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THE MICROLOCAL PROPERTIES OF THE LOCAL 3-D SPECT OPERATOR*

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Abstract. We prove microlocal properties of a generalized Radon transform that integrates over lines in $\mathbb{R}^3$ with directions parallel to a fairly arbitrary curve on the sphere. This transform is the model for problems in slant-hole SPECT and conical-tilt electron microscopy, and our results characterize the microlocal mapping properties of the SPECT reconstruction operator developed and tested by Quinto, Bakhos, and Chung. We show that, in general, the added singularities (or artifacts) are increased as much as the singularities of the function we want to image. Using our microlocal results, we construct a differential operator such that the added singularities are, relatively, less strong than the singularities we want to image.

Key words. emission tomography, radon transform, singular pseudodifferential operators, Fourier integral operators, artifacts

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1. Introduction. Single photon emission computed tomography (SPECT) is a medical diagnostic modality used to detect metabolic processes or body structure. The spatial resolution is not usually as good as with X-ray tomography, but X-ray CT cannot in general detect metabolic processes. These maps of metabolic processes are used to pinpoint tumors, which absorb nutrients faster than the surrounding tissue, and in epilepsy research to map brain activity during a seizure.

Slant-hole SPECT is a novel data acquisition method in which the circular detector array rotates about its center, and the array itself does not need to be moved around the patient. This means that a full data set for this specific geometry can be acquired more quickly than for standard data [23]. This geometry is being used now for detecting breast tumors [15] as well as in conical-tilt electron microscopy [25]. Several algorithms for slant-hole SPECT have been developed [17, 23, 15] if one has full data, but, to the authors' knowledge, there is no algorithm for local data except the one presented in [20]. The local algorithm in [20] adds singularities since it includes a backprojection, and the goal of this article is to understand where it adds singularities and to evaluate their strength.

The slant-hole SPECT operator is a Radon transform that integrates over lines parallel to the cone, $\mathcal{S}_\phi$ (3.7), with opening angle from its axis determined by the tilt of the slant holes. It is a special case of the Radon transform that we will consider in this article, P_m (3.5). The transform P_m integrates over lines parallel to a curve S on the sphere (3.2). It is an admissible line complex in the sense of Gel'fand, and if the weight m in (3.5) is smooth, P_m is a Fourier integral operator (FIO) of order $-1/2$. Guillemin and Sternberg [11] first showed that Radon transforms are FIOs in a very general setting. Greenleaf and Uhlmann [6, 9] have studied the microlocal

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properties of geodesic transforms on admissible line complexes on manifolds. Those authors evaluated the properties of $P_m^*P_m$ in this general setting, and we consider a reconstruction operator $\mathcal{L} = P_{1/m}^* D P_m$ for the case of lines and where D is a well-chosen differential operator. The added singularities and their de-emphasis under the right D are apparently visible in reconstructions in [20], and in section 6 we prove this is the case for Sobolev scales.

FIOs such as our reconstruction operator can move singularities. To understand how, we need to understand the composition calculus of two FIOs since in general the composition of two FIOs is not an FIO. Let X and Y be manifolds, and let $I^m(C)$ be the class of FIOs, $F : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ of order m associated to a canonical relation $C \subset T^*(Y \times X)\backslash\mathbf{0}$ [13], with dim $X =$ dim Y. Under a transversal intersection condition, Hörmander [13] proved that if $F_1 \in I^{m_1}(C_1)$ with $C_1 \subset T^*Y \times T^*Z$ and $F_2 \in I^{m_2}(C_2)$ with $C_2 \subset T^*Z \times T^*X$, then $F_1 \circ F_2 \in I^{m_1+m_2}(C_1 \circ C_2)$, where $C_1 \circ C_2 = \{(x,\xi; y,\eta) \in T^*Y \times T^*X |\ \exists (z,\tau) \in T^*Z; (x,\xi;z,\tau) \in C_1 \text{ and } (z,\tau;y,\eta) \in C_2\}$. Duistermaat and Guillemin [2] and Weinstein [24] extended this calculus to a clean intersection condition and showed that $F_1 \circ F_2 \in I^{m_1+m_2+\frac{e}{2}}(C_1 \circ C_2)$, where the number e is called the excess. When these conditions fail, then the geometry of the projections Π_L, Π_R

$$\begin{array}{ccc} & C & \\ {}^{\Pi_L}\swarrow & & \searrow^{\Pi_R} \\ T^*(Y)\setminus\mathbf{0} & & T^*(X)\setminus\mathbf{0} \end{array} \tag{1.1}$$

in some situations helps to establish a composition calculus.

If either map is a local diffeomorphism, then C is a *local canonical graph*. If at least one of C_1 and C_2 is a canonical graph, then the composition calculus is covered by the transverse intersection condition. If one of the projections Π_R or Π_L is singular (drops rank), then so is the other one. They may have different types of singularities even though they drop rank on the same set:

$$\Sigma = \{(x,\xi,y,\eta) \big|\ \det\ d\Pi_L = 0\} = \{(x,\xi,y,\eta) \big|\ \det\ d\Pi_R = 0\}.$$

In the case of slant-hole SPECT, we will show that the projections drop rank by one and that the singularities they exhibit are *folds* and *blow-downs*, which will be defined in section 2. Under this geometry, the composition operator F^*F is not an FIO anymore. Its kernel belongs to a class of distributions associated to two *cleanly intersecting* Lagrangians $I^{p,l}(\cdot,\cdot)$ which will be described in section 2.

