

# MORERA THEOREMS FOR COMPLEX MANIFOLDS

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ABSTRACT. We prove Morera theorems for the Radon transform integrating on geodesic spheres on complex analytic manifolds of arbitrary dimension. To avoid pathologies, we assume that the radius of each sphere of integration is less than the injectivity radius at its center. The proofs of the main results are local, and they involve the microlocal properties of associated Radon transforms and a theorem of Hörmander, Kawai, and Kashiwara on microlocal singularities. We consider Morera theorems for spheres of fixed radius and spheres of arbitrary radius.

## 1. INTRODUCTION

The classical Morera Theorem states that, if  $\int_C f(z) dz = 0$  for all simple closed curves in a region in the complex plane, then  $f$  is holomorphic in that region. Using harmonic and complex analysis, authors have proven more general Morera theorems that specify subclasses of curves which can be used to determine holomorphy in the plane (see *e.g.*, [BG 1986, BG 1988, BZ, Gl 1989, Gl 1990, Za 1972, Za 1980, BCPZ]).

Authors have generalized some of these results to  $\mathbb{C}^n$  and other complex manifolds [Ag 1978, ABCP, Be, BZ, BG 1986, BG 1988, BP]. Many of the generalizations follow from Pompeiu theorems using Stokes' Theorem. In [Za 1972] the author proves that if the integrals of a function over disks of two well chosen radii is zero, then the function is zero (see also [DL]). This theorem allows one to infer holomorphy of a function  $f$  if one knows that all integrals of  $f$  with respect to constant coefficient  $(n, n-1)$  forms are zero over all spheres of two well chosen radii. This Morera Theorem follows by using Stokes' Theorem to reduce integrals of  $f(*d\bar{z}_j)$  over a sphere to Pompeiu integrals of  $\frac{\partial f}{\partial \bar{z}_j}$  over a disk. Here,  $*$  is the Hodge star operator. Important local versions of this theorem

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are in [Be, BG 1986, BG 1988] and inversion methods are in [BGY]. Morera theorems for constant coefficient  $(n, n-1)$  forms and spheres containing the origin in  $\mathbb{C}^n$  are proven in [GrQ].

In [BZ], the authors prove the analogous Pompeiu theorem on non-compact rank one symmetric spaces. They also prove that if  $M$  is a compact rank-one symmetric space and  $u \in L^1(M)$  and integrals of  $u$  over balls of one well chosen radius are zero, then  $u = 0$  [BZ, Theorem 4]. As in  $\mathbb{C}^n$ , these theorems can give Morera theorems on other spaces. In [BP], the authors prove Morera theorems on the hyperbolic disk for the Möbius group and circles and non-analytic Jordan curves. In [ABC, ABCP], the authors prove Morera theorems on the Heisenberg group. See [Za 1980, BCPZ, Za 1992] for excellent surveys of these problems.

Related results infer holomorphy from holomorphy in directions. For example, a theorem of Forelli says a function in  $\mathbb{C}^n$  which is holomorphic (in one variable) on each complex line through the origin and is  $C^\infty$  at the origin is holomorphic on  $\mathbb{C}^n$  [Ru, Theorem 4.4.5, pp. 60-61].

Our theorems have a different character from the classical theorems. In the spirit of the Morera theorems for  $\mathbb{C}$  in [GlQ], the proofs use microlocal analysis. Our theorems are valid for quite arbitrary complex manifolds and for distributions defined on them. In our proofs, we start with a function (or distribution)  $f$  that is holomorphic on a small starter set  $V$  and that has real-analytic integrals with respect to enough differential forms on enough spheres. The theory of real-analytic Fourier integral operators is used to deduce analytic smoothness of a distribution  $f$  from smoothness assumptions on a Radon transform  $R_{\omega,r}f$  (e.g., Proposition 3.1.1). The del-bar derivative  $\bar{\partial}f$  has the same analytic smoothness as  $f$ , and  $\bar{\partial}f$  is zero on  $V$  (recall on  $\mathbb{C}^n$  that  $\frac{\partial f}{\partial \bar{z}_j} = \frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j}$ ,  $d\bar{z}_j = dx_j - i dy_j$ , and  $\bar{\partial}f = \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$  and the definition of  $\bar{\partial}f$  can be defined invariantly on a complex manifold manifold using complex local coordinates  $(z_1, \dots, z_n)$ ). Then, a theorem of Hörmander, Kawai, and Kashiwara [Hö 1983, Ka] about analytic singularities and support (Lemma 3.1) is used to show  $\bar{\partial}f = 0$  on a larger set using this analytic smoothness of  $\bar{\partial}f$  and the fact  $\bar{\partial}f$  is zero on a starter set. This method allows us to analytically continue  $f$  to the larger set.

Such starter sets are not present in most classical Morera theorems, so we provide counterexamples to our conclusions when the functions are not holomorphic on small sets. Requiring  $f$  to be holomorphic on a starter set is a restrictive additional assumption, but the theorems are valid in great generality. The theorems are also valid if one replaces global forms by forms defined locally (Theorems 2.1.3, 2.2.3).

In Theorem 2.2.2, the starter set is a sphere and  $f$  is assumed to be holomorphic to infinite order on it; in the theorems in §2.1, the starter set is an open set and  $f$  is assumed to be holomorphic on the set.

## 2. THE MORERA THEOREMS

Let  $M$  be a complex analytic manifold of complex dimension  $n$ . Assume  $M$  has a real-analytic Riemannian structure that is consistent with its complex structure. Let  $d(x, y)$  be the geodesic distance between  $x \in M$  and  $y \in M$ . For  $y \in M$  and  $r > 0$ , define  $S(y, r)$  to be the geodesic sphere of radius  $r$  centered at  $y$ ,  $S(y, r) = \{x \in M \mid d(x, y) = r\}$ , and let  $D(y, r)$  denote the closed disk  $D(y, r) = \{x \in M \mid d(x, y) \leq r\}$ . Let  $I_y \in (0, \infty]$  denote the *injectivity radius* of the exponential map at  $y$  ( $I_y$  is the radius of the largest open disk centered at zero in the tangent space  $T_y M$  on which the exponential map is injective). Standard results in differential geometry demonstrate that the map  $y \mapsto I_y$  is a lower semicontinuous function for  $y \in M$ . If  $I_y > r$ , then  $S(y, r)$  is the boundary of  $D(y, r)$  and  $S(y, r)$  is the diffeomorphic image under the exponential map of the Euclidean sphere of radius  $r$  centered at the origin in the tangent space  $T_y M$  [KN, IV 3.4].

**Definition 2.1.** Let  $\mathcal{A} \subset M$  and let  $r > 0$ . Then, we define  $S(\mathcal{A}, r)$  to be the union of the set of spheres parameterized by points in  $\mathcal{A}$ :  $S(\mathcal{A}, r) = \cup_{y \in \mathcal{A}} S(y, r)$ . We define  $D(\mathcal{A}, r)$  to be the union of the set of disks parameterized by points in  $\mathcal{A}$ :  $D(\mathcal{A}, r) = \cup_{y \in \mathcal{A}} D(y, r)$ . If  $\mathcal{B} \subset M \times (0, \infty)$ , then, we define  $S(\mathcal{B})$  to be the union of the set of spheres parameterized by points in  $\mathcal{B}$ :  $S(\mathcal{B}) = \cup_{(y, s) \in \mathcal{B}} S(y, s)$ .

Note that if  $\mathcal{A} \subset M$  is open and  $r \in \mathbb{R}$  satisfies  $r < I_y$  for all  $y \in \mathcal{A}$  then  $S(\mathcal{A}, r)$  and  $D(\mathcal{A}, r)$  are open sets.

Let  $\Lambda^{2n-1}(M)$  be the set of differential forms of degree  $2n - 1$  on  $M$  with complex valued *real-analytic* coefficients and let  $\Lambda^{n, n-1}(M) \subset \Lambda^{2n-1}(M)$  be the subset of  $(n, n - 1)$  forms (those  $2n - 1$  forms that are complex linear in the  $n$  *holomorphic* vector fields  $\frac{\partial}{\partial z_j}$  and complex conjugate linear in the *anti-holomorphic* fields  $\frac{\partial}{\partial \bar{z}_j}$  where  $(z_1, \dots, z_n)$  are complex local coordinates).

This research is based on the pioneering work of Guillemin and Sternberg [Gu, GS] that uses microlocal analysis to understand Radon transforms. Sunada [S] and Tsujishita [Ts] have proven that a transform closely related to the transform in 2.1 is a Fourier integral operator (FIO) (see also [Gu]). Our proofs are in the same spirit as those in [BQ 1993, Q 1980, GIQ].

### 2.1. Spheres of fixed radius.

Let  $\mathcal{A} \subset M$  be open. Assume that  $r < I_y$  for each  $y \in \mathcal{A}$ . Let  $f$  be a continuous function on  $M$  and let  $\omega \in \Lambda^{2n-1}(S(\mathcal{A}, r))$ , then the *Radon transform* of  $f$  is defined for  $y \in \mathcal{A}$  by

$$(2.1.1) \quad R_{\omega, r} f(y) = \int_{z \in S(y, r)} f(z) \omega.$$

This is the integral of the  $2n - 1$  form  $f \omega$  over the geodesic sphere  $S(y, r)$  (which is diffeomorphic to  $S^{2n-1}$  by the assumption that  $r < I_y$ ). It is known that  $R_{\omega, r}$  is a Fourier integral operator (FIO)

[Gu, Su], and this implies that we can extend the definition of this transform to distributions:  $R_{\omega,r} : \mathcal{D}'(M) \rightarrow \mathcal{D}'(\mathcal{A})$ . One can also prove this using an elementary argument under slightly stronger assumptions: if we assume  $I_y > r \forall y \in D(\mathcal{A}, r)$ , then  $R_{\omega,r}^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(D(\mathcal{A}, r))$  is continuous and by duality,  $R_{\omega,r} : \mathcal{D}'(D(\mathcal{A}, r)) \rightarrow \mathcal{D}'(\mathcal{A})$  is also continuous. Of course,  $R_{\omega,r}$  can be extended to domain  $\mathcal{D}'(M)$ . Developing theorems for  $r > I_y$  will be an intriguing continuation of this research.

