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The Mathematics and Physics of Radiation Dose Planning using X-rays

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ABSTRACT. The goal of radiation therapy is to irradiate the body with ionizing radiation coming from different directions and to kill tumors without destroying healthy tissue. When X-rays are the ionizing radiation, we describe some of the physics of the problem, and we give mathematical models which use the dual Radon transform and the exponential dual transform. We present dose planning methods based on these models and give some simple dose plans which stem from these methods. Our mathematically more sophisticated methods exploit inversion formulas for the exponential transform and the null space and singular value decomposition for the dual Radon transform.

1. Introduction.

A common method for treating tumors in the body is to irradiate them with ionizing radiations and X-rays are the most common. The intensity of an X-ray beam falls off roughly exponentially on passing through matter with the unfortunate result that healthy tissue upstream of a tumor receives more radiation than the tumor itself, and healthy tissue downstream of the tumor receives unnecessary radiation. Thus, a single beam of X-rays will provide roughly the same dose (to be defined in §2) to healthy tissue on either side of the tumor as it does to the tumor, and radiation sufficient to damage the tumor can damage

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some surrounding tissue as well. A slightly more sophisticated radiation dose plan uses two beams at right angles, intersecting at the tumor. This will give roughly half the dose to healthy tissues as is given to the tumor. An obvious extension of this attempt to spare healthy tissue led to the development of *rotation therapy* in which an X-ray source can be continuously rotated around the patient. The source emits X-rays in a cone beam from the focal point of the source so intensity falls off approximately as the inverse square of the distance from the focal point. Thick adjustable absorbers with apertures in them control the lateral extent of the beam (the dimensions of the beam perpendicular to the line from the focal point of the source to the tumor). The intensity of the beam can also be varied as a function of the position of the source. In present rotation therapy, beam intensity is relatively constant across each beam with the result that the dose at the tumor is generally quite uniform. While this is satisfactory for some purposes, it is quite unsatisfactory when a critical organ near the tumor has to be spared as much as possible. However, computer control of machines is increasing rapidly enough so that large variations of intensity across the beam could be possible before too long.

Wilson [1946] has suggested that heavy ions be used as the ionizing radiation in radiation therapy. Heavy ions have two advantages over X-rays: (i) a monoenergetic beam of heavy ions passing through matter simply stops at some point, so if that point is chosen to be the downstream edge of the tumor, no tissue beyond that point is irradiated; (ii) heavy ion beams can be tailored to give *more* radiation to the tumor than to the healthy tissues upstream of the tumor. Despite the clear advantages of heavy ion beams over beams of X-rays, the number of facilities for producing such beams is small, and while this number is increasing slowly, the vast majority of candidates for radiation therapy will be treated with X-rays for some time in the future. This is the reason for continuing the study of treatment planning with X-rays.

Radiotherapy planning has a long history. At the beginning of the computer era, say thirty years ago, treatment planning proceeded as follows. The radiotherapist would judge, on the basis of previous experience, that say three beams from three given directions in a plane would be a good initial therapy plan. Isodose charts (which ignored inhomogenities such as those between lung tissue and bone) were superimposed and the doses were summed to produce the resulting dose distribution. This was examined by the radiotherapist and the procedure was modified in either minor or major ways, perhaps several times to achieve the final, acceptable dose plan. "Acceptability" meant giving the tumor the desired dose of radiation while sparing healthy tissue in a sense to be discussed later in this section. Such dose planning techniques are still commonly used. However, computers have enormously increased the speed of calculation of dose plans in

which non-coplanar beams can be used, and several alternative dose plans can be used as starting points in difficult clinical cases. Data from CAT-scans can be used to include in the calculations the inhomogenities in X-ray absorption of the patient. Typical of these treatment planning algorithms are those of Ebert [1977], Goitein [*et al.* 1983, 1988, 1989], and [McShann *et al.*] all of which consider the dose planning problem as a discrete problem. Censor [*et al.*] discretizes the problem completely and solves the resulting inverse problem.