In most cases (see Remark 4.1), our transform is a particular case of FIOs with fold and blow-down singularities which were studied by Greenleaf and Uhlmann [6, 7, 8], Guillemin [10], and Felea [3]. In [6, 7, 8], the canonical relation C they consider has the following geometry: Π_L has a blow-down singularity and Π_R has a fold singularity. Such a canonical relation is called a *fibered folding* canonical relation [8]. This is the case of our operator P_m, as we will show in section 4. In [10] and [3], C has the reverse geometry: Π_L has a fold singularity and Π_R has a blow-down singularity. This makes a difference in the geometry of the composition operator F^*F: in the case of [6, 7, 8] it is shown that $F^*F \in I^{2m,0}(\Delta,\Lambda)$, where Δ is the diagonal and Λ is the flowout of the image of the singular points under Π_R and which intersects Δ cleanly of codimension one. In [3] it is proved that $F^*F \in I^{2m,0}(\Delta, Gr(\psi))$, where ψ is a smooth canonical transformation $\psi : T^*\mathbf{R}^2 \to T^*\mathbf{R}^2$ which is an involution: $\psi^2 = \mathrm{id}$ and the graph $Gr(\psi)$ intersects Δ cleanly of codimension two. For an operator $A \in I^{p,l}(\Delta,\Lambda)$ there are also H^s estimates which we will need in this article: $A : H^s \to H^{s+s_0}$, where $s_0 = \max(p + \frac{1}{2}, p + l)$ [8].

Next, we will mention briefly what happens in a simpler case. If the canonical relation of F satisfies the *Bolker assumption* ($\dim(X) \leq \dim(Y)$ and Π_L is an injective immersion), then Π_R is a submersion and F^*F is covered by the clean intersection condition [2] and is a pseudodifferential operator [11]. This means that the singularities of F^*Ff will be contained in those of f; there will be no added singularities. The extent to which Π_L is not an injective immersion determines how far from being a standard pseudodifferential operator F^*F is. An example of an operator not satisfying the Bolker assumption appears in [4], where Π_L is a cross cap (intuitively an immersion with folds), Π_R is a submersion with folds, and $F^*F \in I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta, C_0)$, where C_0 is a canonical relation having both projections with fold singularities.

Our first theorem is a special case of a result in [9]. In section 4 we prove it from first principles and under the simple geometric assumption Hypothesis 3.1.

THEOREM 1.1 (see [9]). *Let C be the canonical relation of the Radon transform* (3.5) *under Hypothesis* 3.1. *Then the map $\Pi_L : C \to T^*Y_S \setminus \mathbf{0}$ has a blow-down, $\Pi_R : C \to T^*\mathbb{R}^3 \setminus \mathbf{0}$ has a fold singularity, and C is a fibered folding canonical relation.*

Therefore, we can use [8] to conclude that $P_m^* P_m \in I^{-1,0}(\Delta, \Lambda_{\Pi_R(\Sigma)})$, where $\Lambda_{\Pi_R(\Sigma)}$ is the flowout given in (5.2). We also have that $P_m^* P_m : H_c^s \to H_{\text{loc}}^{s+\frac{1}{2}}$ and there is a parametrix for $P_m^* P_m$ mod $I^{-\frac{1}{2}}(\Lambda)$ when P_m^* is elliptic and properly supported [6]. If the Bolker assumption held, then $P_m^* P_m$ would map from H_c^s to H_{loc}^{s+1}, so the composition loses a half derivative smoothness.

We also apply these results to the novel reconstruction operator in [20]: $\mathcal{L}(f) = P_{1/m}^* D P_m f$, where D is a well-chosen second order differential operator and $P_{1/m}^*$ is a backprojection operator (3.6). Using results in [8] and Theorem 1.1, we prove that $\mathcal{L} \in I^{1,0}(\Delta, \Lambda_{\Pi_R(\Sigma)})$. This means that $\mathcal{L} \in I^1(\Delta \setminus \Lambda_{\Pi_R(\Sigma)})$, where Δ is the diagonal in $T^*\mathbb{R}^3$, so $\mathcal{L}(f)$ will reproduce (some) singularities of f at a point x and $\mathcal{L} \in I^1(\Lambda_{\Pi_R(\Sigma)} \setminus \Delta)$, where the flowout $\Lambda_{\Pi_R(\Sigma)}$ represents added singularities (singularities added above x that come from other points in $\mathrm{WF}(f)$). So, in general, $\mathcal{L}$ is the same order on the added singularities as on the singularities of f at x.

The operator $\mathcal{L}$ is local: to recover $\mathcal{L}(f)(x)$, one needs data only over lines near x since the backprojection uses only lines near x (see (3.6)) and the differential operator D is local. The function f is not reconstructible from the local data and therefore the goal needs to be more modest: in our case, to reconstruct the singularities of f. Such reconstructions can show the features of the part of the body the scan is imaging [20].

Therefore, one would want the singularities of $\mathcal{L}(f)$ to be at the same places as the singularities of f. However, because the map Π_L for P_m is not an injective immersion, $\mathcal{L}$ adds singularities. How $P_m^* P_m$ can add singularities was first analyzed in [6] for very general geodesic complexes under certain geometric assumptions. Singularity addition is observed for other tomographic reconstruction operators (which are special cases of the results in [6]) including the ones for cone beam CT, and researchers have developed pseudodifferential [5] and differential operators [16] (see [21] for electron microscopy) that decrease the strength of the added singularities. In a very general setting, electron microscopy over curves [22], one can at least decrease nearby added singularities using a well-chosen differential operator. Our next theorem shows that with the right choice of differentiable operator $D = D_g$ (e.g., (6.1)), we can decrease the strength of the added singularities for our general transform.

THEOREM 1.2. *Under the curvature assumption Hypothesis* 3.1, *if D_g is a second* *order differential operator with symbol* $\sigma(D_g)$ *vanishing on* $\Pi_L(\Sigma)$*, then the reconstruction operator* $\mathcal{L} = P^*_{1/m} D_g P_m \in I^{0,1}(\Delta, \Lambda_{\Pi_R(\Sigma)})$.