Our first theorem is analogous to the classical Morera theorems.

**Theorem 2.1.1.** *Let  $M$  be a complex Riemannian manifold with a real-analytic Riemannian structure and let  $\mathcal{A} \subset M$  be open and connected. Let  $r > 0$  and assume for each  $y \in D(\mathcal{A}, r)$ ,  $I_y > r$ . Let  $R_{\omega,r}$  be a Radon transform on geodesic spheres in  $M$  of radius  $r$ . Let  $f \in \mathcal{D}'(M)$ .*

(2.1.2) *Let  $L \subset \Lambda^{n,n-1}(D(\mathcal{A}, r))$  be a set of **closed** forms such that for each  $y \in \mathcal{A}$  and each  $x \in S(y, r)$ , there is a form in  $L$  that is nondegenerate on  $T_x S(y, r)$ .*

*Assume  $R_{\omega,r} f(y) = \int_{S(y,r)} f \omega = 0$  for all  $y \in \mathcal{A}$  and for all  $\omega \in L$ . Assume, for some  $y_0 \in \mathcal{A}$ ,  $f$  is holomorphic on a neighborhood of the disk  $D(y_0, r)$ . Then  $f$  is holomorphic on  $D(\mathcal{A}, r)$ .*

The converse of this theorem is simple to prove. Namely, let  $f$  be a holomorphic function on  $M$  and let  $\omega$  be a closed  $(n, n-1)$  form on  $D(\mathcal{A}, r)$ . Then, an application of Stokes' theorem shows that  $R_{\omega,r} f(y) = \int_{D(y,r)} \bar{\partial} f \wedge \omega$  since  $d\omega = 0$  on  $D(y, r)$  and  $\omega$  is an  $(n, n-1)$  form. Since  $f$  is holomorphic, this integral is zero.

A counterexample, Example 3.1.3, is given if the hypothesis of Theorem 2.1.1, “ $f$  is holomorphic on a neighborhood of  $D(y_0, r)$ ,” is weakened to become “ $f$  is holomorphic on a neighborhood of  $S(y_0, r)$ .” This also is a counterexample to Theorem 2.1.3 below.

There are manifolds for which no sets of forms  $L \subset \Lambda^{n,n-1}(M)$  satisfying (2.1.2) exist globally; however a basis of closed  $(n, n-1)$  forms on  $\mathbb{C}^n$  and satisfying (2.1.2) is constructed in Example 2.1.2. By using local coordinates this example can be adapted, at least locally, to any complex manifold. Such local forms are sufficient for our more general Morera theorems such as Theorem 2.1.3.

Recall that the *Hodge star* operator,  $*$  on a manifold is defined in terms of an orientation on that manifold. On  $\mathbb{C}^n$  we will choose the orientation  $v = \mathbf{d}\mathbf{x}_1 \wedge \mathbf{d}\mathbf{y}_1 \wedge \cdots \wedge \mathbf{d}\mathbf{x}_n \wedge \mathbf{d}\mathbf{y}_n$  and define  $*$  so that  $\mathbf{d}\bar{\mathbf{z}}_j \wedge *\mathbf{d}\bar{\mathbf{z}}_j = v$ .

**Example 2.1.2.** The set  $L = \{*\mathbf{d}\bar{\mathbf{z}}_j \mid j = 1, \dots, n\} \subset \Lambda^{n,n-1}(\mathbb{C}^n)$  satisfies (2.1.2) and can be used in Theorem 2.1.1. This is true because, restricted to  $S(y, r)$ , the form  $*\mathbf{d}\bar{\mathbf{z}}_j$  is the function  $(\bar{z}_j - \bar{y}_j)/r$  times the standard measure on  $S(y, r)$ . Since  $L$  contains each  $*\mathbf{d}\bar{\mathbf{z}}_j$  for all  $j = 1, \dots, n$ ,  $L$  has a nondegenerate form at each point on each sphere  $S(y, r)$ . The linear span of  $L$  is the set

of constant coefficient  $(n, n - 1)$  forms used in the theorems of [Be, BZ, BG, GrQ].

Theorem 2.1.1 follows immediately from the following theorem.

**Theorem 2.1.3.** *Let  $M$  be a complex Riemannian manifold with a real-analytic Riemannian structure and let  $\mathcal{A} \subset M$  be open and connected. Let  $r > 0$  and assume for each  $y \in D(\mathcal{A}, r)$ ,  $I_y > r$ . Let  $f \in \mathcal{D}'(D(\mathcal{A}, r))$ .*

(2.1.3) *Assume for each  $z \in \mathcal{A}$  there is an open neighborhood of  $z$ ,  $\mathcal{D}_z \subset \mathcal{A}$  and a set  $L_z \subset \Lambda^{2n-1}(S(\mathcal{D}_z, r))$  satisfying the following: for each  $y \in \mathcal{D}_z$  and each  $x \in S(y, r)$  there is an  $\omega \in L_z$  that is nondegenerate on  $T_x S(y, r)$ .*

*Assume, for each  $z \in \mathcal{A}$  and each  $\omega$  in  $L_z$ , that  $R_{\omega, r} f(y) = \int_{S(y, r)} f \omega$  is a real-analytic function for  $y \in \mathcal{D}_z$ . Assume, for some  $y_0 \in \mathcal{A}$ ,  $f$  is holomorphic on a neighborhood of the disk  $D(y_0, r)$ . Then  $f$  is holomorphic on  $D(\mathcal{A}, r)$ .*

Because non-compact Hermitian symmetric spaces and  $\mathbb{C}^n$  both have infinite injectivity radius, our theorems can be applied for spheres of any radius in these spaces.

The next example shows how to construct a neighborhood  $\mathcal{D}_z$  and one form in  $\Lambda^{2n-1}(S(\mathcal{D}_z, r))$  satisfying (2.1.3) for any complex manifold with a real-analytic Riemannian structure. As noted above, Example 2.1.2 can also be used to get a finite set of closed  $(n, n - 1)$  forms satisfying (2.1.3) locally on manifolds.

**Example 2.1.4.** Let  $M$  be an arbitrary complex-analytic manifold with a real-analytic structure. Let  $\mathcal{A} \subset M$  and let  $z \in \mathcal{A}$  with  $I_z > r$ . Define  $\phi(x) = d(z, x)$ . Now,  $d\phi$  is real and nonzero when restricted to tangent spaces above points in  $S(z, r)$ , so  $\bar{\partial}\phi$  must also be nonzero. Therefore,  $*\bar{\partial}\phi$  is nondegenerate above all points of  $S(z, r)$  and so at all points of all sufficiently nearby spheres. By continuity and compactness, there is a small neighborhood  $\mathcal{D}_z$  of  $z$  such that  $*\bar{\partial}\phi$  is nondegenerate above all points in  $S(y, r)$  for all  $y \in \mathcal{D}_z$ .

## 2.2. Spheres with arbitrary radius.

Now, we consider spheres with arbitrary radius. The associated Morera problem is dimensionally overdetermined. For  $f \in C(M)$ ,  $\omega \in \Lambda^{2n-1}(M)$ , we define

$$(2.2.1) \quad R_{\omega} f(y, r) = \int_{z \in S(y, r)} f(z) \omega$$

where  $(y, r) \in M^+ = \{(y, r) \in M \times (0, \infty) \mid r < I_z \ \forall z \in D(y, r)\}$ .

Of course, this is the Radon transform in (2.1.1) but here  $r$  is not fixed. As  $D(y, r)$  is compact and  $z \mapsto I_z$  is lower semicontinuous,  $M^+$  is open in  $M \times (0, \infty)$ . Also, if  $\mathcal{B} \subset M^+$  is open, then  $S(\mathcal{B})$  is open in  $M$ .

Using the assumption that  $R_\omega f$  is defined on  $M^+$  and an elementary duality argument, one can show  $R_\omega$  can be evaluated on distributions.

The set of spheres in §2.1 has the same dimension as  $M$ . Because the set of spheres with arbitrary radius is dimensionally overdetermined—a stronger condition than in §2.1—the theorems in this section are stronger. The function  $f$  is not required to be holomorphic on some open starter set but only to be holomorphic to infinite order on a sphere.

**Definition 2.2.1.** Let  $M$  be a complex-analytic manifold and let  $T \subset M$  be a  $C^2$  submanifold. For  $z \in M$ , let  $d(z, T)$  be the minimum geodesic distance from  $z$  to  $T$ . Let  $f$  be a  $C^1$  function (or distribution) in a neighborhood of  $T$ . We say  $f$  is *holomorphic to infinite order on  $T$*  if and only if  $\bar{\partial}f$  is zero to infinite order on  $T$  (i.e., for each point  $z \in T$  and each set of complex local coordinates  $(z_1, \dots, z_n)$  near  $z$  on  $M$ , all derivatives of  $\frac{\partial f}{\partial \bar{z}_j}(z)$  are functions near  $z$  that are  $\mathcal{O}(d(z, T)^m) \forall m \in \mathbb{N}$ ).

This definition makes sense even if  $T$  is a point.

