf{x}_0 . First, for each $\theta \in]a, b[$ there are the covectors in $\mathcal{B}_\theta(\mathbf{x}_0)$ corresponding to $\Sigma_{\mathcal{C}}$, where Π_L is not an immersion (see Prop. 1, Part 1). Microlocally near such covectors, even a cutoff and microlocalized version of \mathcal{L} is not a regular pseudodifferential differential operator since Π_L does not satisfy the Bolker assumption on $\Sigma_{\mathcal{C}}$. Second are covectors corresponding to where Π_L is not injective that can add (or mask) a singularity to $\mathcal{L}f$ at $(\mathbf{x}_0, \xi d\mathbf{x})$, and they are in $\mathcal{A}(\mathbf{x}_0, \xi)$.

We now construct our “good” differential operator $D_g(\theta, \mathbf{x}_0)$. Let $\mathbf{x}_0 \in \mathbb{R}^3$ and let $\theta \in]a, b[$. Let $\mathbf{v}(\theta, \mathbf{x}_0)$ be the tangent vector to $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ at \mathbf{x}_0

$$\mathbf{v}(\theta, \mathbf{x}_0) := \partial_t (\mathbf{P}_\theta)(t_0, \mathbf{p}_\theta(\mathbf{x}_0)) \quad (43)$$

where t_0 is chosen so that $\mathbf{x}_0 = \mathbf{P}_\theta(t_0, \mathbf{p}_\theta(\mathbf{x}_0))$

and where \mathbf{P}_θ is the map that defines our curves (2). Now define

$$\nu(\theta, \mathbf{x}_0) = \partial_\theta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0) \mathbf{v}(\theta, \mathbf{x}_0) \quad (44)$$

and define D_g to be the second derivative in direction ν

$$D_g = D_g(\theta, \mathbf{x}_0) := (\nabla_\nu)^2 \quad (45)$$

where ∇_ν is the directional derivative in direction ν in the \mathbf{y} coordinate. Our “good” reconstruction operator is

$$\mathcal{L}_g(f)(\mathbf{x}_0) = \mathcal{P}_p^*(D_g(\cdot, \mathbf{x}_0)\mathcal{P}_p f)(\mathbf{x}_0). \quad (46)$$

In Prop. 2, we will show that $D_g(\theta, \mathbf{x}_0)$ is a smooth nonzero operator that annihilates covectors in $\Pi_L(\Sigma_{\mathcal{C}})$. We will show that singularities that can be added above \mathbf{x}_0 from nearby points are near singularities corresponding to $\Sigma_{\mathcal{C}}$, and this will allow us to make the heuristic argument that these nearby added singularities will be de-emphasized by our reconstruction operator \mathcal{L}_g . Our reconstructions in Sect. 4 show they are de-emphasized for those examples. It is an open question whether one can de-emphasize *all* added singularities with such an operator in general.

To justify this, we need more notation. Recall the coordinate map for \mathcal{C} :

$$\begin{aligned} (\theta, \mathbf{x}, \eta) &\in]a, b[\times \mathbb{R}^3 \times (\mathbb{R}^2 \setminus \mathbf{0}) \\ c(\theta, \mathbf{x}, \eta) &= (\theta, \mathbf{p}_\theta(\mathbf{x}), -\eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}) d\theta + \eta d\mathbf{y}; \mathbf{x}, \eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) d\mathbf{x}) \in \mathcal{C} \\ \Pi_L(c(\theta, \mathbf{x}, \eta)) &= (\theta, \mathbf{p}_\theta(\mathbf{x}), -\eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}) d\theta + \eta d\mathbf{y}) \end{aligned}$$

Since the $d\mathbf{y}$ coordinate of $\Pi_L(c(\theta, \mathbf{x}_0, \eta))$ is η , and our operator $D_g(\theta, \mathbf{x}_0)$ acts in the \mathbf{y} coordinate, we study the relation between the “bad” cotangent vectors and the coordinate η . Let $(\theta_0, \mathbf{x}_0) \in]a, b[\times \mathbb{R}^3$ and define $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ to be the set

$$N_{\mathcal{B}}(\theta_0, \mathbf{x}_0) = \{\eta \in \mathbb{R}^2 \mid \eta \partial_\theta \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0) \in \text{span } \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)\}. \quad (47)$$

Prop. 2 part 4 shows that $N_{\mathcal{B}}$ is the set of η corresponding to covectors in $\mathcal{B}_{\theta_0}(\mathbf{x}_0)$.
Let

$$\mathbf{x}_1(\theta_0, t, \mathbf{x}_0) = P_{\theta_0}(t_0 + t, \mathbf{p}_{\theta_0}(\mathbf{x}_0)) \quad (48)$$

where t_0 is chosen to satisfy $P_{\theta_0}(t_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)) = \mathbf{x}_0$.

Then we define

$$N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0) = \{\eta \in \mathbb{R}^2 \mid \eta(\partial_{\theta}(\mathbf{p}_{\theta}(\mathbf{x}_1(\theta_0, t, \mathbf{x}_0)) - \mathbf{p}_{\theta}(\mathbf{x}_0))) = 0\}. \quad (49)$$

Prop. 2 part 5 shows that $N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0)$ is the set of η corresponding to added singularities at \mathbf{x}_0 coming from singularities above $\mathbf{x}_1(\theta_0, t, \mathbf{x}_0)$.