Brahme, Roos and Lax [1982] addressed the problem differently. They asked: given a desired dose distribution, can one use *mathematical analysis* to calculate the distribution of beams necessary to produce that dose distribution? For circularly symmetric dose distributions, they answered this in the affirmative by deriving an integral equation which could be solved for the beam distribution. The most desirable dose distribution is one in which the dose is confined to the tumor and is zero elsewhere. Obviously, this is unachiveable since the X-ray beam has to get into the tumor and will harm tissue beyond the tumor. Application of the Brahme Roos and Lax equation to this "most desirable" distribution results in negative beam intensities, an impossibility. Hence analytical approaches have to be constrained or modified to yield only non-negative X-ray intensities.

The approach of Brahme, Roos, and Lax was extended to certain dose distributions without circular symmetry using techniques involving the dual Radon transform and the exponential dual transform by [Cormack], [Cormack and Cormack], and Cormack and Quinto [1989]. The aim of the present paper is to give an account of some extensions of these methods.

If it is not already possible, it should soon be possible to calculate, by various methods, the distribution of X-ray beams which will produce, say, a dose which is constant in a tumor and which minimizes the *average* dose to healthy tissue in some sense. This is not good enough because real tumors are often imbedded in, or intersected by vital organs, excessive damage to which would jeopardize the life of the patient. We are back to the question of the acceptability of a treatment plan and here still it is clinical experience alone which can favor one dose plan out of several alternatives. The complexity of the problem and the difficulty of reducing this to some kind of numerical cost-effectiveness analysis are discussed by Goitein [1989]. Until such an analysis is available, all that numerical or analytic approaches can do is to offer alternatives from which the radiotherapist will make a choice.

In §2 we will give a brief account of the interaction of X-rays with matter. For further details, the reader should consult [Johns and Cunningham], a classic in the field. We will describe mathematical models for this problem using the

exponential dual Radon transform in §3, and in §4 we will discuss two elementary dose planning methods (4.1-2) and two methods that are mathematically more sophisticated (4.3-4). The latter methods use inversion formulas for the exponential Radon transform and exploit the singular value decomposition and null space characterization of the dual transform. Dose plans from the elementary methods and discussion are in §5.

2. The interaction of X-rays with matter.

Consider a beam of particles travelling precisely parallel to each other in a vacuum. Suppose that N particles pass through unit area perpendicular to their direction of motion each second. The particles are distributed at random in space and time but N is assumed to be sufficiently large so that random fluctuations can be ignored.

Let this beam pass through a thin slab of material normal to the slab. Suppose that the slab has thickness dx and contains many targets with cross-sectional area σ distributed at random through the material such that there are n targets per unit volume on average, n being a large number. Suppose that a particle is absorbed by a target if it strikes it but continues through the material unaffected if it does not strike a target. Then on average, the fractional change in N when the particles pass through a slab of thickness dx is

$$\frac{dN}{N} = -n\sigma dx \ . \tag{2.1}$$

For a slab of finite thickness x and a beam of initial intensity N_o, the number of particles passing through the slab unaffected is found by integrating (2.1) and exponentiating to get

$$N = N_o e^{-n\sigma x} \ , \tag{2.2}$$

which is the exponential law of absorption.

If we write $n\sigma = \mu'$, μ' has the dimensions of an inverse length and Eq. 2 becomes

$$N = N_o e^{-\mu' x} \ . \tag{2.3}$$

This is a good model for the passage of a strictly parallel beam of monoenergetic X-ray photons passing through a slab of matter. Equation (2.3) gives the intensity of X-rays passing through the slab completely *unaffected by the slab*, but it tells us nothing about what is going on in the slab. To find this out, we must look in more detail at the interaction of X-ray photons with atoms.

There are many ways in which photons interact with atoms but for the photons which interest us (photons with energies of 1 MeV to 20 Mev) three interactions

dominate. These are called the photoelectric effect, the Compton effect and pair-production. Each process is characterized by a cross-sectional area of σ which depends on the photon energy and on the atomic number of the atom. Since the human body is largely water and other light elements the atomic numbers are generally small, and for low atomic number elements we can make the following statements.