**Theorem 2.2.2.** *Let  $M$  be a complex Riemannian manifold with a real-analytic Riemannian structure and let  $\mathcal{A}$  be an open, connected subset of  $M^+$ . Let  $f$  be a continuous function on  $S(\mathcal{A})$  (or a distribution).*

(2.2.2) *Let  $L \subset \Lambda^{2n-1}(S(\mathcal{A}))$  be a set of forms such that for each  $(y, r) \in \mathcal{A}$  and each  $x \in S(y, r)$ , there is a form in  $L$  that is nondegenerate on  $T_x S(y, r)$ .*

*Assume for each  $\omega \in L$  that  $R_\omega f(y, r) = \int_{S(y, r)} f \omega$  is real-analytic on  $\mathcal{A}$ . Assume  $f$  is holomorphic to infinite order on  $S(y_0, r_0)$  for some  $(y_0, r_0) \in \mathcal{A}$ . Then  $f$  is holomorphic on  $S(\mathcal{A})$ .*

Theorem 2.2.2 can be applied locally by using it successively in local neighborhoods, even if there are no forms satisfying (2.2.2) globally on  $M$ . Moreover, Example 2.1.2 provides closed  $(n, n-1)$  forms that satisfy (2.2.2) locally. The local theorem is as follows.

**Theorem 2.2.3.** *Let  $M$  be a complex Riemannian manifold with a real-analytic Riemannian structure and let  $\mathcal{A}$  be an open, connected subset of  $M^+$ . Let  $f$  be a continuous function on  $S(\mathcal{A})$  (or a distribution).*

(2.2.3) *Assume for each  $(y_1, r_1) \in \mathcal{A}$  there is an open neighborhood of  $(y_1, r_1)$ ,  $\mathcal{A}_{(y_1, r_1)} \subset \mathcal{A}$  and a set  $L_{(y_1, r_1)} \subset \Lambda^{2n-1}(S(\mathcal{A}_{(y_1, r_1)}))$  satisfying the following: for each  $(y, r) \in \mathcal{A}_{(y_1, r_1)}$  and each  $x \in S(y, r)$  there is an  $\omega \in L_{(y_1, r_1)}$  that is nondegenerate on  $T_x S(y, r)$ .*

*Assume for each  $(y_1, r_1) \in \mathcal{A}$  and each  $\omega \in L_{(y_1, r_1)}$  that  $R_\omega f(y, r) = \int_{S(y, r)} f \omega$  is real-analytic on  $\mathcal{A}_{(y_1, r_1)}$ . Assume  $f$  is holomorphic to infinite order on  $S(y_0, r_0)$  for some  $(y_0, r_0) \in \mathcal{A}$ . Then  $f$  is holomorphic on  $S(\mathcal{A})$ .*

Since the set of spheres surrounding a point is an open connected subset of  $\mathbb{C}^n \times (0, \infty)$ , the following example shows that some holomorphy or smoothness hypothesis is required for the conclusion of Theorem 2.2.2 to be valid. This is Example 2.3 in [GrQ].

**Example 2.2.4.** Let  $k > 0$ ,  $n > 0$  and let  $m > k + n + 1$ . Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  be defined by  $f(z) \equiv f(z_1, \dots, z_n) = z_1^m / \bar{z}_1^n$ . Then,  $f \in C^k(\mathbb{C}^n)$  and  $f$  has vanishing Morera integral over any sphere  $S$  that encloses or contains the origin in  $\mathbb{C}^n$  with respect to each of the  $(n, n-1)$  forms  $\{*\mathbf{d}\bar{z}_j \mid j = 1, \dots, n\}$

The next theorem has a point as a starter set rather than a sphere.

**Theorem 2.2.5.** *Let  $M$  be a complex manifold with a real-analytic Riemannian structure. Let  $\mathcal{A}$  be an open, connected subset of  $M^+$ . Let  $f$  be a continuous function on  $S(\mathcal{A})$  (or a distribution). Let  $L \subset \Lambda^{2n-1}(M)$  satisfy (2.2.2). Assume for each  $\omega \in L$  that  $R_\omega f(y, r) = \int_{S(y,r)} f \omega$  is real-analytic on  $\mathcal{A}$ . Assume there is a  $y_0 \in M$  and an  $r_0 > 0$  such that  $(y, r_0) \in \mathcal{A}$  for each  $y \in D(y_0, r_0)$ . Let  $f$  be holomorphic to infinite order at  $y_0$ . Then  $f$  is holomorphic on  $S(\mathcal{A})$ .*

The hypotheses about  $\mathcal{A}$  and  $D(y_0, r_0)$  in Theorem 2.2.5 guarantee zero integrals over enough spheres near  $y_0$  for  $f$  to be holomorphic. In Remark 3.2.3, weaker assumptions are given on the spheres under which the conclusion of this theorem is true. In the same vein, one can prove similar theorems for other “starter” sets besides  $S(y_0, r_0)$  and  $y_0$ . Furthermore, our techniques give new support theorems (see §3.2). One can state a local version of Theorem 2.2.5 that is analogous to Theorem 2.2.3.

On  $\mathbb{C}^n$ , classical Pompeiu theorems can be used to prove Morera theorems without our assumption that  $f$  is holomorphic on a starter set. Local two sphere Morera theorems follow from Pompeiu theorems (e.g., [Br, BG 1986, BG 1988]) using Stokes’ Theorem as discussed in the introduction. These theorems do not require starter sets, but they are true only for special spaces. Our theorems can be applied in fairly general complex manifolds.

Dr. Y. Zhou has proven local support theorems for the sphere transform with two radii using microlocal techniques. Let  $y \in \mathbb{R}^n$  and let  $r > 0$ . Let  $0 < a < b < a + b < r$  and assume  $a/b$  is not rational. Assume  $f \in \mathcal{D}'(D(y, r))$  has zero integrals over all spheres of radius  $a$  and  $b$  contained in  $D(y, r)$  and assume  $f$  is zero near one sphere of radius less than  $r$  centered at  $y$ . Then  $f$  is zero on  $D(y, r)$ . She proved this theorem for  $S^n$  and  $\mathbb{R}\mathbb{P}^n$  as well [Zh]. More general but related arguments can be used to prove two-radius support theorems on manifolds [ZQ] and two-radius Morera theorems on complex manifolds.

## 3. PROOFS

The real-analytic wave front set,  $\text{WF}_A(f)$ , of a distribution  $f \in \mathcal{D}'(M)$  is defined using real-analytic local coordinates and the definition in [Tr] or [Hö 1983] for Euclidean space. In this section, we give some general microlocal results.

If  $S \subset M$  is a  $C^2$  manifold then the conormal bundle of  $S$ ,  $N^*S \subset T^*M$ , is the set of covectors  $\{(x, \eta) \mid x \in S, \eta \in T_x^*M \text{ and } T_x S \subset \ker \eta\}$ . The following theorem of Hörmander, Kawai, and Kashiwara [SKK, Hö 1983, Theorem 8.5.6] is one key to our proofs.

**Lemma 3.1.** *Let  $M$  be a real-analytic manifold and let  $h \in \mathcal{D}'(M)$ . Let  $S$  be a  $C^2$  hypersurface that divides  $M$  into two disjoint open sets,  $T_1$  and  $T_2$ . Assume  $h$  is zero on  $\text{int } T_1$ . If  $x \in S \cap \text{supp } h$  and  $(x, \eta) \in N^*(S) \setminus 0$ , then  $(x, \eta) \in \text{WF}_A(h)$ .*

Under the assumptions of this lemma,  $h$  cannot be real-analytic near  $x$  because  $x$  is a boundary point of  $\text{supp } h$ . Lemma 3.1 is a strengthening of this simple observation because it provides specific wave front directions above  $x$  that must be in  $\text{WF}_A(h)$ .

For completeness, we give some of the basic calculus of Fourier integral operators. Let  $X, Y$ , and  $Z$  be manifolds. If  $A \subset T^*X \times T^*Y$ , then we define

$$(3.1a) \quad \begin{aligned} A' &= \{(x, y; \xi, -\eta) \mid (x, y; \xi, \eta) \in A\}, \\ A^t &= \{(y, x; \eta, \xi) \mid (x, y; \xi, \eta) \in A\}. \end{aligned}$$

If, in addition,  $B \subset T^*Y$  then

$$(3.1b) \quad A \circ B = \{(x, \xi) \in T^*X \mid \exists (y, \eta) \in B \text{ such that } (x, y; \xi, \eta) \in A\}.$$

Let  $\Gamma \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$  be a Lagrangian manifold and let  $S$  be a Fourier integral operator (FIO) associated to  $\Gamma$ . If  $f \in \mathcal{E}'(X)$  then there is a natural relation between singularities of  $f$  and those of  $Sf$ :

$$(3.2) \quad \text{WF}_A(Sf) \subset (\Gamma^t)' \circ \text{WF}_A(f).$$

[SKK] ([Tr, Theorem 8.5.4] for the  $C^\infty$  category).

If the projection from  $\Gamma$  to  $T^*Y$  is an injective immersion, then we say  $S$  (or  $\Gamma$ ) satisfies the *Bolker Assumption* [GS, pp. 364-365], [Q 1980, equation (9)]. In addition, if  $S$  is real-analytic elliptic and  $f$  is a distribution, then

$$(3.3) \quad \text{WF}_A(f) = \Gamma' \circ \text{WF}_A(Sf).$$



(To prove this, one constructs a FIO  $T$  associated to  $\Gamma^t$ , and shows that  $T \circ S$  is a real-analytic elliptic pseudodifferential operator [SKK]. Because of the injectivity radius assumptions in our theorems,  $T$ , which is related to the dual operator,  $R^*$ , is well behaved. The argument is similar to the one on the bottom of p. 337 below (14) in [Q 1980]. See also [Hö 1971], Theorem 4.2.2 and discussion at the bottom of p. 180 for how to compose FIO.)

### 3.1. Proofs of Morera theorems for spheres with fixed radius.

The set  $\text{WF}_A(\bar{\partial}f)$  can be defined by using complex local coordinates  $(z_1, \dots, z_n)$  by taking the union of the  $\text{WF}_A(\frac{\partial f}{\partial \bar{z}_j})$  for  $j = 1, \dots, n$ . The definition of  $\text{WF}_A(\bar{\partial}f)$  is invariant under complex coordinate changes since they preserve complex structure. In fact, since  $\bar{\partial}$  is real-analytic elliptic,  $\text{WF}_A(\bar{\partial}f) = \text{WF}_A(f)$ . The set  $\text{supp } \bar{\partial}f$  is defined in an analogous way. Here is our microlocal regularity theorem for the sphere transform with fixed radius.