Proposition 2. *Let $(\theta_0, \mathbf{x}_0) \in]a, b[\times \mathbb{R}^3$.*

1. *The vector $\nu(\theta_0, \mathbf{x}_0)$ given in (44) is never zero and is perpendicular to the subspace $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$. The differential operator $D_g(\theta_0, \mathbf{x}_0)$ is nonzero and has smooth coefficients in θ and the parameter \mathbf{x} .*
2. *$N_{\mathcal{B}}(\theta_0, \mathbf{x}_0) \setminus \{\mathbf{0}\}$ is the set of η for which Π_L drops rank at $c(\theta_0, \mathbf{x}_0, \eta)$ (i.e., for which $c(\theta_0, \mathbf{x}_0, \eta) \in \Sigma_C$) and $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ is one-dimensional.*
3. *Let $\eta \in N_{\mathcal{B}}(\theta_0, \mathbf{x}_0) \setminus \{\mathbf{0}\}$. If $\gamma_Y = \Pi_L(c(\theta_0, \mathbf{x}_0, \eta))$, then the symbol of $D_g(\theta_0, \mathbf{x}_0)$ is zero at γ_Y and $\gamma_Y \in \Pi_L(\Sigma_C)$.*
4. *The vector η is in $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0) \setminus \{\mathbf{0}\}$ if and only if $\xi = \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0) \in \mathcal{B}_{\theta_0}(\mathbf{x}_0)$.*
5. *$N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0) \setminus \{\mathbf{0}\}$ is the set of η such that a singularity at*

$$(\mathbf{x}_1(\theta_0, t, \mathbf{x}_0), \eta \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_1(\theta_0, t, \mathbf{x}_0)))$$

can add a singularity to $\mathcal{P}_{\mathbf{p}}f$ at $(\mathbf{x}_0, \eta(\partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0)))$, and this set is a one-dimensional subspace of \mathbb{R}^2 .

6. *As $t \rightarrow 0$, $N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0) \rightarrow N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ in the sense that the limit of the smaller angle between the subspaces $N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0)$ and $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ is zero.*

This proposition justifies the choice of D_g . Part 1 shows that $D_g(\theta_0, \mathbf{x}_0)$ is a smooth nonzero differential operator. Part 3 implies that D_g is zero on covectors in $\Pi_L(\Sigma_C)$. Parts 5 and 6 implies that the added singularities for points on $\gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$ near \mathbf{x}_0 , have coordinates $\eta' \in N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0)$ that are close to $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ —the directions annihilated by the operator $D_g(\theta_0, \mathbf{x}_0)$. This implies that, for fixed θ_0 and \mathbf{x}_0 , the singularities above nearby points on $\gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$ that can add singularities to $\mathcal{L}_g(f)$ near \mathbf{x}_0 have coordinates η' that are close to $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ and so the symbol of $D_g(\theta_0, \mathbf{x}_0)$ is close to zero at those points.

For the line complexes discussed in Section 3.4, \mathcal{L}_g will decrease the strength of the added singularities everywhere as shown in [11]. However, for general curves, our operator de-emphasizes added singularities only from points near \mathbf{x}_0 .

Proof. We first prove part 1. Since \mathbf{P}_{θ} is a diffeomorphism, its derivative with respect to t , $\mathbf{v}(\theta_0, \mathbf{x}_0)$ in (43), is nonzero. By definition, $\mathbf{v}(\theta_0, \mathbf{x}_0)$ is tangent to $\gamma_{\theta, \mathbf{p}_{\theta}(\mathbf{x}_0)}$ at \mathbf{x}_0 . Moreover, $\text{span } \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0)$ is the normal space to $\gamma_{\theta, \mathbf{p}_{\theta}(\mathbf{x}_0)}$ at \mathbf{x}_0 . By definition, $\eta \in N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ if and only if

$$\eta \partial_{\theta} \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0) \in \text{span } \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0)$$

if and only if

$$0 = (\eta \partial_{\theta} \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0)) \mathbf{v}(\theta_0, \mathbf{x}_0) = \eta (\partial_{\theta} \partial_{\mathbf{x}} \mathbf{p}_{\theta}(\mathbf{x}_0) \mathbf{v}(\theta_0, \mathbf{x}_0)) = \eta \nu(\theta_0, \mathbf{x}_0) \quad (50)$$

where we have used the definition of ν , (44). Finally, note that

$$\nu(\theta_0, \mathbf{x}_0) \neq \mathbf{0} \quad \forall (\theta_0, \mathbf{x}_0) \in]a, b[\times \mathbb{R}^3. \quad (51)$$

This is true because \mathbf{v} is perpendicular both rows of the matrix $\partial_{\mathbf{x}}\mathbf{p}_{\theta}(\mathbf{x}_0)$ (which span the normal space to $\gamma_{\theta, \mathbf{p}_{\theta}(\mathbf{x}_0)}$ to \mathbf{x}_0) and therefore cannot be normal to both rows of the matrix $\partial_{\theta}\partial_{\mathbf{x}}\mathbf{p}_{\theta}(\mathbf{x}_0)$ by the maximum rank assumption, Hyp. 1(d). Therefore, $\partial_{\theta}\partial_{\mathbf{x}}\mathbf{p}_{\theta}(\mathbf{x}_0)\mathbf{v}(\theta_0, \mathbf{x}_0)$ is not the zero matrix and $\nu(\theta_0, \mathbf{x}_0) \neq \mathbf{0}$. This proves $D_g(\theta_0, \mathbf{x}_0)$ is not the zero operator.

To show D_g has smooth coefficients, first note that the number t_0 in (48) is the projection on the first coordinate of the map $(\theta, \mathbf{x}_0) \mapsto \mathbf{P}_{\theta}^{-1}(\mathbf{x}_0)$ and is therefore a smooth function of (θ, \mathbf{x}_0) . Now, since $\mathbf{v}(\theta, \mathbf{x}_0)$ is the derivative of \mathbf{P}_{θ} in t at t_0 , ν is a smooth function from Z to \mathbb{R}^2 . Therefore $D_g(\theta, \mathbf{x}_0)$ is a smooth differential operator in both θ and the parameter \mathbf{x}_0 .

Part 2 in the proposition uses the last assertion in Theorem 3.1: that λ given by (17) is in Σ_C if and only if $\eta\partial_{\theta}\partial_{\mathbf{x}}\mathbf{p}_{\theta}(\mathbf{x}) \in \text{span}(\partial_{\mathbf{x}}\mathbf{p}_{\theta}(\mathbf{x}))$. This is exactly the definition for η to be in $N_{\mathcal{B}}$, (47). The set $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ is one dimensional since the fiber of Σ_C is one dimensional by Theorem 3.1.