In the photoelectric effect (cross-section σ_{PE}) the photon disappears and is replaced by an electron of slightly lower energy. These photoelectrons travel very roughly the same distance before coming to rest and they are emitted with a spread of angles about some characteristic angle relative to the direction of the incident beam. In the Compton effect (cross-section σ_C) the photon does not disappear, but is scattered with reduced energy at some angle and the remaining energy appears as the energy of an emitted electron, something like a billiard-ball collision between the photon and an electron in the atom. So in the Compton effect the original photon is replaced by a lower energy photon and an electron which takes up the rest of the energy. In pair-production (cross-section σ_P) the original photon disappears and is replaced by an electron-positron pair. In order to create an electron-positron pair the photon must have at least the energy necessary to create the rest-energy of the pair which is 1.02 MeV. So pair creation is impossible for energies less than 1.02 MeV. The net result of a beam of X-rays passing through material is the production of secondary electrons of limited range and some secondary photons which may produce further secondary electrons. Since the three processes are statistically independent, the cross-sections, which are really measures of the probabilities of the processes, are additive and the σ of (2.1) and (2.2) is given by

$$\sigma = \sigma_{PE} + \sigma_C + \sigma_P \ . \tag{2.4}$$

It is the secondary electrons and not the photons which produce biological effects in tissue through the ionization which they produce by virtue of their charge. The precise mechanisms by which the ionization damages and kills cells has been a subject of debate which we shall not enter into. Beyond all debate is the fact that this ionization kills rapidly multiplying cells (such as those found in tumors) more effectively than it kills normal cells. So the whole object of radiotherapy is to maximize the damage to abnormal tissue while minimizing the damage to normal tissue and normal organs which may be contained in abnormal tissue.

While many refinements are possible we shall simply take the harmful effects of X-rays to be measured by the amount of energy deposited per unit mass by the secondary electrons and refer to this as the *dose*.

What happens when a fine beam of X-rays uncontaminated by electrons passes through a slab of tissue normal to the slab? First, the intensity of photons which have not interacted with atoms will fall off exponentially, as in (2.3). The energy dissipated in the tissue by those photons which have interacted with it will be distributed as dose with circular symmetry about the line of the incident beam. The isodose surfaces will thus be symmetrical about the beam. The dose at the leading edge of the tissue, $x = 0$, will be zero because, by hypothesis, there are no electrons in the incident beam. Only when the beam has penetrated into the tissue will the dose build up from zero to some maximum (in a region called the "buildup region") after which it will decline following roughly the exponential law. The isodose surfaces will start out at zero radius at $x = 0$, will increase in radius to the maximum and then decrease in radius with increasing x. If μ' is small (*e.g.* for energies above 1 MeV) the exponential beyond the build-up region will be roughly linear and the isodose surfaces will be cones.

To achieve uniformity of dose it is common to use equal intensity beams oppositely directed along the same line. In this case the superposition of two sets of conical isodose surfaces will result in isodose surfaces which are cylinders about the direction of the beams, at least between the maxima of the individual beams. For low energies these isodose curves are close to the line of the incident beams, and the basis of the approximations discussed in §3 is that they are so close to the line of the beams that they can be considered to be confined to the line of the beams. At higher energies this is not true but the symmetry of the isodose cylinders is a justification to treat dose given by the beam as a convolution.

We conclude this section with some useful, approximate, absorption coefficients which are valid in the radiotherapy of tumors not near the skin. While there are many ^{60}Co machines around, their use seems to be declining and they are being replaced by electron linear accelerators, the energies of which have been increasing over the years. The gamma-rays from ^{60}Co machines are nearly monoenergetic with a mean energy of 1.25 MeV. The photons from linear accelerators have continuous energy spectra, but the cross-sections σ in (2.4) at these energies vary slowly with energy, so some simple average of these energies is a reasonable approximation. The numbers given below are for monoenergetic photons, but they are not far off in general. The quantity μ' in (2.3) varies quite widely with the atomic number of the atom but if μ' is divided by the density, ρ, the variation with atomic number is much slower and so more commonly given in tables. Here we write (2.3) as

$$N = N_o e^{(-\mu'/\rho)(\rho x)} = e^{-\mu t} \tag{2.5}$$

where

$$\frac{\mu'}{\rho} = \mu \quad and \quad t = \rho x.$$

Hence in c.g.s. units μ is measured in cm^2/gm and the "thickness" t is measured in gm/cm^2. Now for human tissue $\rho \approx 1 gm/cm^2$ (most of us just float in water!), hence μ can be thought of as μ', and a "thickness" of t gm/cm^2 is nearly the actual thickness of tissue in cm.