**Proposition 3.1.1.** *Let  $M$  be an  $n$ -dimensional complex manifold with a real-analytic Riemannian structure and let  $\mathcal{A} \subset M$  be open. Let  $r > 0$  and let  $A = D(\mathcal{A}, r)$ . Assume for each  $y \in A$ ,  $I_y > r$ . Let  $f \in \mathcal{D}'(A)$  and let  $y_1 \in \mathcal{A}$ . Let  $\omega \in \Lambda^{2n-1}(S(\mathcal{A}, r))$  and assume  $R_{\omega, r}f(y)$  is real-analytic for all  $y$  in an open neighborhood of  $y_1$ . Let  $x$  and  $x_a$  be geodesically antipodal points in  $S(y_1, r)$ . Assume  $\omega$  is nondegenerate on  $T_x(S(y_1, r))$ . Then,  $N_{x_a}^*S(y_1, r) \cap \text{WF}_A(f) = \emptyset$  implies  $N_x^*S(y_1, r) \cap \text{WF}_A(\bar{\partial}f) = \emptyset$ .*

This proposition can be applied if  $f$  is holomorphic on an open neighborhood of  $x_a$ , because, in this case,  $f$  has no real-analytic wave front set above  $x_a$ . Also, there is an exact correspondence between covectors: under the hypotheses of the proposition, for each  $\xi \in N_x^*S(y_1, r) \setminus \text{WF}_A(f)$ , there is a specific direction  $\sigma(\xi) \in N_{x_a}^*S(y_1, r)$  such that  $(x_a, \sigma(\xi)) \notin \text{WF}_A(f)$  if and only if  $(x, \xi) \notin \text{WF}_A(f)$  ( $\xi$  and  $\sigma(\xi)$  are the two preimages under  $\pi_2$  of a specific covector above  $y_1$ , see (3.1.2) and (3.1.7)).

*Proof.* The proof is closely related to the proof in [Q 1993]. In [*ibid.*], the measure was assumed to be nowhere zero. However, we need to prove the conclusion is valid as long as the measure in (2.1.1) is nonzero near  $x$ .

We first outline the proof and then provide the microlocal details. Since  $R_{\omega, r}$  is a real-analytic FIO associated to the diagram (3.1.1) [Gu, Su, Q 1993], wave fronts of  $f$  at both  $x$  and  $x_a$  conormal to  $S(y_1, r)$  contribute to  $\text{WF}_A(R_{\omega, r}f)$  above  $y_1$  in two specific directions  $(y_1, \pm\eta_1)$  and they are the only wave front directions for  $f$  that can give wave front of  $R_{\omega, r}f$  in these directions (see (3.1.2) and (3.1.7)). By the smoothness assumptions on  $f$  near  $x_a$ ,  $f$  has no real-analytic wave front at  $x_a$  in a direction that will contribute to wave front of  $R_{\omega, r}f$  in direction  $(y_1, \pm\eta_1)$ . Because the measure of  $R_{\omega, r}$  is nonzero near  $x$  and  $S(y_1, r)$ ,  $R_{\omega, r}$  is microlocally elliptic near  $x$  above  $N^*S(y, r)$ . Therefore,  $R_{\omega, r}f$  will have wave front in direction  $(y_1, \pm\eta_1)$  if and only if  $f$  has wave front above  $x$

conormal to  $S(y_1, r)$ . As  $R_{\omega, r}f$  is real-analytic at  $y_1$ , this implies  $f$  has no wave front conormal to  $S(y_1, r)$  above  $x$ . Now, since  $\bar{\partial}$  is a real-analytic differential operator  $\text{WF}_A(\bar{\partial}f) \cap N_x^*S(y_1, r) = \emptyset$ . This finishes the outline of the proof.

Here are the details. We assume that  $A = D(\mathcal{A}, r)$  and that  $\omega \in \Lambda^{2n-1}(S(\mathcal{A}, r))$ . By the injectivity radius assumptions, the operator  $R_{\omega, r} : \mathcal{D}'(A) \rightarrow \mathcal{D}'(\mathcal{A})$  (and  $R_{\omega, r}^* : \mathcal{D}'(\mathcal{A}) \rightarrow \mathcal{D}'(A)$ ) is continuous. The incidence relation [He] (the support of the Schwartz Kernel [GS, Q 1980]) of  $R_{\omega, r}$  is the manifold  $Z = \{(x, y) \in A \times \mathcal{A} \mid d(x, y) = r\}$  [He], and  $Z$  is a good manifold with well-behaved projections to  $A$  and  $\mathcal{A}$  because of the injectivity radius assumptions. The Radon transform can be described by the *double fibration*:

$$\begin{array}{ccc} Z & \xrightarrow{p_2} & \mathcal{A} \\ \downarrow p_1 & & \\ A & & \end{array}$$

where  $p_1$  and  $p_2$  are the projections onto the respective factors. Note that  $S(\mathcal{A}, r)$  in Definition 2.1 is  $p_1(p_2^{-1}(\mathcal{A}))$ .

The Lagrangian manifold of  $R_{\omega, r}$  is  $\Gamma = N^*Z \setminus 0$  [GS, Q 1980]. The relevant microlocal diagram is the diagram on the cotangent level corresponding to this double fibration:

$$(3.1.1) \quad \begin{array}{ccc} \Gamma & \xrightarrow{\pi_2} & T^*(\mathcal{A}) \setminus 0 \\ \downarrow \pi_1 & & \\ T^*(A) \setminus 0 & & \end{array}$$

where  $\pi_1$  and  $\pi_2$  are the natural projections. It is known [Gu, Su top paragraph and remark on p. 488] that  $R_{\omega, r}$  is a FIO associated with the Lagrangian manifold  $\Gamma$ . It is shown in [Q 1993] (see also [Gu, Su]) that

(3.1.2)  $\pi_2$  is a two to one local diffeomorphism that maps corresponding covectors in  $\Gamma$  that lie above antipodal points in  $S(y_1)$  to the same point in  $T^*\mathcal{A}$ . If the covectors in  $\Gamma$  are  $(x, \xi, y_1, \eta_1)$ , and  $(x_a, \xi', y_1, \eta_1)$ , then  $x$  and  $x_a$  are antipodal points in  $S(y_1, r)$  and  $(x, \xi)$  and  $(x_a, \xi')$  are in  $N^*S(y_1, r)$ .

Using the hypotheses of the proposition, we choose a small open disk  $\mathcal{U} \subset \mathcal{A}$  centered at  $y_1$  and a small open disk,  $V$  centered at  $x$ , and we write  $f = f_x + f_a$  in such a way that

(3.1.3) the radii of  $\mathcal{U}$  and  $V$  are both less than  $r/2$

(3.1.4)  $\text{supp } f_x \subset V$ ,

(3.1.5)  $V \subset S(\mathcal{U}, r)$ , and

(3.1.6)  $\omega|_{(T_z S(y, r))}$  is nondegenerate for each  $y \in \mathcal{U}$  and  $z \in S(y, r) \cap V$ .

Note that (3.1.6) can be made to hold by the assumption that  $\omega$  is nondegenerate when restricted to  $T_x S(y_1, r)$  and by continuity of  $\omega$ .

Use (3.1.2) to choose

$$(3.1.7) \quad \begin{aligned} \xi \in N_x^* S(y, r) \setminus 0, \quad \xi' \in N_{x_a}^* S(y_1, r) \setminus 0 \quad \text{and} \quad \eta_1 \in T_{y_1}^* \mathcal{A} \setminus 0 \quad \text{so that} \\ (x, y_1; \xi, -\eta_1) \in \Gamma, \quad (x_a, y_1; \xi', -\eta_1) \in \Gamma. \end{aligned}$$

We now show

$$(3.1.8) \quad (y_1, \eta_1) \notin \text{WF}_A(R_{\omega, r} f_a).$$

By (3.1.4),  $f_a$  is zero near  $x$  and so  $(x, \xi) \notin \text{WF}_A(f_a)$ . Now, since  $(x_a, \xi') \notin \text{WF}_A(f)$ , and  $f = f_a$  near  $x_a$ ,  $(x_a, \xi') \notin \text{WF}_A(f_a)$ . Since both  $(x, \xi)$  and  $(x_a, \xi')$  are not in  $\text{WF}_A(f_a)$ ,  $(y_1, \eta_1) \notin (\Gamma^t)' \circ \text{WF}_A(f_a)$ . The microlocal fact (3.2) shows that  $\text{WF}_A R_{\omega, r} f_a \subset (\Gamma^t)' \circ \text{WF}_A(f_a)$ , and this proves (3.1.8).

As  $R_{\omega, r} f$  is real-analytic near  $y_1$ ,  $(y_1, \eta_1) \notin \text{WF}_A(R_{\omega, r} f)$ . By linearity of  $R_{\omega, r}$  and (3.1.8),  $(y_1, \eta_1) \notin \text{WF}_A(R_{\omega, r} f_x)$ .

Let  $\mathcal{U}' = \{y \in \mathcal{U} \mid S(y, r) \cap V \neq \emptyset\}$ . By the choice of  $\mathcal{U}'$  and  $V$ , (3.1.2), and (3.1.4),  $\pi_2$  is an injective immersion above  $V \times \mathcal{U}'$  (if  $x \in V$ , then no antipodal point to  $x$  in any sphere of radius  $r$  is in  $V$ ). So, by definition [GS, Q 1980],  $R_{\omega, r}$  satisfies the Bolker Assumption for distributions supported in  $V$ . Because the measure for  $R_{\omega, r}$  is nowhere zero above  $V$  by (3.1.6),  $R_{\omega, r}$  is an elliptic FIO for functions supported in  $V$  [GS, Q 1980].