We now use the definition of \mathcal{B}_{θ} and equation (21) in Theorem 3.1 to justify the characterization of D_g in Part 3. Let $\eta \in N_{\mathcal{B}}(\theta_0, \mathbf{x}_0) \setminus \{\mathbf{0}\}$. Then $c(\theta_0, \mathbf{x}_0, \eta) \in \Sigma_C$ by part 2. By definition of the map c , $\lambda_Y = \Pi_L(c(\theta_0, \mathbf{x}_0, \eta))$ has $d\mathbf{y}$ coordinate η and is in $\Pi_L(\Sigma_C)$. Since $\nu(\theta_0, \mathbf{x}_0)$ is perpendicular to η by part 1 of this proposition, the symbol of $D_g(\theta_0, \mathbf{x}_0)$ is zero at λ_Y .

Part 4 follows from the definition of \mathcal{B}_{θ_0} , (27): $(\mathbf{x}_0, \xi) \in \mathcal{B}_{\theta_0}(\mathbf{x}_0)$ if and only if ξ can be written $\xi = \eta\partial_{\mathbf{x}}\mathbf{p}_{\theta_0}(\mathbf{x}_0)$ where $\eta \neq \mathbf{0}$ and $\eta\partial_{\theta}\partial_{\mathbf{x}}\mathbf{p}_{\theta_0}(\mathbf{x}_0) \in \text{span}\partial_{\mathbf{x}}\mathbf{p}_{\theta_0}(\mathbf{x}_0)$. This condition is true if and only if $\eta \in N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$.

Part 5 holds by the noninjectivity condition for Π_L , (19) in Theorem 3.1.

The argument for Part 6 follows from calculating a derivative as a limit. The vector $\eta \in N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0)$ iff

$$\eta \left(\frac{\partial_{\theta}\mathbf{p}_{\theta}(\mathbf{x}_1(\theta_0, t, \mathbf{x}_0)) - \partial_{\theta}\mathbf{p}_{\theta}(\mathbf{x}_0)}{t} \right) = 0.$$

As $t \rightarrow 0$, this equation becomes

$$[\eta\partial_{\mathbf{x}}\partial_{\theta}\mathbf{p}_{\theta}(\mathbf{x}_0)]\partial_t\mathbf{x}_1(\theta_0, 0, \mathbf{x}_0) = 0. \quad (52)$$

Equation (52) shows that the vector η in that equation is in $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ because $\partial_t\mathbf{x}_1(\theta_0, 0, \mathbf{x}_0)$ is a tangent vector at \mathbf{x}_0 to $\gamma_{\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0)}$, and being perpendicular to this tangent vector means $\eta\partial_{\mathbf{x}}\partial_{\theta}\mathbf{p}_{\theta}(\mathbf{x}_0)$ is in the span of the rows of $\partial_{\mathbf{x}}\mathbf{p}_{\theta}(\mathbf{x}_0)$. Because \mathbf{p}_{θ} and \mathbf{x}_1 are C^{∞} functions, this argument shows that the subspace $N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0)$ approaches $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ in the sense that the line $N_{\mathcal{A}}(\theta_0, t, \mathbf{x}_0)$ approaches the line $N_{\mathcal{B}}(\theta_0, \mathbf{x}_0)$ in \mathbb{RP}^1 . \square

Remark 6. Note that since $D_g(\theta, \mathbf{x})$ depends on \mathbf{x} as well as θ , D_g is not a regular differential operator on Y but an operator that depends on the parameter \mathbf{x} . However, the operator \mathcal{L}_g satisfies microlocal regularity condition (30) in Theorem 3.3 as well as Theorem 3.4. Therefore, possible added singularities for $\mathcal{L}_g f(\mathbf{x})$ are as given in Theorem 3.5.

These claims are true for the following reason. The reconstruction operator \mathcal{L}_g has Schwartz kernel \mathcal{R} that can be written

$$\begin{aligned} \mathcal{R}(h) &= \int_Z D_g(\theta, \mathbf{x})h(\theta, \mathbf{y}, \mathbf{x})\phi(\theta)dA \\ &= \int_{]a, b[\times \mathbb{R}^3} D_g(\theta, \mathbf{x})h(\theta, \mathbf{p}_{\theta}(\mathbf{x}), \mathbf{x})\phi(\theta)w(\theta, \mathbf{x})d\theta d\mathbf{x} \end{aligned}$$

where h is in $\mathcal{D}(Y \times \mathbb{R}^3)$ and w is a smooth nowhere zero Jacobian factor. Thus \mathcal{R} has wavefront set contained in $N^*(Z) \setminus \{\mathbf{0}\}$ since it is the composition of the operator D_g on $Y \times \mathbb{R}^3$ with integration on Z , which is a conormal distribution with wavefront set $N^*(Z) \setminus \{\mathbf{0}\}$. Thus, one can use the Hörmander-Sato Lemma and the other arguments in the proof of (30) of Theorem 3.3 to prove (30) for \mathcal{L}_g . Theorems 3.4 and 3.5 are true for \mathcal{L}_g because their proofs depend only on (30) and Hyp. 1.

3.4. ET over lines. Now, we consider the set of lines parallel to a curve $\mathbf{c} :]a, b[\rightarrow S^2$ given in Example 2.1. We will denote the set of lines in this complex by Y_c . This is a special type of line complex that is admissible in the sense of Gelfand [15]. For admissible line complexes on manifolds, Greenleaf and Uhlmann show precisely how $\mathcal{P}_p^* \mathcal{P}_p$ adds singularities and they prove microlocal mapping properties of $\mathcal{P}_p^* \mathcal{P}_p$ (e.g., [19, 21]). In almost all cases the transform of Example 2.1 satisfies their assumptions. However, if C is an equator, then the transform does not satisfy their assumptions [11, Remark 4.1].