The quantities given in Table 1 may be useful in estimating the validity of approximations. They also show: (a) the slow variation of μ with energy above 1 MeV, and (b) the insensitivity of μ to type of tissue.

Table 1.

Absorption Coefficient, μ, $\mathrm{cm}^2/\mathrm{gm}$

Photon Energy in MeV	Bone	Muscle	Water
1	0.068	0.070	0.071
4	0.033	0.034	0.034
8	0.024	0.024	0.024
15	0.020	0.019	0.019
(ρ =	1.65 $\mathrm{gm/cm}^3$	1.04 $\mathrm{gm/cm}^3$	1.00 $\mathrm{gm/cm}^3$)

3. Mathematical models.

Because normal rotation therapy units rotate in a single plane around the patient, we will consider the problem in a planar cross-section of the body. In order to simplify the formulas, we will assume the cross-section of the body to be irradiated is convex. We will assume that body attenuation in the cross-section is constant and we will ignore scatter. Thus we assume the effects of X-rays and secondary electrons occur only on the line the X-rays traverse. We call this the *first-order approximation*, and the attenuation is assumed to be zero in the *zero-order approximation.*

For convenience, we will use a parallel beam parameterization of the set of lines. Let $\theta \in [0, 2\pi]$ and let $\overline{\theta} = (\cos\theta, \sin\theta)$, and $\theta^{\perp} = (-\sin\theta, \cos\theta)$, a vector perpendicular to $\overline{\theta}$. The line $L(\theta, p)$ that is perpendicular to $\overline{\theta}$ and p directed units from the origin is parameterized by $\overline{s}(t) = p\overline{\theta} + t\theta^{\perp}$ for $t \in \mathbb{R}$. Let l and r be the values of t at the "left" and "right" endpoints of the part of $\overline{s}(t)$ inside

the body.[1] These endpoints are defined by $r \leq l$ and they are known *a priori*.

Let the attenuation be denoted by μ and let $G(\theta, p)$ be the radiation intensity for photons irradiated along the line $x \cdot \overline{\theta} = p$ in direction $\theta^{\perp}$. According to the results of §2, the total intensity at point $\overline{s}(t)$ from the opposite directions $\theta^{\perp}$ and $(\theta + \pi)^{\perp}$ is

$$G(\theta, p) \exp(-\mu(l - t)) + G(\theta + \pi, -p) \exp(-\mu(t - r)) \ . \tag{3.1}$$

Figure 1: Parameterization of $L(\theta, p)$ including the endpoints l and r.

Define the auxiliary function, $F(\theta, p)$ by $F(\theta, p) = 2G(\theta, p) \exp(-\mu r)$ and assume $F(\theta, p) = F(\theta + \pi, -p)$; of course, this restricts G. Then the intensity given in (3.1) is

$$F(\theta, p) \cosh(\mu t) \ . \tag{3.2}$$

We will call F the *radiation intensity profile*. When $\mu = 0$, we have $F = G$.

The total radiation absorbed at the point x in the cross-section being irradiated by the rotation therapy machine is denoted $D(x)$, and the function D is called the *radiation dose distribution*. The value of $D(x)$ is the total intensity at x of X-rays from all lines passing through x–the integral of (3.1) or (3.2) over

[1] We suppress dependence of l and r on (θ, p). Figure 1 and the fact that $L(\theta + \pi, -p)$ is $L(\theta, p)$ parameterized in the opposite direction show why $r(\theta + \pi, -p) = -l(\theta, p)$.

all lines that pass through x. So, in appropriate units,

$$D(x) = R_\mu^* F(x) := \int_{\theta=0}^{2\pi} F(\theta, x \cdot \overline{\theta}) \cosh(\mu x \cdot \theta^\perp) d\theta \tag{3.3}$$

where R_μ^* is the dual exponential Radon transform [Tretiak]. For $\mu = 0$ the transform in (3.3) is the classical dual Radon transform; in this case, it is the same as the spherical mean transform of [Cormack 1963, Cormack and Quinto 1980].