Let  $\tilde{\Gamma}$  be the fibers of  $\Gamma$  above  $V \times \mathcal{U}'$ . As  $R_{\omega, r}$  satisfies the Bolker assumption above  $V \times \mathcal{U}'$  (on  $\tilde{\Gamma}$ ), we can use (3.3) to conclude: since  $(y_1, \eta_1) \notin \text{WF}_A(R_{\omega, r} f_x)$ ,  $(\tilde{\Gamma})' \circ \{(y_1, \eta_1)\} = (x, \xi) \notin \text{WF}_A(f_x)$ . Therefore,  $\text{WF}_A(f_x) \cap N_x^* S(y_1, r) = \emptyset$ . Since  $f_a$  is zero near  $x$ ,  $\text{WF}_A(f) \cap N_x^* S(y_1, r) = \emptyset$ . Since  $x \in S(y_1, r)$  can be arbitrary (with possibly different choices of  $\omega \in L$  for different  $x$ ) and since  $\text{WF}_A(f) = \text{WF}_A(\bar{\partial} f)$ , this completes the proof.  $\square$

*Proof of Theorem 2.1.1.* This follows as a corollary of Theorem 2.1.3.  $\square$

To eat away at  $\text{supp } \bar{\partial} f$  in the proof of Theorem 2.1.3, we need a simple geometric lemma.

**Lemma 3.1.2.** *Let  $\mathcal{A} \subset M$  be open and connected, and let  $B \subset M$  be a closed nonempty set. Let  $r > 0$  and  $\epsilon > 0$ , and assume  $\forall y \in \mathcal{A}$ ,  $r < I_y$ . Assume there is a disk  $D(y_0, r)$  for  $y_0 \in \mathcal{A}$  that is disjoint from  $B$  but  $S(\mathcal{A}, r) \cap B \neq \emptyset$ . Then, there is an  $\epsilon > 0$  and a  $y_1 \in \mathcal{A}$  such that  $(\text{int } D(y_1, r + \epsilon)) \cap B = \emptyset$  but  $(\text{bd } D(y_1, r + \epsilon)) \cap B \neq \emptyset$ . Furthermore,  $\epsilon$  can be chosen so  $D(y_1, \epsilon) \subset \mathcal{A}$  and  $r + \epsilon < I_{y_1}$ .*

*Proof of Lemma 3.1.2.* Assume the conclusion of the lemma is false. Let  $y_2 \in \mathcal{A}$  be such that  $D(y_2, r) \cap B \neq \emptyset$ , and let  $p : [0, 1] \rightarrow \mathcal{A}$  be a continuous path from  $y_0$  to  $y_2$ . Now, choose  $\epsilon > 0$  so that:

(3.1.9) if  $y \in M$  and  $d(y, p([0, 1])) \leq \epsilon$ , then  $y \in \mathcal{A}$  and  $r + \epsilon < I_y$ ;

(3.1.10) if  $d(x, y_0) \leq r + \epsilon$ , then  $x \notin B$ .

Note that (3.1.9) can be satisfied for some  $\epsilon$  because  $\mathcal{A}$  is open,  $p([0, 1])$  is compact,  $r < I_y \forall y \in \mathcal{A}$ , and the function  $I_y$  is lower semicontinuous. Because  $B$  is closed, (3.1.10) can be made to hold for some  $\epsilon > 0$ .

If  $t \in [0, 1]$  define  $T(t)$  to be the closed ball centered at  $p(t)$  of radius  $r + \epsilon$ . By (3.1.10),  $T(0)$  is disjoint from  $B$  and by assumption,  $T(1)$  meets  $B$ . Let  $t_1$  be the smallest value of  $t \in [0, 1]$  such that  $T(t)$  meets  $B$ . Because  $T(0) \cap B = \emptyset$ ,  $t_1 > 0$ . By the choice of  $t_1$ ,  $T(t_1)$  meets  $B$  only on the boundary,  $\text{bd}T(t_1)$ . Let  $y_1 = p(t_1)$ . Condition (3.1.9) shows that  $D(y_1, \epsilon) \subset \mathcal{A}$ .  $\square$

*Proof of Theorem 2.1.3.* We want to show  $\text{supp } \bar{\partial}f$  is disjoint from  $\cup_{y \in \mathcal{A}} D(y, r)$ . We assume the set of forms  $L$  is defined on all of  $S(\mathcal{A}, r)$ . If not, a finite covering of the path in Lemma 3.1.2 from  $y_0$  to  $y_2$  and a local version of the proof below can be used to prove the theorem.

We can apply Lemma 3.1.2 with  $B = \text{supp } f$  to come up with an  $\epsilon > 0$  and a  $y_1 \in \mathcal{A}$  such that the disk  $T_1 = D(y_1, r + \epsilon)$  meets  $\text{supp } f$  only on its boundary. Let  $x \in \text{bd}T_1 \cap \text{supp } \bar{\partial}f$ . Then because  $D(y_1, \epsilon) \subset \mathcal{A}$ , the point,  $y$ , that is  $\epsilon$  units from  $y_1$  on the geodesic between  $y_1$  and  $x$  is in  $\mathcal{A}$ . The sphere  $S(y, r)$  is contained in  $T_1$ , and by Gauss' Lemma [KN, IV 3.3] and (3.1.9),  $S(y, r)$  is tangent to  $\text{bd}T_1$  at  $x$ . Let  $\xi \in N_x^* S(y, r) \setminus 0$ . Recall that  $f$  is holomorphic near the antipodal point in  $S(y, r)$  to  $x$  because this antipodal point is in the interior of  $T_1$ . Therefore,  $y$  and  $S(y, r)$  satisfy the hypotheses of Proposition 3.1.1 and

$$(3.1.11) \quad (x, \xi) \notin \text{WF}_A(\bar{\partial}f).$$

Because  $S(y, r)$  and  $\text{bd}T_1$  are tangent at  $x$ ,  $\xi \in N_x^*(\text{bd}T_1)$ . Now, by Lemma 3.1 and (3.1.11),  $x \notin \text{supp } \bar{\partial}f$ . But, this contradicts the assumption that  $T(1)$  meets  $\text{supp } \bar{\partial}f$ . Therefore,  $\bar{\partial}f$  is zero on  $D(\mathcal{A}, r)$ .  $\square$

**Example 3.1.3.** The conclusion of Theorem 2.1.1 is false, in general, if one assumes  $f$  is zero on a neighborhood of  $S(y_0, r)$ . In  $\mathbb{C}^n$ , we let  $L = \{*\mathbf{d}\bar{z}_j \mid j = 1, \dots, n\}$  and we construct a radial function such that  $R_{(*\mathbf{d}\bar{z}_j)}f \equiv 0$  for  $j = 1, \dots, n$  and  $f$  is zero on a neighborhood of  $S(0, r)$ , but  $f$  is not identically zero.

*Construction.* This proof follows from arguments that are similar to those in Example 3.2 in [Q 1993]. The proof will be given up to the point it draws directly from the proof in that article. For convenience, we will assume  $r = 1$ , and if  $z \in \mathbb{C}^n$ , and we will let  $z_j$  be the  $j^{\text{th}}$  complex coordinate of  $z$ .

First, note that if  $f$  is a radial function in  $C^\infty(\mathbb{C}^n)$ ,  $\int_{S(y,1)} f(*d\bar{z}_j) = \int_{S(y,1)} f(z)(\bar{z}_j - \bar{y}_j)dA$  where  $dA$  is the standard measure on the sphere  $S(y,1)$ . Now,  $(\bar{z}_j - \bar{y}_j)$  is a spherical harmonic that is homogeneous of degree 1 on the sphere  $S(y,1)$ . So, we can use the Funk-Hecke theorem (the uniqueness of spherical functions) to show for  $|y| \geq 1$  that

$$(3.1.12) \quad R_{(*d\bar{z}_j)}f(y) = \bar{a}_j \text{Area}(S^{2n-2}) 2^{2-2n} |y|^{1-2n} \int_{w-2}^w K(s,w)(w-s)^{\frac{2n-3}{2}} f(s) ds$$

where  $a = y/|y|$ ,  $w = |y| + 1$ , and

$$K(s,w) = s(s^2 - w^2 + 2w - 2)[(w+s)(s^2 - (w-2)^2)]^{(2n-3)/2}.$$

Note that  $K(w,w)$  is never zero for  $w > 1$ . We use the value of the Gegenbauer polynomial of degree one,  $C_1^\lambda(t) = 2\lambda t$  where  $\lambda = (n-1)$ . First, one defines an even, nonzero, smooth function  $f(t)$  with support in  $[-1 + \epsilon, 1 - \epsilon]$ . Then, one uses the solvable integral equation (3.1.12) and the values of the integral of the part of  $f$  on  $[-1 + \epsilon, 1 - \epsilon]$  to extend  $f$  (with zero integrals) to  $[1 + \epsilon, 3 + \epsilon/2]$ . In this case, the lower limit of integration in (3.1.12) can be changed to  $1 + \epsilon$  because  $f$  is zero in  $[1 - \epsilon, 1 + \epsilon]$ . Finally, one continues as in the proof of Example 3.2 in [Q 1993], successively defining  $f$  on more of the real line by solving (3.1.12) on more of the line.  $\square$

This example shows that the hypotheses of Theorem 2.1.1 that  $D(y_0, r)$  is disjoint from  $\text{supp } \bar{\partial}f$  cannot be weakened to become “ $S(y_0, r)$  is disjoint from  $\text{supp } \bar{\partial}f$ .” The fundamental reason is that the hypothesis about antipodal points in Proposition 3.1.1 is necessary. We have chosen an  $\epsilon \in (0, 1)$  and constructed the function  $f$  such that  $\bar{\partial}f$  is zero on  $A = \{x \in \mathbb{C}^n \mid 1 - \epsilon < |x| < 1 + \epsilon\}$ , but  $f$  satisfies  $\text{bd } A \subset \text{supp } \bar{\partial}f$ . Therefore, if  $|y| = \epsilon$ , then  $S(y, 1)$  is tangent to  $\text{bd}(\text{supp } \bar{\partial}f)$  at antipodal points  $x$  and  $x_a$ . Furthermore, covectors in  $\text{WF}_A(f)$  at  $x_a$  cancel covectors in  $\text{WF}_A(f)$  at  $x$  to make  $R_{(*d\bar{z}_j)}f$  real-analytic (in fact, zero) near  $y$ , even though  $f$  is not real-analytic in the conormal directions to  $\text{bd } \text{supp } f$  at either  $x$  or at  $x_a$ . A related counterexample, [BG 1986, Theorem 10], is given to their main theorem for the Radon transform on disks of two radii in area measure. Furthermore, John [Jo, p. 115] constructs a function  $f \in C(\mathbb{R}^3)$  with zero integrals over  $S(y, 1)$  in area measure and for which the interior of  $D(0, 1)$  is disjoint from  $\text{supp } f$ .