Thus, using this good D implies that $\mathcal{L} \in I^1(\Delta \setminus \Lambda_{\Pi_R(\Sigma)})$, and so it is order one on $\Delta \setminus \Lambda_{\Pi_R(\Sigma)}$. However, $\mathcal{L} \in I^0(\Lambda_{\Pi_R(\Sigma)} \setminus \Delta)$, and so it is order zero on $\Lambda_{\Pi_R(\Sigma)} \setminus \Delta$. With an arbitrary D, the added singularities (those on $\Lambda_{\Pi_R(\Sigma)} \setminus \Delta$) would be increased by one order in Sobolev scale, but with this good D, their order is not increased. Therefore, the added singularities are weaker than they would be with arbitrary D, but the "real" singularities of $\mathcal{L}(f)$ above x are as strong as for general D. In Remark 6.1, we construct a pseudodifferential operator $\mathcal{D}$ that is close to the D such that the operator $P^*_{1/m} \mathcal{D} P_m$ is a classical pseudodifferential operator.

The article is organized as follows: Section 2 contains the definitions of fold and blow-down singularities and a description of the $I^{p,l}$ classes. In section 3 we describe the operator P_m, and we prove Theorem 1.1 in section 4. Section 5 contains the microlocal properties of the operator $\mathcal{L}$. In section 6 we will describe a differential operator that decreases the strength of the added singularities and give the proof of Theorem 1.2

2. Wavefront set, singularities, and $I^{p,l}$ classes. To understand what our operators do to singularities, we need to understand what singularities are. Practically, they can be density (absorption) jumps such at boundaries between regions in the body. Mathematically, they are where a function is not smooth, and we can characterize smoothness using the Fourier transform.

Definition 2.1. *Let* $s \in \mathbb{R}$ *and let* f *be a distribution such that its Fourier transform,* $\mathcal{F}f$*, is a locally integrable function. Then* $f \in H^s(\mathbb{R}^n)$ *if*

$$\|f\|_s = \left(\int_{\mathbb{R}^n} \left|\mathcal{F}f(\xi)\right|^2 (1 + \|\xi\|^2)^s d\xi \right)^{1/2}$$

is finite and f *is in* $H^s_{loc}(\mathbb{R}^n)$ *if for each compactly supported smooth function* φ, $\varphi f \in H^s(\mathbb{R}^n)$.

If s is a natural number, then a distribution in $H^s(\mathbb{R}^n)$ has L^2 derivatives up to order s.

Our concept of singularity, the wavefront set, specifies points and directions (in cotangent space) in which distributions are not smooth. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Associated to the tangent vector $\xi_1 \partial_{x_1} + \xi_2 \partial_{x_2} + \xi_3 \partial_{x_3}$ is its dual cotangent vector, $\xi \mathbf{dx} := \xi_1 \mathbf{dx}_1 + \xi_2 \mathbf{dx}_2 + \xi_3 \mathbf{dx}_3$; we will identify covectors with their coordinates, so $\xi_1 \mathbf{dx}_1 + \xi_2 \mathbf{dx}_2 + \xi_3 \mathbf{dx}_3$ will be identified with (ξ_1, ξ_2, ξ_3).

Definition 2.2. *Let* $s \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$*, and* $\xi_0 \in \mathbb{R}^n \setminus 0$*. The function* f *is in* H^s *at* x_0 *in direction* ξ_0 *if there exists a cutoff function* φ *near* x_0 *and an open cone* V *with* $\xi_0 \in V$ *such that*

$$\int_{\xi \in V} \left|\mathcal{F}\big(\varphi f\big)(\xi)\right|^2 (1 + \|\xi\|^2)^s \, d\xi \tag{2.1}$$

is finite.

On the other hand, if f *is* not *in* H^s *at* x_0 *in direction* ξ_0*, then we say the covector* (x_0, ξ_0) is in the wavefront set $\mathrm{WF}^s(f)$.

Note that if $f \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$, then $\mathrm{WF}^s(f) = \emptyset$. The wavefront set can be defined for functions on manifolds (such as Y_S; see (3.2)) using coordinates: WF^s is local since it is defined by cutoff functions, and WF^s transforms contravariantly under coordinate change. This is why WF^s is normally defined as a subset of a cotangent space.

Next, we define the fold and blow-down singularities.

DEFINITION 2.3 (see [8]). *Let M and N be manifolds of dimension n and let $f : M \to N$ be C^∞.*

1. *f is a* Whitney fold *if near each $m_0 \in M$, f is either a local diffeomorphism or df drops rank simply by one at m_0 so that $L = \{m \in M \,\big|\, \operatorname{rank} df = n-1\}$ is a smooth hypersurface at m_0 and $\ker df(m_0) \not\subset T_{m_0}L$.*
2. *f is a* blow-down *along a smooth hypersurface $L \subset M$ if f is a local diffeomorphism away from L and df drops rank simply by one at L, where the Hessian of f is equal to zero and $\ker df \subset T(L)$, so that $f\big|_L$ has one-dimensional fibers.*

DEFINITION 2.4 (see [8]). *The canonical relation C in* (1.1) *is a (nonradial)* fibered folding canonical relation *if the following conditions hold.*

1. *$\Pi_R : C \to T^*(X)\backslash\mathbf{0}$ is a Whitney fold with fold hypersurface Σ and $\Pi_R(\Sigma)$ is an embedded nonradial hypersurface.*
2. *$\Pi_L : C \to T^*(Y)\backslash\mathbf{0}$ is a blow-down along Σ, $\Pi_L(\Sigma)$ is embedded, nonradial, and symplectic, and $\Pi_L : (C \setminus \Sigma) \to T^*(Y)$ is injective.*

Remark 2.5. A submanifold $M \subset T^*X$ is *nonradial* if $\rho \notin (TM)^\perp$, where $\rho = \sum \xi_i \partial_{\xi_i}$. We conclude this section by defining $I^{p,l}$ classes. They were first introduced by Melrose and Uhlmann [19] and Guillemin and Uhlmann [12].