The *forward problem*, finding $D(x)$ from $F(\theta, p)$ is solved by (3.3). Finding $F(\theta, p)$ from $D(x)$ is the *inverse problem*. As we will see in §4.3, the inversion formulas in [Tretiak and Metz, Hazou and Solmon] give a function F for which $R_\mu^* F = D$. However, even if D is non-negative, F can have negative values. This physical impossibility is the fundamental subtlety of the inverse problem. Thus a realistic inverse problem can be phrased:

Given a desired dose distribution, D, find a radiation intensity profile F that is **non-negative** *and such that $R_\mu^* F$ is close to D.*

The measure of closeness can best be decided by the algorithm developers in conjunction with radiotherapists and physicists. A mathematically natural measure of closeness is $||D - R_\mu^* F||_{L^1}$, but effects of X-rays are not exactly proportional to any power of intensity. A beam of intensity X might do very little harm, but a beam of intensity $4X$ might kill all the tissue it passes through. Thus, measures using intensity threshholds or L^p norms for large p might be more appropriate than L^1 norms.

4. Dose planning methods.

The present analytic dose planning methods are fairly rudimentary. In [Brahme, Roos and Lax] an inversion formula is developed to solve $R_\mu^* F = D$ for radial functions D. Their solution to the problem is, given radial desired dose distribution, D, to solve (3.3) for F and take the positive part of this radial solution as the radiation intensity profile. Cormack and Cormack [1987] have solved the problem for somewhat more complicated dose distributions D. Here we describe the following methods.

4.1. All-none method.

Often doctors want to irradiate one specific area, K, uniformly. We assume the zero-order approximation, attenuation $\mu = 0$. An elementary dose plan specifies the intensity F to be constant along lines of radiation passing through K and zero on lines not intersecting K. This methods irradiates K uniformly but

also irradiates the convex hull, C, of K with the same intensity as K. The dose falls off as $1/|x|$ for points x outside C. For K a disc, this corresponds exactly to the dose plan of [Brahme *et al.*]]. This method can be modified if there is a sensitive region, L to avoid; just block the radiation that would pass through L. However, the tumor will not necessarily be irradiated uniformly (see [Cormack 1987]). Dose distributions resulting from the basic and modified all-none method are evaluated in §5.

4.2. Backprojection method.

Let R_μ be the attenuated Radon transform [Tretiak]. In [Cormack and Quinto 1988] we suggest the following plan that can be applied for any desired positive dose distribution, D:

Choose a desired dose distribution $D \geq 0$ and let $F = R_\mu D$, then F is non-negative and $R_\mu^* F = D * \frac{2\cosh \mu|x|}{|x|}$ is an approximation to D. (F can be multiplied by a scaling factor to improve the approximation.)

The resulting dose distribution $R_\mu^* F$ is generally similar to D and is non-negative.

A refinement of this method is useful if certain regions must not be irradiated. Let $[a, b] \subset [0, \pi]$ and $A = [a, b] \cup [a + \pi, b + \pi]$. If $F = \chi_A(\theta) R_\mu D$ is the radiation intensity profile, then no radiation will be given for angles θ outside of A. In this case, $R_\mu^* F = D * \frac{2C(x)\cosh \mu|x|}{|x|}$ where $C(r\overline{\theta})$ is one for $\theta + \pi/2 \in A$ and zero otherwise; the resulting radiation dose distribution is D convolved with a function that is zero outside of a "bow tie" region. In §5 we test another refinement of this method in which dose is given by the formula for F above for lines that intersect the tumor but not the region that must not be irradiated and F is zero for all other lines. This is analogous to the modified all-none method. Dose distributions resulting from the backprojection method are evaluated in §5.

4.3 Direct inversion.

Any inversion formula for R_μ^* gives a candidate for F. For example, filtered back projection [Tretiak, Tretiak and Metz] can be used to find a non-positive candidate:

$$\textit{given a desired } D, \textit{ let } F = \Lambda_\mu R_\mu \ D \textit{ then } R_\mu^* F \cong D \tag{4.1}$$

where Λ_μ is the Riesz potential in the inversion formula of Tretiak and Metz or an approximation thereto [Hazou and Solmon]. The simplest positive radiation intensity profile related to F is be the positive part of F. Such a plan will be evaluated in a future article.