### 3.2. Proofs of the Morera theorems for arbitrary spheres.

First we give the microlocal regularity theorem for the sphere transform with variable radius.

**Proposition 3.2.1.** *Let  $M$  be a complex manifold with a real-analytic Riemannian structure. Let  $\mathcal{A}$  be an open subset of  $M^+$  and let  $A = S(\mathcal{A})$ . Let  $f \in \mathcal{D}'(A)$  and assume  $R_\omega f$  is real-analytic on  $\mathcal{A}$  for each  $\omega$  in a set  $L$  given in (2.2.2). Then  $\text{WF}_A(\bar{\partial}f) \cap N^*(S(y_1, r_1)) = \emptyset \forall (y_1, r_1) \in \mathcal{A}$ .*

*Proof.* We prove that the Radon transform  $R_\omega$  is a Fourier integral operator that satisfies the Bolker Assumption. This will imply the conclusion of the theorem by general microlocal arguments. To do

this, we will calculate the Lagrangian manifold,  $\Gamma$  associated to this operator. Then, the conclusion of Proposition 3.2.1 follows from the theory of Fourier integral operators.

Since the transform is defined locally, we must localize to  $A$  and  $\mathcal{A}$ . The incidence relation of this Radon transform [He] is the set  $Z = \{(x, y, r) \in A \times \mathcal{A} \mid d^2(y, x) - r^2 = 0\}$ . Since  $\mathcal{A} \subset M^+$ ,  $Z$  is an imbedded submanifold of  $M \times M \times (0, \infty)$ . Let  $\Gamma$  be the conormal bundle of  $Z$  in  $T^*(A \times \mathcal{A})$  with the zero section removed. We will show that  $R_\omega$  is a Fourier integral operator associated to the Lagrangian manifold  $\Gamma$  [GS, Q 1980]. Let  $\pi_2$  be the projection on the second factor,  $\pi_2 : \Gamma \rightarrow T^*(\mathcal{A})$ .

We first calculate  $\Gamma$  using local coordinates on  $M$  and then use this to show that  $R_\omega$  satisfies the Bolker Assumption. We then use microlocal regularity theorems for such FIO to finish the proof of the theorem. The proof is much like the proof outlined above and in [Q 1993] for spheres of fixed radius, but since it is not in the literature, it will be given. We make the calculation in local real-analytic geodesic normal coordinates, and we write the distance on  $M$  locally in terms of the Euclidean distance on a tangent space.

Let  $(y_1, r_1) \in \mathcal{A}$ . We can choose  $\delta > 0$  such that  $s = r_1 + \delta \in (r_1, I_{y_1})$  (recall that  $r_1 < I_{y_1}$  because  $(y_1, r_1) \in \mathcal{A} \subset M^+$ ). Let  $B$  be the open ball centered at  $0 \in T_{y_1}M$  of radius  $s$ . Then in geodesic coordinates on  $B$ ,  $\exp = \exp_{y_1} : B \rightarrow M$  is a diffeomorphism onto the open ball  $\mathcal{B} \subset M$  centered at  $y_1$  of radius  $s$  [KN, IV 3.4]. Let  $U \subset B$  be an open ball of radius  $s' > 0$  centered at zero such that  $B$  is a normal neighborhood of each vector in  $U$  and such that  $s - s' > r_1$ . This is possible because  $y \mapsto I_y$  is lower semicontinuous. Let  $\mathcal{U} = \exp_{y_1} U$ . Perhaps by making  $\delta, s$ , and  $s'$  smaller, we can assume  $\mathcal{U} \times (r_1 - \delta, s) \subset \mathcal{A}$ . In this case,  $R_\omega f$  is defined above all points in  $\mathcal{U} \times (r_1 - \delta, s)$  for all  $\omega \in L$ .

By [KN, III 8.3 and IV 3.4], the shortest geodesic in  $M$  between each point  $y \in \mathcal{U}$  and each point  $x \in \mathcal{B}$  lies in  $\mathcal{B}$ , and the proof of [KN, IV 3.6] shows that the square of the distance function,  $d^2(y, x)$ , is real-analytic on  $\mathcal{U} \times \mathcal{B}$ . If  $Y \in U$  and  $X \in B$ , let  $y = \exp(Y)$  and  $x = \exp(X)$ . In this case, we will prove that the distance function on  $\mathcal{U} \times \mathcal{B}$  can be written in terms of the Euclidean distance on  $T_{y_1}M$  as

$$(3.2.1) \quad \begin{aligned} d^2(y, x) &= \|Y - X\|^2 + c(Y, X) \\ &\text{for some real-analytic function } c \text{ satisfying} \\ \nabla_X c(0, X) &= \nabla_Y c(0, X) = \frac{\partial^2 c}{\partial x_i \partial y_j}(0, X) = 0 \quad \forall X \in B, \quad i, j \in \{1, \dots, 2n\} \end{aligned}$$

where  $\nabla_X$  and  $\nabla_Y$  are the (real) gradients in the respective variables and the partial derivatives are real partial derivatives on the tangent space.

We now prove (3.2.1). By [KN, IV 3.4],  $\nabla_X c(0, X) \equiv 0$  for  $X \in B$ . To see  $\nabla_Y c(0, X) \equiv 0$  first note that the segment between  $0$  and  $X \in B \setminus 0$  corresponds to the geodesic between  $y_1$  and  $\exp(X)$ . Let  $S$  be the inverse image under  $\exp_{y_1}$  of the geodesic sphere of radius  $\|X\|$  centered

at  $\exp(X)$ ;  $S = \{Z \in B \mid d(\exp(Z), \exp(X)) = \|X\|\}$  and  $0 \in S$ . A simple argument using Gauss' lemma [KN, IV 3.3] shows that the segment is perpendicular to  $S$  (if  $S$  does not lie entirely in  $B$ , then one can use this argument on a small geodesic sphere tangent to  $S$  at  $0$ ).

Furthermore, the Euclidean sphere  $S' = \{Y \in B \mid \|Y - X\| = \|X\|\}$  is also perpendicular at  $Y = 0$  to this segment. Therefore,  $S$  and  $S'$  are tangent at  $Y = 0$  and so directional derivatives of  $c$  at  $Y = 0$  tangent to  $S$  are zero. The directional derivative of  $c$  at  $Y = 0$  in the perpendicular direction (in the direction of  $X$ ) is zero because geodesics through  $y_1$  correspond to straight lines through the origin in  $B$  in the Euclidean distance. Now, the first two derivative equalities in (3.2.1) imply the third equality. These coordinates give local coordinates on  $N^*Z$  in which the needed properties of  $\pi_2$  can be checked.

As  $Z$  is defined by the equation  $d^2(y, x) - r^2 = 0$ , the differential of this equation gives a basis for the fibers of  $\Gamma$ . Therefore,

$$(3.2.2) \quad \Gamma = \{(X, Y, r; \alpha[2(Y - X) + \nabla_X c(Y, X)]d\mathbf{X} - \alpha[2(Y - X) + \nabla_Y c(Y, X)]d\mathbf{Y} - 2\alpha r d\mathbf{r}) \\ \mid (X, Y, r) \in B \times U \times (r_1 - \delta, r_1 + \delta), \alpha \neq 0, d^2(\exp(Y), \exp(X)) - r^2 = 0\}.$$

Here,  $X = (X_1, \dots, X_{2n}) \in B$  and  $X d\mathbf{X} = X_1 d\mathbf{X}_1 + \dots + X_{2n} d\mathbf{X}_{2n}$ .

Therefore,

$$(3.2.3) \quad \pi_2(X, Y, r; \alpha[2(Y - X) + \nabla_X c(Y, X)]d\mathbf{X} - \alpha[2(Y - X) + \nabla_Y c(Y, X)]d\mathbf{Y} - 2\alpha r d\mathbf{r}) \\ = (Y, r; -\alpha[2(Y - X) + \nabla_Y c(Y, X)]d\mathbf{Y} - 2\alpha r d\mathbf{r})$$

Equation (3.2.1) is used to show  $\pi_2$  is an immersion when  $Y = 0$  (it is easiest to parameterize  $\Gamma$  using  $(X, Y, \alpha)$  and letting  $r = d(\exp(X), \exp(Y))$  in the calculation). So, by continuity,  $\pi_2$  is an immersion in a neighborhood of  $Y = 0$ . Equations (3.2.3) and (3.2.1) are used to show that  $\pi_2$  is injective when  $Y = 0$ . Since  $y_1 = \exp(0)$  is arbitrary, this shows  $\pi_2$  satisfies the Bolker Assumption everywhere.

Therefore,  $R_\omega$  is a real-analytic FIO associated to  $\Gamma$ .