Proposition 3. *Let $\mathbf{c} :]a, b[\rightarrow S^2$ be a differentiable curve and $\mathbf{c}(\theta) \cdot \mathbf{c}'(\theta) \neq 0$ for all $\theta \in]a, b[$. Let Y_c be a complex of lines parallel to directions in \mathbf{c} . Let $\mathbf{x}_0 \in \mathbb{R}^3$ let $\theta \in]a, b[$. Let \mathbf{x} be in the line $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$. Then, the set of cotangent vectors in $T^*(Y_c)$ above which Π_L is not an immersion (that is $\Pi_L(\Sigma_C)$) are the ones above which Π_L is not injective, and $\Pi_L : \mathcal{C} \setminus \Sigma_C \rightarrow T^*(Y)$ is an injective immersion.*

The proof for this case was given in [40, Proof of Theorem A.1, below (A.16)] and it is a special case of results in [19]. One can also prove this proposition using the fact that $\mathbb{S}_{\mathbf{x}_0}$ is a cone and so along each line, ℓ , in the cone, conormals to the cone are the same. This so called cone condition is actually a condition for a line complex to be admissible. Since the added singularities at \mathbf{x}_0 are conormal to $\mathbb{S}_{\mathbf{x}_0}$, the added singularities above points in ℓ are all in the same co-directions.

In [11], the authors show that the differential operator $D_g(\theta, \mathbf{x}_0)$ de-emphasizes all added singularities by one degree in Sobolev scale (using results in [19, 20]). This is the advantage of having the set Σ_C on which Π_L is not an immersion the same set on which Π_L is not injective.

4. Numerical examples. In this section, we present reconstructions from our operator \mathcal{L}_g using the helices described in Example 2.2. We consider two cases, pitch 20π , which is realistic in electron microscopy (private communication Albert Lawrence) and pitch π . The notation is as in that example. The examples are for full data, $\theta \in [0, 2\pi]$ so \mathbf{p}_θ can be extended periodically so we do not need a cut off in the definition of \mathcal{P}_p^* (see Remark 1). We use 140 angles in $[0, 2\pi]$ and a 201×201 detector grid on $[-1, 1]^2$.

The images on the left show the reconstruction computed with “good” differential operator given in section 3.3 and the derivative in the perpendicular direction. Note that the perpendicular direction corresponds to values of η in $N_{\mathcal{B}}(\theta, \mathbf{x}_0)$, and these correspond to images under Π_L of covectors in \mathcal{C} at which Π_L drops rank .

4.1. One ball centered at origin. The first example shows our reconstructions of a ball with diameter 1 and center at the origin. In this example, one sees that the added singularities are on the planes $x_1 = \pm 0.5$ which are the planes predicted by the theory as will be justified in our next proposition.

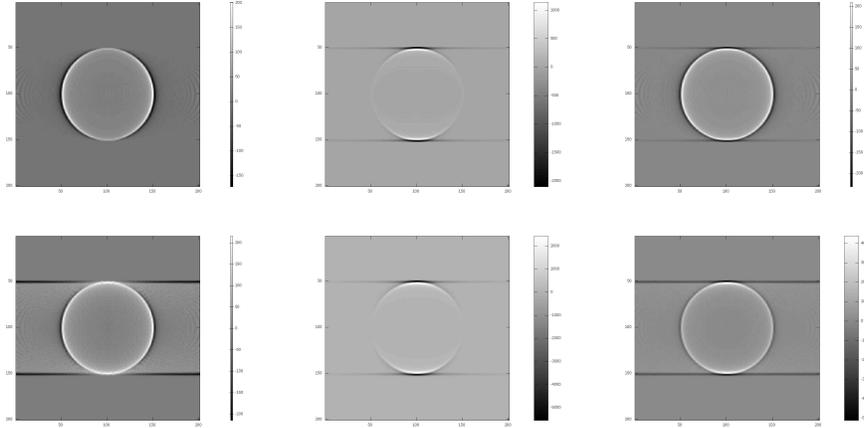


FIGURE 1. The section $x_3 = 0$ of the reconstruction of the characteristic function of one ball centered at the origin. The axis of rotation is the vertical axis (the x_1 axis in Example 2.2). The top row is for data over helices of pitch 20π , and the bottom row is for helices of pitch π . The left-hand reconstruction is with D_g , the middle reconstruction is with a derivative in the perpendicular direction, and the right-hand reconstruction has a linear combination of the two (the left reconstruction plus $0.1 \times$ the middle one). Note the scales are much higher on the middle reconstructions. The horizontal lines, which are easily visible on the bottom reconstructions, correspond to the intersection of the added singularities, Q (see (53)), with the reconstruction plane.

Let f be a distribution of compact support, and let $\mathbf{x} \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Let

$$\mathcal{A}'(\mathbf{x}, \xi) = \mathcal{A}(\mathbf{x}, \xi) \setminus \{(\mathbf{x}, \tau\xi d\mathbf{x}) \mid \tau > 0\}.$$

By Corollary 1, $\mathcal{A}'(\mathbf{x}, \xi)$ is the set of singularities that can be added to $\mathcal{L}(f)$ at points besides \mathbf{x} from a singularity of f at $(\mathbf{x}, \xi d\mathbf{x})$.

Proposition 4. *Let f be the characteristic function of a disk, \mathcal{D} , of radius $r > 0$ centered at the origin let Y be the complex of helices given in Example 2.2. Let $\mathcal{Q} = \{(\mathbf{z}, \tau d\mathbf{z}_1) \mid \mathbf{z} \in \mathbb{R}^3, z_1 = \pm r, (z_2, z_3) \neq \mathbf{0}, \tau \neq 0\}$. Then, the union $\cup_{\mathbf{x} \in \text{bd}(\mathcal{D})} \mathcal{A}'(\mathbf{x}, \pm \mathbf{x}) = \mathcal{Q}$.*

This proposition shows that the added singularities will be conormal to the set

$$Q = \{(\pm r, x_2, x_3) \mid (x_2, x_3) \in \mathbb{R}^2\}. \quad (53)$$

The proof is fairly technical, and it is in the appendix. The basic idea is to show that the only helices that are tangent to the boundary of the disk \mathcal{D} are ones with radius r from their axis and that the added singularities corresponding to those points are above the points where the helices meet Q .

4.2. Two balls with centers along x_1 axis. The second example shows our reconstruction of two balls with diameter 0.5 and centers at $(\pm 0.25, 0, 0)$. Here the artifacts occur along four surfaces, two surfaces associated with each ball.