4.4. The null space and singular value decomposition.

A more refined method than that proposed in §4.3 can be developed using the null space, N_μ, of R_μ^*. Such a method will come up with an inverse, F, using an inversion formula such as (4.1). Then the method will choose a function $\tilde{F}$ in the convex set $F + N_\mu$ that is close[2] to being positive and take *its* positive part as the radiation intensity profile. The potential advantage of this proposal over just truncating F is that $\tilde{F}$ should be more positive than F.

For $\mu \neq 0$ the null space is not known but should be derivable from the range characterization of the exponential Radon transform [Natterer, Kuchment and L'vin].

The null space is well known for attenuation $\mu = 0$; Helgason's range theorem [1984] provides a complete characterization of N_0. As in the general case, any inversion method can be used to find a candidate for F. The positive part of the function in $F + N_0$ "closest" to being positive will then be the radiation intensity profile.

A specific proposal using a singular value decomposition for R_0^* ([Louis 1984] or [Cormack 1964]) is now given. We assume the cross-section of the body is contained in the disc $|x| \leq 10$ so we are only interested in values of $D(x)$ for $|x| \leq 10$ and $F(\theta, p)$ for $|p| \leq 10$. Given desired dose distribution D, the rough plan is:

1. *use the singular value decomposition to find the unique function $F_R \in N_0^\perp$ that satisfies $R_0^* F_R = D$;*

2. *add an appropriate linear combination of null singular functions, F_N, to F_R to get a solution $F_R + F_N$ to $R_0^*(F_R + F_N) = D$ that is "most" positive (use the basis of the null space to find this most positive function in $F_R + N_0$);*

then

3. *let the radiation intensity profile F be the positive part of the function $F_R + F_N$ chosen in 2.*

The simplest dose plan, using item 1. only, will take the positive part of F_R as the radiation intensity profile. As F_R is the L^2 minimum norm solution to $R_0^* F = D$, it should not be "too negative" itself. Also using this "fairly positive" solution F_R as a start for the algorithm for 2. and 3. rather than the solution from formula (4.1) might give the closest positive function to $F_R + N_0$ more easily.

[2]The measure of closeness will be determined from both mathematical and medical considerations.

The authors' dose plan algorithms using the complete method 1.-3. are still in preliminary stages. Item 2 will be the main focus of research. Simple methods to choose F_N could involve trial and error; more complicated ones could involve iterative procedures to select candidates for F_N for which $F_R + F_N$ is close to the convex set of positive functions.

5. Examples.

Dose plans for specific tumors using methods 4.1 and 4.2 are now analyzed. The tumors sit in circular cross-sections of the body of radius 10. Tumor A is a half-disc of radius 1 with center at the origin of the cross-section. The analytic dose plan methods in [Brahme, Roos, and Lax] and [Cormack and Cormack] cannot be applied to tumor A because it is not sufficiently symmetric. Tumor B is an annulus with outer radius 1 and inner radius 0.8 and centered at the origin. We assume tumor B surrounds a sensitive region with radius 0.75. For this comparison, units are chosen so that 1 unit is the minimum dose required to kill the tumor. Therefore, in each case, the desired dose distribution for each tumor is one unit of dose to points in the tumor and zero units outside the tumor.

Tests were done with both $\mu = 0$ and $\mu = 0.024$. The displays of analogous dose plans for $\mu = 0$ and $\mu = 0.024$ look almost identical, and the analogous numerical data vary by about 1%. The data discussed below all are for $\mu = 0.024$.

The results on the left of figure 2 use the all-none dose plan method of §4.1. For tumor A, the basic all-none method is used: constant radiation is given to the body cross-section on lines intersecting this tumor and zero radiation on lines not intersecting the tumor. Tumor A is irradiated evenly and sufficiently to kill it. Although the basic version of this method gives exact dose in the tumor only for attenuation $\mu = 0$, the dose distributions in the tumor for both $\mu = 0$ and $\mu = 0.024$ agree almost exactly and each varies about 3% throughout the tumor.[3] Radiation intensity goes rapidly to zero (see table 2).