We assume that  $R_\omega f$  is real-analytic near  $(y_1, r_1)$  for all  $\omega \in L$ . We now prove that

$$(3.2.4) \quad \text{WF}_A(f) \cap \left[ \Gamma' \circ \left( T_{(y_1, r_1)}^* \mathcal{A} \right) \right] = \emptyset.$$

First,  $R_\omega$  satisfies the Bolker Assumption so we can use (3.3). However, since the measure associated to  $\omega$  could be zero somewhere, we need to localize as we did in the proof of Proposition 3.1.1.

Let  $(x, \xi) \in N^*S(y_1, r_1) \setminus 0$ . Let  $((y_1, r_1), \eta)$  be the covector in  $T_{(y_1, r_1)}^* M^+$  corresponding to  $(x, \xi)$  in  $\Gamma$ . Assume  $\omega \in L$  is nondegenerate near  $x$  on  $S(y_1, r_1)$ . We write  $f = f_x + f_0$  where  $f_x$  is supported in a small neighborhood of  $x$  on which  $\omega$  is nondegenerate (on tangent spaces to

spheres near  $S(y_1, r_1)$ , see (3.1.6)). First,  $f_0$  is zero and hence real-analytic near  $x$  and  $\pi_2$  is an injection, so by (3.2) for  $R_\omega$ ,  $((y_1, r_1), \eta) \notin \text{WF}_A R_\omega f_0$ . Since  $R_\omega f$  is real-analytic near  $(y_1, r_1)$ ,  $((y_1, r_1), \eta) \notin \text{WF}_A R_\omega f_x$ . By (3.3), equation (3.2.4) is true if  $f$  is replaced by  $f_x$ : the measure for  $R_\omega$  is nowhere zero in a neighborhood of the support of  $f_x$  for spheres near  $S(y_1, r_1)$  by the definition of  $f_x$ , and  $R_\omega$  satisfies the Bolker Assumption globally on  $\Gamma$ . However, near  $x$ ,  $f_0$  is zero, so (3.2.4) holds for  $f$  at  $x$ . Therefore, equation (3.2.4) is true for  $f$ , at least above  $x$ .

Since, for each  $x \in S(y_1, r_1)$  there is an  $\omega \in L$  that is nondegenerate above points near  $x$  (on tangent spaces to spheres near  $S(y_1, r_1)$ ), (3.2.4) holds for all  $x \in S(y_1, r_1)$ . However, the expression in brackets in (3.2.4) is  $N^*S(y_1, r_1) \setminus 0$ . To finish the proof, we just need to observe that, since  $\bar{\partial}$  is real-analytic elliptic,  $\text{WF}_A(\bar{\partial}f) = \text{WF}_A(f)$ .  $\square$

We need a geometric lemma for the proof of Theorem 2.2.2.

**Lemma 3.2.2.** *Let  $\mathcal{A}$  be a open, connected subset of  $M^+$ , and let  $B \subset M$  be closed. Assume for some  $(y_0, r_0) \in \mathcal{A}$ , the sphere  $S(y_0, r_0)$  is disjoint from  $B$ , and assume for some  $(y_2, r_2) \in \mathcal{A}$ ,  $S(y_2, r_2)$  meets  $B$ . Then, there is a  $(y_1, r_1) \in \mathcal{A}$  such that  $S = S(y_1, r_1) \cap B \neq \emptyset$  and for each  $x \in S \cap B$ , there is an open neighborhood  $U$  of  $x$  such that  $S \cap U$  divides  $U$  into two disjoint connected open sets and  $f$  is zero on one of these sets.*

*Proof.* The proof is similar to the proof of Lemma 3.1.2 and it will be sketched. Let  $p : [0, 1] \rightarrow \mathcal{A}$  be a continuous path from  $(y_0, r_0)$  to  $(y_2, r_2)$  and let  $t_1$  be the smallest number in  $[0, 1]$  such that  $S(p(t)) \cap B \neq \emptyset$ . As  $S(y_0, r_0) \cap B = \emptyset$ ,  $t_1 > 0$ . One uses continuity of  $p$  to show that the point  $(y_1, r_1) = p(t_1)$  is the desired point.  $\square$

*Proof of Theorem 2.2.2.* Since  $R_\omega f(y, r)$  is real-analytic for  $(y, r) \in \mathcal{A}$ , we use Proposition 3.2.1 and the hypotheses of Theorem 2.2.2 to conclude

$$(3.2.5) \quad \text{WF}_A(\bar{\partial}f) \cap N^*S(y, r) = \emptyset \quad \forall (y, r) \in \mathcal{A}.$$

Here, we use the non-degeneracy assumption about  $L$ , (2.2.2). Therefore,  $\text{WF}_A(\bar{\partial}f) \cap N^*S(y_0, r_0) = \emptyset$ . By an assumption of Theorem 2.2.2,  $\bar{\partial}f$  is zero to infinite order on  $S(y_0, r_0)$ . Because of these two facts, a theorem of Boman [Bo] can be used to conclude: as  $\bar{\partial}f$  is zero to infinite order on  $S(y_0, r_0)$ , and  $\text{WF}_A(\bar{\partial}f) \cap N^*S(y_0, r_0) = \emptyset$ , then  $\bar{\partial}f = 0$  in a neighborhood,  $V$ , of  $S(y_0, r_0)$ .

We continue the proof assuming  $f$  is holomorphic on  $V$ . Assume  $f$  is not holomorphic on  $S(\mathcal{A})$ . Then, we can use Lemma 3.2.2 to find a  $(y_1, r_1) \in \mathcal{A}$  and  $x \in S(y_1, r_1) \cap \text{supp} \bar{\partial}f$  and a neighborhood  $U$  of  $x$  such that  $S = S(y_1, r_1)$  divides  $U$  into two disjoint open sets and  $f$  is zero on one of these sets. Let  $\xi \in N_x^*S$ . By Lemma 3.1,  $(x, \xi) \in \text{WF}_A(\bar{\partial}f)$ . However, (3.2.5) implies that  $(x, \xi) \notin \text{WF}_A(\bar{\partial}f)$ . This contradiction shows that  $x \notin \text{supp} \bar{\partial}f$ , and it proves the theorem.  $\square$



*Proof of Theorem 2.2.3.* This theorem follows directly by using Theorem 2.2.2 locally and using compactness on the path between  $(y_0, r_0)$  and any other point in  $\mathcal{A}$ . Specifically, choose a point  $(y, r) \in \mathcal{A}$  and let  $\gamma$  be a path in  $\mathcal{A}$  from  $(y_0, r_0)$  to  $(y, r)$ . By using Theorem 2.2.2 locally on this path and using compactness of this path, we see  $S(y_1, r_1)$  is disjoint from  $\text{supp } \bar{\partial}f$ .  $\square$

*Proof of Theorem 2.2.5.* For the same reasons as in the proof of Theorem 2.2.2,  $N^*S(y, r) \setminus 0 \cap \text{WF}_A(f) = \emptyset \quad \forall (y, r) \in \mathcal{A}$ . Let  $\mathcal{B} = \{(y, r_0) \mid d(y, y_0) \leq r_0\}$ . Since  $\mathcal{B} \subset \mathcal{A}$ , for each  $r \in [0, r_0]$ , and each  $x \in S(y_0, r)$  there is a sphere  $S(y, r_0)$  with  $(y, r_0) \in \mathcal{A}$  that is tangent to  $S(y_0, r)$  at  $x$ . Therefore, (3.2.4) implies

$$(3.2.6) \quad \text{WF}_A(f) \cap N^*(S(y_0, r)) = \emptyset \quad \forall r \in [0, r_0].$$

Assume  $f$  is holomorphic to infinite order at  $y_0$ . Applying (3.2.6) with  $r = 0$  shows that  $T_{y_0}^*M \cap \text{WF}_A(f) = \emptyset$ . This means that  $f$  (and therefore  $\bar{\partial}f$ ) is real-analytic in a neighborhood of  $y_0$ . Now, since  $\bar{\partial}f$  is zero to infinite order at  $y_0$ , we can conclude  $\bar{\partial}f$  is zero in a neighborhood,  $V$ , of  $y_0$ .

Now, let  $E = \{r \in [0, r_0] \mid S(y_0, r) \cap \text{supp } \bar{\partial}f \neq \emptyset\}$ . If  $E = \emptyset$ , we can use Theorem 2.2.2 to finish the proof.

Finally, assume  $E \neq \emptyset$ . Since  $\bar{\partial}f$  is zero in  $V$ ,  $r_E = \inf E > 0$ . Now, use (3.2.6) with  $r = r_E$  and Proposition 3.1 to conclude  $\bar{\partial}f$  is zero on a neighborhood of  $S(y_0, r_E)$ . This contradicts the choice of  $r_E$  and shows  $E = \emptyset$ .  $\square$

*Remark 3.2.3.* The key to the proof of Theorem 2.2.5 is not specifically that  $\mathcal{B} = \{(y, r_0) \mid d(y, y_0) \leq r_0\} \subset \mathcal{A}$ , but that for each sphere  $S(y_0, r)$  for  $0 \leq r \leq r_0$ , there are spheres in  $\mathcal{A}$  tangent to  $S(y_0, r)$  at each point on  $S(y_0, r)$ . One can generalize these theorems to  $f$  being holomorphic to infinite order on other sets, too, if  $\mathcal{A}$  is large enough. For example, the following theorem is an enjoyable exercise. Let  $M = \mathbb{C}^n$  and  $\mathcal{A} = \mathbb{C}^n \times (a, b)$  and let  $S$  be a smooth surface that is the boundary of a star-shaped region  $R$ . If integrals of  $f(*d\bar{z}_j)$  over all spheres in  $\mathcal{A}$  are real-analytic and  $f$  is holomorphic to infinite order on  $S$ , then  $f$  is holomorphic on  $\mathbb{C}^n$ .