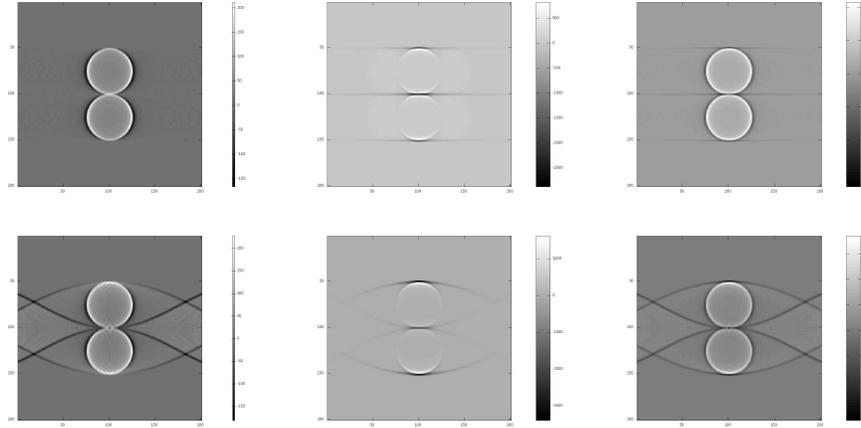


FIGURE 2. The section $x_3 = 0$ of the reconstruction of the characteristic function of two balls centered on the axis of rotation of the specimen (the x_1 axis in Example 2.2) which is the vertical axis in the figures. The top row is for data over helices of pitch 20π , and the bottom row is for helices of pitch π . The left-hand reconstruction is with D_g , the middle reconstruction is with a derivative in the perpendicular direction, and the right-hand reconstruction has a linear combination of the two (the left reconstruction plus $0.1 \times$ the middle one). Note the scales are much higher on the middle reconstructions.

Acknowledgements: The authors are indebted to Albert (Rick) Lawrence (Center for Research in Biological Systems, University of California, San Diego) for informing us of this problem in electron tomography and enthusiastically encouraging us to look at the microlocal analysis. We thank Gunther Uhlmann (University of Washington) for stimulating discussions and letting us know about [5]. We thank Plamen Stefanov (Purdue University) for his perspective and helpful comments about these results. We thank Allan Greenleaf and Raluca Felea for stimulating discussions about the microlocal analysis of operators with folds and blowdowns. We appreciate Ozan Öktem for his perspective and for introducing us to ET. We are indebted to Venky Krishan for his thoughtful comments about the article, conormal distributions, and \mathcal{L}_g in particular. We appreciate the comments of the referee that improved the article and helped make its main points clearer, and we appreciate the reference to [26]. Finally, we thank Jan Boman (emeritus, University of Stockholm) for his unwavering support for this work and stimulation for our research.

5. **Appendix.** In this appendix, we give the proofs of Theorem 3.3 and Proposition 4.

Proof of Theorem 3.3. This follows from basic theorems about FIOs and wavefront sets. The most elementary proofs of (29) and (30) use only the properties of the wavefront set of distributions and operators associated to them. The Schwartz kernel, \mathcal{S} of \mathcal{P}_p as a distribution on $\mathbb{R}^3 \times Y$ is integration over Z in the smooth nowhere zero measure defined by arc length on each $\gamma_{\theta, \mathbf{y}}$ and $d\mathbf{y}d\theta$ (e.g.,

[39, Proposition 1.1]). With this information, it is straightforward to show that $\text{WF}(\mathcal{S}) = N^*(Z) \setminus \mathbf{0} = \mathcal{C}'$ [28, Example 8.2.8, p. 266]. Note that the projections

$$\text{WF}(\mathcal{S})_{\mathbb{R}^3} := \{(\mathbf{x}, \xi) \in T^*(\mathbb{R}^3) \mid \exists(\theta, \mathbf{y}) \in Y, (x, \xi; \theta, \mathbf{y}, 0) \in \text{WF}(\mathcal{S})\}$$

and

$$\text{WF}(\mathcal{S})_Y := \{(\theta, \mathbf{y}, \tau) \in T^*(Y) \mid \exists \mathbf{x} \in \mathbb{R}^3, (\mathbf{x}, 0; \theta, \mathbf{y}, \tau) \in \text{WF}(\mathcal{S})\}$$

are both empty because neither Π_R nor Π_L map to the zero section, (Theorem 3.1). This allows us to use [28, Theorem 8.2.13] to conclude (29). Next, the Schwartz kernel of $\mathcal{P}_{\mathbf{p}}^*$ is an integration along Z (but the measure is non-zero only for $\theta \in]c', d'[/math>). Since D is a smooth differential or pseudodifferential operator $\text{WF}(Dg) \subset \text{WF}(g)$ for $g \in \mathcal{E}'(Y)$. Because of the cutoff ϕ , we can compose $\mathcal{P}_{\mathbf{p}}^*$ and $D\mathcal{P}_{\mathbf{p}}$ and we can use [28, Theorem 8.2.13] again to conclude $\text{WF}(\mathcal{P}_{\mathbf{p}}^*D\mathcal{P}_{\mathbf{p}}(f)) \subset \mathcal{C}^t \circ (\mathcal{C} \circ \text{WF}(f))$. Note that the equalities in (29) and (30) follow from (24).$

We now prove the ellipticity condition in the theorem, (32), using the calculus of FIOs. Assume (31) holds for $(\mathbf{x}_0, \xi_0 d\mathbf{x})$, and let θ_0 be the unique angle in $\Theta(\mathbf{x}_0, \xi_0)$. As noted in Proposition 1, Part 3, $(\mathbf{x}_0, \xi_0, \theta_0)$ determine a unique $\eta_0 \in \mathbb{R}^2$ so that $\xi_0 = \eta_0 \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)$. Consider the point in \mathcal{C} above $(\mathbf{x}_0, \xi_0 d\mathbf{x})$ and its projection to $T^*(Y)$

$$\begin{aligned} \lambda &:= (\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0), -\eta_0 \partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{x}_0) d\theta + \eta_0 d\mathbf{y}; \mathbf{x}_0, \eta_0 \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)) \\ \Pi_L(\lambda) = \lambda_Y &:= (\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0), -\eta_0 \partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{x}_0) d\theta + \eta_0 d\mathbf{y}). \end{aligned}$$