DEFINITION 2.6. *Two submanifolds M and N* intersect cleanly *if $M \cap N$ is a smooth submanifolds and if $T(M \cap N) = TM \cap TN$.*

If M and N intersect tangentially on a lower dimensional submanifold, then they do not intersect cleanly.

We will consider Lagrangian submanifolds in the product space $T^*X \times T^*Y$. It is proved that any two pairs of cleanly intersecting Lagrangians $(\tilde\Lambda_0, \tilde\Lambda_1)$ and (Λ_0, Λ_1) are equivalent. This means that we can find microlocally a canonical transformation χ which takes (Λ_0, Λ_1) into $(\tilde\Lambda_0, \tilde\Lambda_1)$. Thus, let us consider the following model case: $\tilde\Lambda_0 = \Delta_{T^*R^n} = \{(x,\xi;x,\xi) | x \in R^n,\ \xi \in R^n\backslash 0\}$, which is the diagonal in $T^*R^n \times T^*R^n$ and $\tilde\Lambda_1 = \{(x', x_n, \xi', 0; x', y_n, \xi', 0) | x' \in R^{n-1},\ \xi' \in R^{n-1} \setminus 0\}$. Notice that $\tilde\Lambda_0$ intersects $\tilde\Lambda_1$ cleanly in codimension 1. Next we will define the class of *product-type* symbols $S^{p,l}(m,n,k)$.

DEFINITION 2.7. *$S^{p,l}(m,n,k)$ is the set of all functions $a(z,\xi,\sigma) \in C^\infty(\mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^k)$ such that for every $K \subset \mathbf{R}^m$ and every $\alpha \in \mathbf{Z}^n_+, \beta \in \mathbf{Z}^n_+, \gamma \in \mathbf{Z}^k_+$ there is $c_{K,\alpha,\beta}$ such that*

$$|\partial_z^\alpha \partial_\xi^\beta \partial_\sigma^\gamma a(z,\xi,\sigma)| \le c_{K,\alpha,\beta}(1+|\xi|)^{p-|\beta|}(1+|\sigma|)^{l-|\gamma|} \quad \forall (z,\xi,\tau) \in K \times \mathbf{R}^n \times \mathbf{R}^k.$$

For the model case $(\tilde\Lambda_0, \tilde\Lambda_1)$ the $I^{p,l}$ class is defined as follows.

DEFINITION 2.8 (see [12]). *Let $I^{p,l}(\tilde\Lambda_0, \tilde\Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2$ with $u_1 \in C_0^\infty$ and*

$$u_2(x,y) = \int e^{i((x'-y')\cdot\xi' + (x_n - y_n - s)\cdot\xi_n + s\cdot\sigma)} a(x,y,s;\xi,\sigma) d\xi d\sigma ds$$

with $a \in S^{p',l'}$, where $p' = p - \frac{n}{2} + \frac{1}{2}$ and $l' = l - \frac{1}{2}$.

At this point we can define the $I^{p,l}(\Lambda_0, \Lambda_1)$ class for any two cleanly intersecting Lagrangians in codimension 1.

DEFINITION 2.9 (see [12]). *Let $I^{p,l}(\Lambda_0, \Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2 + \sum v_i$, where $u_1 \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, $u_2 \in I^p(\Lambda_1 \setminus \Lambda_0)$, the sum $\sum v_i$ is locally finite, and $v_i = Fw_i$, where F is a zero order FIO associated to χ^{-1}, the canonical transformation from above, and $w_i \in I^{p,l}(\tilde\Lambda_0, \tilde\Lambda_1)$.*

This class of distributions is invariant under FIOs associated to canonical transformations which map the pair (Λ_0, Λ_1) to itself. If $F \in I^{p,l}(\Lambda_0, \Lambda_1)$, then $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ and $F \in I^p(\Lambda_1 \setminus \Lambda_0)$ [12].

The pair (Λ_0, Λ_1) we consider in section 4 is (Δ, Λ_Γ), where Δ is the diagonal in $T^*X \times T^*Y$ and Λ_Γ is a *flowout* defined now.

DEFINITION 2.10. *Let* $\Gamma = \{(x,\xi) | p_i(x,\xi) = 0,\ 1 \le i \le k\}$ *be an involutive subset in* T^*X*. Then the flowout of* Γ *is given by*

$$\left\{(x,\xi;y,\eta) \in T^*X \times T^*X | (x,\xi) \in \Gamma,\ (y,\eta) = \exp\left(\sum_{i=1}^{k} t_i H_{p_i}\right)(x,\xi), t \in R^k\right\},$$

where H_{p_i} *is the Hamiltonian flow for* p_i*.*

For example, $\tilde{\Lambda}_1$ from the model case is the flowout of $\Gamma = \{(x,\xi) | \xi_n = 0\}$.

3. The generalized X-ray transform and SPECT. Now we define our general Radon transform and relate it to SPECT. Let I be an interval in $\mathbb{R}$ and let S be the simple curve parametrized by the C^∞ function $\theta : I \to S^2$ that is regular (θ' different from zero). If S is closed, we will assume that I is closed and $\theta : \mathbb{R} \to S^2$ is periodic (with period the length of I). If S is not closed, then we assume that I is an open interval. We let $\mathcal{S}$ be the symmetric cone generated by S,

$$\mathcal{S} = \{t\theta(a) \,\big|\, a \in I, t \in \mathbb{R}\}. \tag{3.1}$$

The following *curvature conditions* will be required for our proofs.