Using the correspondence between microlocal proofs of Morera theorems and support theorems, we get the following new support theorem. Define

$$(3.2.7) \quad R_\mu f(y, r) = \int_{z \in S(y, r)} f(z) \mu(z, y, r) ds,$$

where  $\mu(z, y, r)$  is a continuous weight.

**Theorem 3.2.4.** *Let  $M$  be a real-analytic manifold and let  $\mathcal{A}$  be an open, connected subset of  $M^+$ . Let  $f \in \mathcal{D}'(S(\mathcal{A}))$ . Assume the weight,  $\mu$ , in (3.2.7) is nowhere zero and real-analytic on*

$M \times M^+$ . Assume  $R_\mu f(y, r) = 0$  for all  $(y, r) \in \mathcal{A}$ . If  $f$  is zero to infinite order on  $S(y_0, r_0)$  for some  $(y_0, r_0) \in \mathcal{A}$ , then  $f(z) = 0$  for all  $z \in S(\mathcal{A})$ .

*Proof of Theorem 3.2.4.* Proposition 3.2.1 holds for any Radon transform on spheres  $S(y, r)$  that has nowhere zero real-analytic weight. So, since  $R_\mu f(y, r)$  is real-analytic on  $\mathcal{A}$  (it is zero)  $\text{WF}_A(f) \cap N^*(S(y, r)) = \emptyset \forall (y, r) \in \mathcal{A}$ . We assume  $f$  is not identically zero on  $S(\mathcal{A})$ . Now, we use the argument in the last two paragraphs of the proof of Theorem 2.2.2 applied to  $f$  instead of  $\bar{\partial}f$  to prove that  $f$  is zero in  $S(\mathcal{A})$ .  $\square$

Theorem 3.2.4 answers a conjecture of Helgason [KE p. 174, §6, # 1] in many cases. Let  $M$  be a complete simply connected Riemannian manifold of negative curvature and  $B$  a closed ball in  $M$ . Let  $f \in C_c^\infty(M)$  have zero integrals over all geodesic spheres enclosing  $B$ . Helgason conjectures that  $f$  must be zero outside of  $B$ . If  $M$  is *real-analytic*, then his conjecture follows from Theorem 3.2.4. This is true because, by Theorem 1.33 page 36 [CE], any negatively curved simply connected manifold has infinite injectivity radius. However, a stronger theorem follows from [Q 1993]:

**Theorem 3.2.5.** *let  $M$  be a real-analytic manifold and  $B$  a closed geodesic ball in  $M$ . Let  $f \in \mathcal{E}'(M)$  and let  $\rho$  be the radius of the smallest disk containing  $\text{supp } f \cup B$ . Assume the injectivity radius,  $I_M$  of  $M$  is larger than  $\rho$ . Let  $r \in (\rho, I_M)$ . Assume  $f$  has zero integrals on all spheres of radius  $r$  that enclose  $B$ . Then  $f$  is zero outside of  $B$ .*

*Proof outline.* Let  $D$  be the smallest disk containing  $\text{supp } f \cup B$ . We assume  $\text{supp } f \not\subset B$ , so there is a point  $x \in \text{supp } f \cap \text{bd } D$  that is not in  $B$ . To show  $x \notin \text{supp } f$ , we use a sphere  $S$  of radius  $r$  containing  $D$  and tangent to  $D$  at  $x$ . We can use Proposition 3.1.1 on  $S$  because the antipodal point to  $x$  in  $S$  is not in  $D$ .  $\square$

## REFERENCES

- Ag 1978. Agranovsky, M., *Fourier transform on  $SL_2(\mathbb{R})$  and Morera type theorems*, Soviet Math. Dokl. **19** (1978), 1522–1526.
- ABC. Agranovsky, M., Berenstein, C.A., and Chang, D-C., *Morera Theorem for holomorphic  $H^p$  spaces in the Heisenberg group*, J. reine angew. Math. **443** (1993), 49–89.
- ABCP. Agranovsky, M., Berenstein, C.A., Chang, D-C., Pascuas, D., *A Morera type theorem for  $L^2$  functions in the Heisenberg group*, J. Analyse Math. **57** (1991), 282–296.
- Be. Berenstein, C.A., *A test for holomorphy in the unit ball of  $\mathbb{C}^n$* , Proc. Amer. Math. Soc. **90** (1984), 88–90.
- BCPZ. Berenstein, C.A., Chang, D-C., Pasucas, D., and Zalzman, L., *Variations on the theorem of Morera*, Contemp. Math. **137** (1992), 63–78.
- BG 1986. Berenstein, C.A. and Gay, R., *A local version of the two circles theorem*, Israel J. Math. **55** (1986), 267–288.
- BG 1988. ———, *Le Problème de Pompeiu local*, J. Analyse Math. **52** (1988), 133–166.
- BGY. Berenstein, C.A., Gay R., and Yger, A., *Inversion of the local Pompeiu transform*, J. Analyse Math. **54** (1990), 259–287.

- BP. Berenstein, C.A. and Pascuas, D., *Morera and Mean-Value Type theorems in the Hyperbolic Disk*, Israel J. Math. **86** (1994), 61–106.
- BZ. Berenstein, C.A. and Zalzman, L., *Pompeiu's problem on symmetric spaces*, Comment. Math. Helv. **55** (1980), 593–621.
- Bo. Boman, J., *A local vanishing theorem for distributions*, C. R. Acad. Sci. Paris **315** (1992), 1231–1234.
- BQ 1987. Boman, J. and Quinto, E.T., *Support theorems for real analytic Radon transforms*, Duke Math. J. **55** (1987), 943–948.
- BQ 1993. ———, *Support theorems for real analytic Radon transforms on line complexes in  $\mathbb{R}^3$* , Trans. Amer. Math. Soc. **335** (1993), 877–890.
- CE. J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, American Elsevier, New York, 1975.
- DL. Delsarte, J. and Lions, J.L., *Moyennes généralisées*, Comment. Math. Helv. **33** (1959), 59–69.
- GI 1989. Globevnik, J., *Integrals over circles passing through the origin and a characterization of analytic functions*, J. Analyse Math. **52** (1989), 199–209.
- GI 1990. ———, *Zero integrals on circles and characterizations of harmonic and analytic functions*, Trans. Amer. Math. Soc. **317** (1990), 313–330.
- GIQ. Globevnik, J., and Quinto E., *Morera Theorems and microlocal analysis*, J. Geometric Anal. **6** (1996), 19–30.
- GrQ. Grinberg, E. and Quinto E., *Morera Theorems on Spheres through a point in  $\mathbb{C}^n$* , to appear, Proceedings of 1997 ISAAC conference, Kluwer (R. Gilbert, ed.), 1998.
- Gu. Guillemin, V., *Some remarks on integral geometry*, 1975, unpublished.
- GS. Guillemin, V. and Sternberg, S., *Geometric Asymptotics*, Amer. Math. Soc., Providence, RI, 1977.
- He. Helgason, S., *Differential operators on homogeneous spaces*, Acta Math. **102** (1959), 239–299.
- Hö 1971. Hörmander, L., *Fourier Integral Operators, I*, Acta Math. (1971), 79–183.
- Hö 1983. ———, *The Analysis of Linear Partial Differential Operators I*, Springer, New York, 1983.
- Jo. John, F., *Plane Waves and Spherical Means*, Interscience, New York, 1966.
- Ka. Kaneko, A., *Introduction to Hyperfunctions*, Kluwer, New York, 1989.
- KE. Kobayashi, S. and Ellis, J., *Problems in Differential Geometry*, Proceedings of the United States-Japan Seminar in Differential Geometry (1965), Kyoto, Japan.
- KN. Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry, Vol. I*, Interscience, New York, 1963.
- Q 1980. Quinto, E.T., *The dependence of the generalized Radon transform on defining measures*, Trans. Amer. Math. Soc. **257** (1980), 331–346.
- Q 1993. ———, *Pompeiu transforms on geodesic spheres in real analytic manifolds*, Israel J. Math. **84** (1993), 353–363.
- Ru. Rudin, W., *Function Theory on the Unit Ball in  $\mathbb{C}^n$* , vol. 241, Springer Grundlehren, Berlin, New York, 1980.
- SKK. Sato, M., Kawai, T., and Kashiwara, M., *Hyperfunctions and pseudodifferential equations*, Lecture Notes in Math., vol. 287, Springer Verlag, New York, 1973, pp. 265–529.
- Su. Sunada, T., *Spherical means and geodesic chains on a Riemannian manifold*, Trans. Amer. Math. Soc. **267** (1981), 483–501.
- Tr. Treves, F., *Introduction to Pseudodifferential and Fourier integral operators I*, Plenum Press, New York, 1980.
- Ts. Tsujishita, T., *Spherical means on Riemannian manifolds*, Osaka J. Math. **13** (1976), 591–597.
- Za 1972. Zalzman, L., *Analyticity and the Pompeiu problem*, Arch. Rat. Mech. Anal. **47** (1972), 237–254.
- Za 1980. ———, *Offbeat Integral Geometry*, Amer. Math. Monthly **87** (1980), 161–175.
- Za 1992. ———, *A bibliographic survey of the Pompeiu problem*, in “Approximation by Solutions of Partial Differ-

ential Equations," B. Fuglede, M. Goldstein, W. Haussmann, W. K. Hayman, and L. Rogge, Editors, Vol. 365, Series C: Mathematics and Physical Sciences, NATO ASI Series, Kluwer Academic, Boston, 1992, 185–194.

Zh. Yiying Zhou, *Two Radius Support Theorem for the Sphere Transform*, submitted (1998).

ZQ. Yiying Zhou and Eric Todd Quinto, *Two-radius Support Theorems for Spherical Radon transforms on Manifolds*, Proceedings of Conference Honoring Leon Ehrenpreis, Contemporary Math. (1998) (to appear).

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