We first show that $\gamma_Y \notin \text{WF}(\mathcal{P}_{\mathbf{p}}f)$. Since $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \notin \mathcal{B}_{\theta_0}(\mathbf{x}_0)$ and by the Inverse Function Theorem, there are neighborhoods U of θ_0 , V of \mathbf{x}_0 and W of η_0 such that $\Pi_L \circ c$ is injective for $\theta \in U$, $\mathbf{x} \in V$, and $\eta \in W$ where c is the coordinate map from (17). Because Π_L is an injective immersion at γ , $\dim(\mathbb{R}^3) = \dim(Y)$, and \mathcal{C} is a canonical relation, Π_R is also an immersion at γ [27, Proposition 4.1.3]. Therefore, we may assume U , V and W are chosen so small that both $\Pi_R \circ c$ and $\Pi_L \circ c$ are injective on $U \times V \times W$. Note that we may assume that W is conic since c is linear in η and Π_R , and Π_L are linear in the cotangent coordinates. Let $\varphi \in C_c^\infty(V)$ be equal to one in a smaller neighborhood of \mathbf{x}_0 . Let $a(\theta, \mathbf{y}, \alpha d\theta + \beta d\mathbf{y})$ be a smooth symbol homogeneous of degree zero that is supported in the open set $\{(\theta, \mathbf{p}_{\theta}(\mathbf{x}), -\eta \partial_{\theta} \mathbf{p}_{\theta}(\mathbf{x}) d\theta + \eta d\mathbf{y}) \mid \theta \in U, \mathbf{x} \in V, \eta \in W\}$ and equal to one on a smaller conic neighborhood of λ_Y . Let A be the pseudodifferential operator on Y with symbol a . Then,

$$\mathcal{P}_{\mathbf{p}}f = A\mathcal{P}_{\mathbf{p}}(\varphi f) + (1 - A)\mathcal{P}_{\mathbf{p}}(\varphi f) + \mathcal{P}_{\mathbf{p}}((1 - \varphi)f). \quad (54)$$

Since $c(U \times V \times W) \subset \mathcal{C}$ was constructed so that the Bolker Assumption holds on it, $c(U \times V \times W)$ is a local canonical graph. The symbol of $A\mathcal{P}_{\mathbf{p}}\varphi$ is supported in this set and is elliptic near λ_Y , and $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(f)$. Therefore, $\lambda_Y \in \text{WF}(A\mathcal{P}_{\mathbf{p}}(\varphi f))$. Since $(1 - A)$ is microlocally smoothing near λ_Y , $\lambda_Y \notin \text{WF}((1 - A)\mathcal{P}_{\mathbf{p}}(\varphi f))$.

Now we show $\lambda_Y \notin \text{WF}(\mathcal{P}_{\mathbf{p}}(1 - \varphi)f)$. If $\lambda_Y \in \text{WF}(\mathcal{P}_{\mathbf{p}}(1 - \varphi)f)$ then

$$\lambda_Y \in \Pi_L(\Pi_R^{-1}(\text{WF}(1 - \varphi)f))$$

by (29). As there is only one angle, θ_0 , in $\Theta(\mathbf{x}_0, \xi_0)$ we have

$$\begin{aligned} \mathcal{A}(\mathbf{x}_0, \xi_0) &= \{(\mathbf{x}_1, \tau \eta_0 \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_1) d\mathbf{x}) \mid \mathbf{x}_1 \in \gamma_{\theta_0, \mathbf{p}_{\theta_0}}(\mathbf{x}_0), \tau > 0, \\ &\quad \eta_0(\partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{x}_1) - \partial_{\theta} \mathbf{p}_{\theta_0}(\mathbf{x}_0)) = 0\} \end{aligned} \quad (55)$$

where we have used $\xi_0 = \eta_0 \partial_{\mathbf{x}} \mathbf{p}_{\theta_0}(\mathbf{x}_0)$. The conditions in (55) are exactly the conditions in Theorem 3.1, (18) and (19) for covectors in \mathcal{C} to map to the same

covector under Π_L , in this case to γ_Y . That is, the only covectors $(\mathbf{x}, \boldsymbol{\xi} d\mathbf{x})$ such that $\gamma_Y \in \Pi_L(\Pi_R^{-1}(\{(\mathbf{x}, \boldsymbol{\xi} d\mathbf{x})\}))$ are those in $\mathcal{A}(\mathbf{x}_0, \xi_0)$. By (29), these are the only covectors that can add singularities to $\mathcal{P}_p(1-\varphi)f$ at γ_Y . Since $\mathcal{A}(\mathbf{x}_0, \xi_0) \cap \text{WF}(f) = \{(\mathbf{x}_0, \tau \xi_0 d\mathbf{x}) \mid \tau > 0\}$ and $(1-\varphi)$ is zero in a neighborhood of \mathbf{x}_0 , that shows that $\gamma_Y \notin \text{WF}(\mathcal{P}_p((1-\varphi)f))$. Because λ_Y is in the wavefront set of only one term in the sum (54),

$$\lambda_Y \in \text{WF}(\mathcal{P}_p f). \quad (56)$$

Since λ_Y has nonzero projection onto the $d\mathbf{y}$ coordinate, and D is elliptic in \mathbf{y} , D is elliptic in co-direction λ_Y . Now, by (56),

$$\lambda_Y \in \text{WF}(g) \text{ where } g := D\mathcal{P}_p f.$$

To show $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(\mathcal{P}_p^* D\mathcal{P}_p f)$ we break up

$$\mathcal{P}_p^* D\mathcal{P}_p f = \varphi \mathcal{P}_p^* A g + \varphi(\mathbf{x}) \mathcal{P}_p^*(1-A)g + (1-\varphi) \mathcal{P}_p^* g. \quad (57)$$