HYPOTHESIS 3.1. *Let* I *be an interval and let* $\theta : I \to S^2$ *parametrize the smooth regular simple curve* S*. Define* $\beta(a) = \theta(a) \times \theta'(a)$*. We assume the following conditions on the function* θ*.*

(a) $\forall a \in I,\ \theta''(a) \cdot \theta(a) \neq 0$.

(b) $\forall a \in I,\ \beta'(a) = \theta(a) \times \theta''(a) \neq \mathbf{0}$.

(c) *The curve* $a \mapsto \beta(a)$ *is a regular simple curve.*

If Hypothesis 3.1(a) holds, then the transform P_m is an FIO, as was proven in [20]. Hypotheses 3.1(b) and (c) are needed to show Π_R is a fold (see section 4). Hypothesis 3.1(a) is always satisfied by the slant-hole SPECT transform, and Hypotheses 3.1(b) and (c) hold for the slant-hole SPECT transform if $\phi \neq \pi/2$. When $\phi = \pi/2$, $\beta(a) = e_3$ and so Hypotheses 3.1(b) and (c) both fail. We discuss the microlocal properties of this case in Remark 4.1.

It should be pointed out that if the simplicity assumptions in Hypothesis 3.1 for the curves S and $\beta(I)$ do not hold but the curves intersect in a finite number of points, one can apply our theorems on subintervals of I on which the curves are simple to get the same results.

The set of lines in our data set are described in the following way. For each $a \in I$ define the plane through the origin perpendicular to $\theta(a)$:

$$\theta^\perp = \theta^\perp(a) := \{x \in \mathbb{R}^3 \,\big|\, x \cdot \theta(a) = 0\}.$$

We integrate over all lines that are parallel to directions in S. This set is given as

$$Y_S := \{(y, \theta(a)) \,\big|\, a \in I,\ y \in \theta^\perp\}. \tag{3.2}$$

Then for each $(y,\theta) \in Y_S$ the line containing y in direction θ is denoted by

$$L(y,\theta) := \{(y + t\theta) \,\big|\, t \in \mathbb{R}\}. \tag{3.3}$$

The *incidence relation* is the set of lines in Y_S and points on those lines:

$$Z_S = \{(y,\theta,x) \,\big|\, (y,\theta) \in Y_S, x \in L(y,\theta)\}. \tag{3.4}$$

The generalized parallel beam Radon transform on Y_S is defined for $f \in C_c(\mathbb{R}^3)$ by

$$P_m f(y,\theta) = \int_{x \in L(y,\theta)} f(x) m(y,\theta,x) dx_L, \tag{3.5}$$

where $m(y,\theta,x)$ is a smooth nowhere zero weight on Z and dx_L is the arc length measure on the line $L(y,\theta)$. $P_m f(y,\theta)$ integrates f over the line through y in direction θ in weight m.

As noted in the introduction, the operator $\mathcal{L}$ is local in the following sense. To calculate $\mathcal{L}(f)$ at x, we differentiate $P_m f$ at lines near x and then backproject–integrate over lines through x. So, to calculate $\mathcal{L}(f)$ we need only data over lines near x.

We now give the specific weights that model SPECT. If $\mu(x)$ is the attenuation factor of the body at x, then the *attenuated Radon transform*, R_μ, is the transform (3.5) with weight

$$m(y,\theta,x) = \exp\left(\int_0^\infty \mu(x+t\theta)dt\right).$$

The *exponential Radon transform*, E_ν, is defined as (3.5) with exponential weight $m(y,\theta,x) = \exp(\nu x \cdot \theta)$ for constant $\nu > 0$. This transform is essentially R_μ with constant attenuation.

Now, we define the dual operator P_m^*. Let $\theta \in S$ and let $p_{\theta(a)}(x)$ be the orthogonal projection from $\mathbb{R}^3$ to $\theta^\perp$:

$$p_{\theta(a)}(x) = x - (x \cdot \theta(a))\theta(a).$$

If S is a closed curve, then we let $\varphi : I \to \mathbb{R}$ be the function 1, and if S is not closed, then we choose a subinterval $[c,d] \subset I$ and let $\varphi : I \to [0,1]$ be smooth, equal to one on $[c,d]$, and supported in I. This allows us to define

$$P_m^* g(x) = \int_{a \in I} \varphi(a) g(p_{\theta(a)}(x), \theta(a)) m(p_{\theta(a)}(x), \theta(a), x) da. \tag{3.6}$$

In the following we will consider the backprojection operator $P_{1/m}^*$ so that the effect of the weight m in the P_m is mitigated by the effect of $1/m$ in $P_{1/m}^*$. This choice is made in [20] and for a related problem in [1].

Example 3.1 (slant-hole SPECT line complex). Let $\phi \in (0,\pi/2)$. We let $\mathcal{S}_\phi$ be the union of all lines through the origin with angle ϕ from the x_3-axis,

$$\mathcal{S}_\phi = \{x \in \mathbb{R}^3 \,\big|\, x \cdot e_3 = \pm\|x\| \cos(\phi)\}. \tag{3.7}$$