Because tumor B surrounds a sensitive organ, the modified all-none method (see §4.1) is used: constant radiation is given to lines that intersect the tumor but not the sensitive region, zero radiation is given over all other lines. Thus, the sensitive region inside the the circle of radius 3/4 is not irradiated at all. Again, the tumor is irradiated with enough energy to kill it. Because radiation is not given over some lines passing through the tumor (those also passing through the sensitive organ), the radiation through the tumor varies (by a factor of 2), with highest dose at the outer boundary of the tumor.[4] Therefore, some material outside the tumor is irradiated as much as the tumor is (at radius $r = 1.1$ from the center of the cross-section, the dose is 1.09, at $r = 1.2$, the dose is 0.87).

Dose outside the tumor can be decreased using this method by not irradiating along lines near the outer boundary of the tumor. For example, by irradiating

[3]Because the dose distributions are almost identical for $\mu = 0$ (when the dose should be uniform) and for $\mu = 0.024$, dose variation must be due primarily to round off and discretization error.

[4]Radiation given on each line intersecting the tumor is constant, and more lines through the tumor pass through points on the outer boundary.

only along lines $L(\theta, p)$ with $p \in [0.75, 0.93]$, our calculations show that dose at the inside and outside boundaries of the tumor are 1.0, and dose varies by only 70% in the tumor. Furthermore, dose decreased more rapidly than if radiation is given on all lines with $p \in [0.75, 1]$; dose becomes 0.5 at $r = 1.3$ and 0.25 at $r = 2.2$.

The results on the right of figure 2 are from the backprojection dose plan method of §4.2. Radiation is given over lines intersecting the convex hull of the support of the tumor (those in the support of $R_\mu D$ where D is the desired dose distribution). As opposed to the method of §4.1, the radiation is not constant, but the dose given along line ℓ is a multiple of $R_\mu D(\ell)$. The multiplication factor is chosen to guarantee sufficent radiation to kill the entire tumor. With this method, some parts of the tumor can receive much more than enough radiation to kill them. In tumor A the maximum dose is about twice the minimum dose. Dose in tumor B is fairly constant, but the sensitive organ inside also is irradiated. The average radiation in the sensitive organ is 0.77 and the minimum radiation is 0.66 (at the center) and radiation is 0.88 at $r = 0.7$.

If the sensitive region cannot be irradiated at all, the backprojection method is undesirable. If the backprojection method is used but dose is given only over lines passing through the tumor and not through the sensitive region, our calculations show that dose in the tumor varies by 650%, and about 24.5 times as much radiation is given to the surrounding region as in the modified all-none method when the tumor is irradiated enough to kill it.

In table 2, the resulting dose distributions using these methods are compared to the desired distributions (1 in the tumor 0 elsewhere) for attenuation $\mu = 0.024$. The first set of data give percent variation between maximum and minimum dose in the tumor ([max dose-min dose]/min dose). The second set of results are values of the radius r at which dose is 0.5 units, half of the dose that is desired *in the tumor*. The third set give radii at which dose is 0.25 units.

Table 2.

	All-None (§4.1)	Back-projection (§4.2)
	Percent dose variation inside the tumor for $\mu = 0.024$.	
Tumor A	3.0%	90.7%
Tumor B	105.9%	7.0%
	Radius at which dose is 0.5 units.	
Tumor A	1.4	1.8
Tumor B	1.7	1.4
	Radius at which dose is 0.25 units.	
Tumor A	2.7	3.1

Tumor B	2.8	2.5

The pictures in figure 2 are the resulting dose distributions from the dose plan methods of §4.1 and 4.2. They are presented using a linear grey scale with 256 shades of grey from black to white and 100×100 pixels. Black represents zero dose. The grey scales are the same in all pictures.

Figure 2A Basic all-none method (left) and backprojection method (right) for tumor A.

Figure 2B Modified all-none method (left) and backprojection method (right) for tumor B.

For tumor A, the all-none method might be better because this method gives almost constant dose in the tumor and lower dose overall. In contrast, the back projection method provides over 90% more dose to parts of the tumor than is necessary and so more dose overal. For tumor B, the modified all-none method might be better if the sensitive organ should receive no dose. If the sensitive organ can receive substantial dose (but less dose than the tumor), then the backprojection method gives less dose overall.

In general, the all-none method might be appropriate for simple convex tumors, but the backprojection method could be useful for more complicated tumors. Final determination of the utility of such methods rests with the radiotherapists.

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