As in the proof of (56), since Π_R is an injective immersion on $c(U \times V \times W)$, we know that $\mathcal{C}^t \circ \{\lambda_Y\} = \Pi_R(\Pi_L^{-1}(\{\lambda_Y\})) = (\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(\varphi \mathcal{P}_p^* A \mathcal{P}_p f)$. Clearly $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \notin \text{WF}((1-\varphi) \mathcal{P}_p^* D\mathcal{P}_p f)$ since $1-\varphi$ is zero near \mathbf{x}_0 . Now we show

$$(\mathbf{x}_0, \xi_0 d\mathbf{x}) \notin \text{WF}(\varphi \mathcal{P}_p^*(1-A)g). \quad (58)$$

As noted in the first part of the proof, there is only one angle in $\Theta(\mathbf{x}_0, \xi_0)$ and there is a unique $\eta_0 \in \mathbb{R}^2$ satisfying $\xi_0 = \eta_0 \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0)$ and a unique $\lambda = (\theta_0, \mathbf{p}_{\theta_0}(\mathbf{x}_0), -\eta_0 \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0) + \eta_0 d\mathbf{y}; \mathbf{x}_0, \xi_0 d\mathbf{x}) \in \mathcal{C}$ with $\Pi_R(\lambda) = (\mathbf{x}_0, \xi_0 d\mathbf{x})$. Therefore, if $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(\varphi \mathcal{P}_p^*(1-A)g)$, then $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \mathcal{C}^t \circ \{\tilde{\lambda}_Y\}$ for some $\tilde{\lambda}_Y \in \text{WF}((1-A)g)$. This means that $(\tilde{\lambda}_Y; \mathbf{x}_0, \xi_0 d\mathbf{x}) \in \mathcal{C}$. However, the only covector in \mathcal{C} above $(\mathbf{x}_0, \xi_0 d\mathbf{x})$ is λ (since there is only one angle θ in $\Theta(\mathbf{x}_0, \xi_0)$), and $\Pi_L(\lambda) = \lambda_Y \notin \text{WF}((1-A)g)$ (since the symbol of $(1-A)$ is zero near λ). Therefore (58) is valid. Since $(\mathbf{x}_0, \xi_0 d\mathbf{x})$ has been shown to be in only one term of (57), the ellipticity condition in the theorem, (32), is valid. \square

Proof of Proposition 4. The outline is as follows. We first show that for each $\mathbf{x} \in \text{bd}(\mathcal{D})$ with $\mathbf{x} \neq (\pm r, 0, 0)$, there is a unique helix $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x})}$ tangent to $\text{bd}(\mathcal{D})$ at \mathbf{x} . Let ℓ be the axis of this helix, then the great circle of $\text{bd} \mathcal{D}$ perpendicular to ℓ contains \mathbf{x} . We then show that $\mathcal{A}'(\mathbf{x}, \pm \mathbf{x} d\mathbf{x})$ is the set of conormals to Q at points on this helix $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x})}$ that intersect Q . As \mathbf{x} is moved around this great circle, the points of intersection sweep out the two lines on Q that are parallel to the axis ℓ . Then, for $\theta \in [0, \pi]$, these lines sweep out Q . This finishes the outline of the proof.

To start the proof, we fix $\theta \in [0, \pi]$. If $\mathbf{z} \in \mathbb{R}^3$, we use the notation that

$$\tilde{\mathbf{z}} = A_\theta \mathbf{z}$$

where A_θ is the rotation matrix (10) given in Example 2.2.

We now explicitly compute the derivatives of the map \mathbf{p}_θ . Using the notation of Example 2.2, $\mathbf{p}_\theta(\mathbf{x}) = p(A_\theta \mathbf{x})$ is defined by (10) to (12). Then by the chain rule

$$\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}) = \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}) A_\theta \quad (59)$$

and

$$\partial_\theta \mathbf{p}_\theta(\mathbf{x}) = \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}) (\partial_\theta A_\theta) \mathbf{x} = \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}) (\partial_\theta A_\theta) A_\theta^T \tilde{\mathbf{x}} \quad (60)$$

where

$$\partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}) = \begin{pmatrix} \cos(\omega \tilde{x}_3) & -\sin(\omega \tilde{x}_3) & \omega(-\tilde{x}_1 \sin(\omega \tilde{x}_3) - x_2 \cos(\omega \tilde{x}_3)) \\ \sin(\omega \tilde{x}_3) & \cos(\omega \tilde{x}_3) & \omega(\tilde{x}_1 \cos(\omega \tilde{x}_3) - x_2 \sin(\omega \tilde{x}_3)) \end{pmatrix}. \quad (61)$$

We will use the notation for coordinates

$$\mathbf{x}_0 = (x_{0,1}, x_{0,2}, x_{0,3}) \quad \text{and} \quad \mathbf{x}_1 = (x_{1,1}, x_{1,2}, x_{1,3}).$$

If $(\mathbf{x}_0, \xi_0 d\mathbf{x}) \in \text{WF}(f)$ then $\mathbf{x}_0 \in \text{bd}(\mathcal{D})$ and $\xi_0 = c\mathbf{x}_0^T$ for some $c \neq 0$ where we use \mathbf{x}_0^T because of the convention that covectors are row vectors and points are column vectors. Let $\theta \in \Theta(\mathbf{x}_0, \xi_0)$. Since $\xi_0 = c\mathbf{x}_0^T d\mathbf{x}$, if $\eta \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $\eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_0) = \xi_0$, then $\eta \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}_0) A_\theta = \xi_0$, or equivalently

$$\eta \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}_0) = \xi_0 A_\theta^T = c\tilde{\mathbf{x}}_0^T. \quad (62)$$

Multiplying on the right by the vector $(\omega\tilde{x}_{0,2}, -\omega\tilde{x}_{0,1}, 1)^T$ annihilates the left hand side of the last equation (as this vector is orthogonal to the rows of the matrix (61) when $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0$), while the right hand side becomes $c\tilde{x}_{0,3}$, so if $\theta \in \Theta(\mathbf{x}_0, \xi_0)$ then $\tilde{x}_{0,3} = 0$. In particular, this means that $\tilde{\mathbf{x}}_0$ is on the plane $\tilde{x}_3 = 0$ and $\tilde{\mathbf{x}}_0 \in \text{bd}(\mathcal{D})$ because the disk \mathcal{D} is rotation invariant. Therefore, the helix $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ has radius r since it intersects the plane $\tilde{x}_3 = 0$ at the point, $\tilde{\mathbf{x}}_0$ that is r units from the origin. Furthermore, the axis ℓ of the helix is perpendicular to the plane $\tilde{x}_3 = 0$. Now, we plug $\tilde{x}_3 = 0$ into (61) and we see $\eta = c(\tilde{x}_{0,1}, \tilde{x}_{0,2})$.