We let $S_\phi = \{\theta \in S^2 \,\big|\, \theta \cdot e_3 = \cos(\phi)\}$. Then S_ϕ is a latitude circle on S^2 and the top half of $\mathcal{S}_\phi \cap S^2$. We define

$$Y_\phi := Y_{S_\phi} = \{(y,\theta) \,\big|\, \theta \in S_\phi,\ y \in \theta^\perp\} \tag{3.8}$$

and call it the *slant-hole SPECT line complex*. Y_ϕ is the set of lines in the slant-hole SPECT data set, and the set of lines in Y_ϕ are parallel to the cone $\mathcal{S}_\phi$. Equivalently, these lines have directions on the latitude circle S_ϕ. The curvature condition of Hypothesis 3.1(a) is easily seen to hold for the slant-hole geometry. For the exponential weight, $P^*_{1/m} P_m f = f * I_S$ where I_S is the integral over the cone $\mathcal{S}_\phi$ [20]:

$$I_S(f) = \int_{y \in \mathcal{S}_\phi} \frac{f(x) \exp(\nu \sec(\phi) x \cdot e_3)}{\|x\|}.$$

Notice that $P^*_{1/m} P_m$ is a convolution operator supported on $\mathcal{S}_\phi$, and the wavefront set of such a distribution is the conormal bundle of $\mathcal{S}_\phi \setminus \{0\}$ (union covectors $(0, \xi \mathbf{dx})$ with ξ in the dual cone to $\mathcal{S}_\phi$ [14]). In section 5 we will show this is true in general.

The case $\phi = \pi/2$ is special in that the transform does not satisfy Hypothesis 3.1 (see Remark 4.1). Although this model is not a type of slant-hole SPECT, it comes up in a standard data acquisition method in electron microscopy when one rotates the specimen along one axis, so-called single-axis tilt (see, e.g., [21]).

4. Our proof of Theorem 1.1. We now give a proof of Theorem 1.1 from first principles under the explicit assumptions in Hypothesis 3.1. The criteria in this hypothesis are simple to check, and this theorem is not true for the case discussed in Remark 4.1. A general version of our theorem on manifolds is in [9].

For convenience in the calculations in this section, and without loss of generality, we will now assume that the parametrization θ is chosen such that $\|\theta'\| \equiv 1$.

We use the following notation:

$$\alpha(a) = \theta'(a), \quad \beta(a) = \theta(a) \times \alpha(a). \tag{4.1}$$

Note that $\alpha(a)$, $\beta(a)$, $\theta(a)$ form an orthonormal basis of $\mathbb{R}^3$, and $\alpha(a)$ and $\beta(a)$ form an orthonormal basis of $\theta^\perp(a)$. Also, $\alpha'(a) = \theta''(a)$; $\beta'(a) = \theta(a) \times \theta''(a)$. Finally, note that $\alpha(a)$ is tangent to S at $\theta(a)$, and $\beta(a)$ is normal to both $\alpha(a)$ and $\theta(a)$, and so $\beta(a)$ is normal to the cone $\mathcal{S}$ at $\theta(a)$. We will use coordinates on Y_S (3.2):

$$\begin{gathered} \mathcal{Y}_S = \mathbb{R}^2 \times (a_1, a_2) \ni (r, s, a) \mapsto (r\alpha(a) + s\beta(a), \theta(a)) \in Y_S, \\ L(r, s, a) = L(r\alpha(a) + s\beta(a), \theta(a)), \end{gathered} \tag{4.2}$$

and we identify covectors with their coordinates, so $\eta_r \mathbf{dr} + \eta_s \mathbf{ds} + \eta_a \mathbf{da}$ will be identified with (η_r, η_s, η_a).

In these coordinates, our X-ray transform becomes

$$\mathcal{P}_m(f)(r, s, a) = \int_{x \in L(r,s,a)} f(x) m\big((r, s, a), x\big) dx_L, \tag{4.3}$$

where m is smooth and nowhere zero. Of course, both E_ν and R_μ fit into this framework with $Y_S = Y_\phi$ as long as, for R_μ, the attenuation μ is smooth. The canonical relation of P_m is given in Lemma A.2 in [20]:

$$\begin{aligned} C = \Big\{ &\big(r, s, a, \eta_r, \eta_s, x \cdot \big(\eta_r \alpha'(a) + \eta_s \beta'(a)\big); \\ &x, \eta_r \alpha(a) + \eta_s \beta(a)\big) \Big| \, (\eta_r, \eta_s) \neq \mathbf{0},\ x \cdot \alpha(a) - r = 0,\ x \cdot \beta(a) - s = 0 \Big\}. \end{aligned} \tag{4.4}$$

We have

$$\begin{array}{ccc} & C & \\ {}^{\Pi_L}\swarrow & & \searrow^{\Pi_R} \\ T^*(\mathcal{Y}_S) \setminus \mathbf{0} & & T^*(\mathbb{R}^3) \setminus \mathbf{0} \end{array} . \tag{4.5}$$

Proof of Theorem 1.1. We will use the coordinate map

$$(x, a, \eta_r, \eta_s) \mapsto \big(r, s, a, \eta_r, \eta_s, x \cdot \big(\eta_r \alpha'(a) + \eta_s \beta'(a)\big); x, \eta_r \alpha(a) + \eta_s \beta(a)\big)$$

on C. In these coordinates Π_L and Π_R are given by

$$\Pi_L(x, a, \eta_r, \eta_s) = \big(x \cdot \alpha(a), x \cdot \beta(a), a; \eta_r, \eta_s, x \cdot (\eta_r \alpha'(a) + \eta_s \beta'(a))\big) \tag{4.6}$$

and

$$\Pi_R(x, a, \eta_r, \eta_s) = \big(x; \eta_r \alpha(a) + \eta_s \beta(a)\big). \tag{4.7}$$