Suppose now that \mathbf{x}_1 is another point on the curve $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ and that $\eta(\partial_\theta \mathbf{p}_\theta(\mathbf{x}_1) - \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0)) = 0$ (i.e., $(\mathbf{x}_1, \eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}_1)) \in \mathcal{A}(\mathbf{x}_0, \pm \xi_0)$). Now, using (62) we see that

$$\eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0) = \eta \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}_0) (\partial_\theta A_\theta) A_\theta^T \tilde{\mathbf{x}}_0 = c\tilde{\mathbf{x}}_0^T (\partial_\theta A_\theta) A_\theta^T \tilde{\mathbf{x}}_0.$$

A simple calculation shows that

$$(\partial_\theta A_\theta) A_\theta^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad (63)$$

and since $\tilde{x}_{0,3} = 0$ it follows that $\eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0) = 0$. Since \mathbf{x}_1 is on the curve $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ and $\tilde{x}_{0,3} = 0$, we can use equation (11) to assert that

$$\tilde{x}_{0,1} = \tilde{x}_{1,1} \cos(\omega\tilde{x}_{1,3}) - \tilde{x}_{1,2} \sin(\omega\tilde{x}_{1,3}) \quad (64)$$

$$\tilde{x}_{0,2} = \tilde{x}_{1,1} \sin(\omega\tilde{x}_{1,3}) + \tilde{x}_{1,2} \cos(\omega\tilde{x}_{1,3}) \quad (65)$$

or taking the inverse rotation

$$\tilde{x}_{1,1} = \tilde{x}_{0,1} \cos(\omega\tilde{x}_{1,3}) + \tilde{x}_{0,2} \sin(\omega\tilde{x}_{1,3}) \quad (66)$$

$$\tilde{x}_{1,2} = -\tilde{x}_{0,1} \sin(\omega\tilde{x}_{1,3}) + \tilde{x}_{0,2} \cos(\omega\tilde{x}_{1,3}). \quad (67)$$

Using this, it follows that

$$\partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}_1) \partial_\theta A_\theta A_\theta^T = \begin{pmatrix} 0 & -\omega\tilde{x}_{0,2} & \sin(\omega\tilde{x}_{1,3}) \\ 0 & \omega\tilde{x}_{0,1} & -\cos(\omega\tilde{x}_{1,3}) \end{pmatrix} \quad (68)$$

which implies that

$$\begin{aligned} \eta \partial_\theta \mathbf{p}_\theta(\mathbf{x}_1) &= \eta \partial_{\tilde{\mathbf{x}}} p(\tilde{\mathbf{x}}_1) \partial_\theta A_\theta A_\theta^T \tilde{\mathbf{x}}_1 \\ &= (\tilde{x}_{0,1} \sin(\omega\tilde{x}_{1,3}) - \tilde{x}_{0,2} \cos(\omega\tilde{x}_{1,3})) \tilde{x}_{1,3} \\ &= -\tilde{x}_{1,2} \tilde{x}_{1,3}. \end{aligned}$$

It follows that $\eta(\partial_\theta \mathbf{p}_\theta(\mathbf{x}_1) - \partial_\theta \mathbf{p}_\theta(\mathbf{x}_0)) = 0$ precisely if $\tilde{x}_{1,2} = 0$ or $\tilde{x}_{1,3} = 0$. If $\tilde{x}_{1,3} = 0$ then $\mathbf{x}_1 = \mathbf{x}_0$ (as $\tilde{\mathbf{x}}_0$ is the only point on the helix and the plane $\tilde{x}_3 = 0$), while $\tilde{x}_{1,2} = 0$ for a point on $\gamma_{\theta, \mathbf{p}_\theta(\mathbf{x}_0)}$ if and only if $x_{1,1} = \tilde{x}_{1,1} = \pm r$. Note that there are an infinite number of values for $\tilde{x}_{1,3}$, each differing by any integral multiple of the period for \tilde{x}_3 , $\frac{2\pi}{\omega}$ in equation (11). This proves that the points below added singularities for $\mathcal{L}(f)$ are as described at the start of this proof.

Finally, we compute $\xi_1 = \eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_1) d\mathbf{x}$. Using $\partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_1) = \partial_{\tilde{\mathbf{x}}} p(\mathbf{x}_1) A_\theta$ and the relations between \mathbf{x}_0 and \mathbf{x}_1 , it follows that $\eta \partial_{\mathbf{x}} \mathbf{p}_\theta(\mathbf{x}_1) = \eta \partial_{\tilde{\mathbf{x}}} p(\mathbf{x}_1) A_\theta = \tau(\tilde{x}_{1,1}, \tilde{x}_{1,2}, 0) A_\theta = (\tau r, 0, 0)$ for some $\tau \neq 0$.

To summarize, we have shown that $\mathcal{A}(\mathbf{x}_0, \pm \mathbf{x}_0 d\mathbf{x})$ consists of all $(\mathbf{x}_1, \tau d\mathbf{x}_1)$ such that \mathbf{x}_0 and \mathbf{x}_1 lie on the same helix of radius r , $\tau \neq 0$, and $x_{1,1} = \pm r$. Clearly the union of all $\mathcal{A}(\mathbf{x}_0, \mathbf{x}_0 d\mathbf{x})$ is the set \mathcal{Q} . By Corollary 1, Part 1, these are the covectors at which singularities at $(\mathbf{x}_0, \pm \mathbf{x}_0 d\mathbf{x})$ can add singularities. \square

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Received xxxx 20xx; revised xxxx 20xx.

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