Notice that

$$d\Pi_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_r \alpha_1' + \eta_s \beta_1' & \alpha_1 & \beta_1 \\ 0 & 0 & 0 & \eta_r \alpha_2' + \eta_s \beta_2' & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & \eta_r \alpha_3' + \eta_s \beta_3' & \alpha_3 & \beta_3 \end{pmatrix}$$

and that $\det d\Pi_R = (\alpha \times \beta) \cdot (\eta_r \alpha' + \eta_s \beta') = \theta \cdot (\eta_r \theta'' + \eta_s \theta \times \theta'') = \eta_r \theta \cdot \theta''$ Thus, the set where $d\Pi_R$ drops rank by one is

$$\begin{aligned} \Sigma &= \{\gamma \in C \,\big|\, \eta_r = 0\} \\ &= \{(x \cdot \alpha(a), x \cdot \beta(a), a, 0, \eta_s, x \cdot (\eta_s(\theta(a) \times \theta''(a)); x, 0, \eta_s \beta(a))) \\ &\qquad \big|\, x \in \mathbb{R}^3, a \in I, \eta_s \neq 0\} \end{aligned} \tag{4.8}$$

since $\theta \cdot \theta'' \neq 0$ by the curvature condition Hypothesis 3.1(a).

Next we find the kernel of $d\Pi_R$. From the matrix $d\Pi_R$, we see members of its kernel are of the form $(0, 0, 0, \delta_a, \delta_{\eta_r}, \delta_{\eta_s}) = \delta_a \partial_a + \delta_{\eta_r} \partial_{\eta_r} + \delta_{\eta_s} \partial_{\eta_s}$ and we solve for δ_a, etc. We have $\eta_s \beta' \delta_a + \alpha \delta_{\eta_r} + \beta \delta_{\eta_s} = 0$. We rewrite this expression as $\eta_s(\theta \times \theta'')\delta_a + \theta' \delta_{\eta_r} + (\theta \times \theta')\delta_{\eta_s} = 0$ and solve for δ_{η_r} and δ_{η_s} in terms of δ_a. After taking the inner product with θ' and using $\theta' \cdot \theta' = 1$, we see $\delta_{\eta_r} = -\eta_s \theta' \cdot (\theta \times \theta'')\delta_a \neq 0$ by Hypothesis 3.1(b). Furthermore,

$$\delta_{\eta_s} = -\eta_s (\theta \times \theta'') \cdot (\theta \times \theta')\delta_a = 0. \tag{4.9}$$

So if $c = -\eta_s \theta' \cdot (\theta \times \theta'')$, then Ker $d\Pi_R$ is the span of $(0, 0, 0, 1, -c\eta_s, 0) = \partial_a - c\eta_s \partial_{\eta_r}$ and $(\delta_a - c\eta_s \delta_{\eta_r})\eta_r = -c\eta_s \neq 0$. So Π_R has a fold singularity by Definition 2.4.

Note that

$$\Pi_R(\Sigma) = \{(x, \eta_s \beta(a)) \,\big|\, x \in \mathbb{R}^3, a \in I, \eta_s \neq 0\}. \tag{4.10}$$

Since β is a simple curve by Hypothesis 3.1(c), $\Pi_R(\Sigma)$ is embedded. To check whether $\Pi_R(\Sigma)$ is a nonradial hypersurface we explain how we can describe $\beta(I)$ as the zero-set of a smooth function $F : S^2 \to [0, \infty)$.[1] Extend F to $\mathbb{R}^3$ homogeneous of order one

[1]In this proof sketch, we will assume S is closed. If S is not closed, then there is some interval $[c, d] \subset I$ such that $\operatorname{supp}(\varphi) \subset [c, d]$, and we use our argument on this closed interval. By Hypothesis 3.1(c) the image $B = \beta(I)$ is a regular curve that is topologically closed. Since the set $B \subset S^2$ is compact and the curve is regular, we can use the implicit function theorem locally to get a finite number of open sets in S^2, $\{U_j \,|\, j = 1, \dots, N\}$ and a finite number of smooth functions $F_j : U_j \to [0, \infty)$ such that $F_j(\theta) = 0$ if and only if $\theta \in B \cap U_j$. Let $U_0 = S^2 \setminus B$ and let $F_0 = 1$. Let $\{\varphi_j \,|\, j = 0, \dots, N\}$ be a partition of unity on S^2 subordinate to $\{U_j \,|\, j = 0, \dots, N\}$. The desired function is $F = \sum_{j=0}^N \varphi_j F_j$.

and then

$$\Pi_R(\Sigma) = \{(x,\xi) \big| x \in \mathbb{R}^3, F(\xi) = 0\}.$$

Using Remark 2.5, notice that ρ is different from $H_F = \sum \partial_{\xi_i} F \partial_{x_i}$, which spans $(T\Pi_R(\Sigma))^\perp$. Thus $\Pi_R(\Sigma)$ is nonradial.

Now we consider Π_L and calculate the derivative matrix

$$d\Pi_L = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \beta_2 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \eta_r\alpha_1' + \eta_s\beta_1' & \eta_r\alpha_2' + \eta_s\beta_2' & \eta_r\alpha_3' + \eta_s\beta_3' & x \cdot (\eta_r\alpha'' + \eta_s\beta'') & x\cdot\alpha' & x\cdot\beta' \end{pmatrix}$$

with $\det d\Pi_L = \eta_r\theta \cdot \theta''$. Therefore, $d\Pi_L$ drops rank on Σ and a straightforward calculation shows that Ker $d\Pi_L$ is spanned by $\theta(a) \cdot (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$, and this vector is tangent to Σ. Therefore Π_L has a blow-down singularity.

It is straightforward to show that $\beta'(a)$ is parallel $\alpha(a)$, so for some scalar function $c(a)$,

$$c(a)\alpha(a) = \beta'(a). \tag{4.11}$$

By Hypothesis 3.1(b) $c(a) \neq 0$ $\forall a \in I$, and c is smooth. A calculation shows that

$$\Pi_L(\Sigma) = \{(y,\eta) \in T^*Y_S \big| \eta_1 = 0 = \eta_3 - c(y_3)y_1\eta_2\}, \tag{4.12}$$

where we use coordinates $(y_1, y_2, y_3) = (r, s, a)$. To show $\Pi_L(\Sigma)$ is symplectic, one calculates the Poisson bracket of the defining functions of $\Pi_L(\Sigma)$ and sees

$$\{\eta_1, \eta_3 - c(y_3)y_1\eta_2\} = c(y_3)\eta_2 \neq 0.$$

This shows that Π_L satisfies the conditions in Definition 2.4.

Thus by Definition 2.4 the canonical relation C is a fibered folding canonical relation. This concludes our proof of Theorem 1.1. $\square$

It should be pointed out that by [8], $P_m^* P_m \in I^{-1,0}(\Delta, \Lambda_{\Pi_R(\Sigma)})$, where the flowout $\Lambda_{\Pi_R(\Sigma)}$ is given by (5.2). We also have that $P_m^*P_m : H_c^s \to H_{\text{loc}}^{s+\frac{1}{2}}$, and when P_m^* is elliptic and proper (when S is closed) there is a parametrix for $P_m^*P_m$ mod $I^{-\frac{1}{2}}(\Lambda)$ [6].

In [20] it was shown by different methods that Π_L is an injective immersion except on the subset of C where $\eta_r = 0$. This is exactly our Σ. This set corresponds to the bad cotangent directions, those that cause added singularities.

Remark 4.1. We now investigate the properties of the slant-hole transform (Example 3.1) when $\phi = \pi/2$. In this case Hypotheses 3.1(b) and (c) do not hold since $\beta = e_3$ and $\beta' = 0$. Using the calculations above, we see $\ker \Pi_R = \text{span}(\partial_a)$, which is tangent to Σ and $\Pi_R(\Sigma) = \{(x,\xi) \big| \xi = \eta_s\beta\} = \{(x,\xi)\big| \xi_1 = 0 = \xi_2\}$, which is involutive. Similarly, $\Pi_L(\Sigma) = \{(y,\eta)\big|\eta_1 = 0 = \eta_3\}$, which is also involutive. Therefore, by [18], $\mathcal{L} \in I^{1,0}(\Delta, \Lambda_{\Pi_R(\Sigma)})$ in this case, too.

5. Microlocal properties of $\mathcal{L}$ in general. In this section we describe precisely what $\mathcal{L} = P_{1/m}^* D P_m$ does to singularities where D is a second order differential or pseudodifferential operator.

By the calculus of FIOs (see, e.g., [13, 8] or [22, section 3.2])

$$\mathrm{WF}(\mathcal{L}(f)) \subset \Pi_R(C),$$

and so $\mathcal{L}$ can reproduce only singularities in $\Pi_R(C)$—so-called *visible* singularities.

Note that singularities of f can be added to $\mathcal{L}(f)$ at "bad" covectors that are in $\Pi_R(\Sigma)$. Furthermore, P_m satisfies the Bolker assumption above the "good" covectors (those in $\Pi_R(C \setminus \Sigma)$).

Recall that $\mathcal{S}$ is the symmetric cone generated by the curve S. Let $\mathcal{S}' = \mathcal{S} \setminus \{0\}$. Let $x \in \mathbb{R}^3$ and define $N_x = N^*(x + \mathcal{S}')$. Then

$$N_x = \{(x + t\theta(a), \eta_s\beta(a)) \,\big|\, a \in I, t \neq 0, \eta_s \neq 0\}. \tag{5.1}$$

This is true for the following reason. A vector ξ is normal to $\mathcal{S}' = \{t\theta(a) \,\big|\, a \in I\}$ if and only if it is normal, for some $a \in I$, to both $\theta(a)$ and $\alpha(a)$ (and so is parallel $\beta(a)$).

Our next theorem shows how wavefront is affected by $\mathcal{L}$.

THEOREM 5.1. *Let $f \in \mathcal{E}'(\mathbb{R}^3)$ and let S be a curve satisfying Hypothesis* 3.1. *Then*

$$\mathrm{WF}(\mathcal{L}(f)) \subset (\mathrm{WF}(f) \cap \Pi_R(\Sigma)) \cup \{(x,\xi) \,\big|\, \textit{for some } y \in x + \mathcal{S}', (y,\xi) \in N_x \cap \mathrm{WF}(f)\}.$$

If S is closed and D is elliptic, then $\mathrm{WF}^s(\mathcal{L}(f)) \cap \Pi_R(C \setminus \Sigma) = \mathrm{WF}^{s+1}(f) \cap \Pi_R(C \setminus \Sigma)$.

Thus, $\mathcal{L}$ can add singularities above x that come from singularities of f in $N_x \cap \mathrm{WF}(f)$. Those singularities are added in $\Pi_R(\Sigma) \cap T^*_x\mathbb{R}^3$ since the covectors in $\Pi_R(\Sigma)$ are conormal to $\mathcal{S}'$. Furthermore, $\mathcal{L}$ can show the singularities of f in $\Pi_R(C \setminus \Sigma)$.

Proof. The results in [6, 7] can be applied to the composition $\mathcal{L}$ to conclude that it is in $I^{1,0}(\Delta, \Lambda_{\Pi_R(\Sigma)})$, and by Definition 2.10 the flowout is

$$\Lambda_{\Pi_R(\Sigma)} = \{(x,\xi,y,\xi) \,\big|\, x \in \mathbb{R}^3, (y,\xi) \in N_x\}. \tag{5.2}$$

Now, by [9] note that

$$C^t \circ C = \Delta \cup \Lambda_{\Pi_R(\Sigma)}. \tag{5.3}$$

The second statement in the theorem follows from (5.3) because, in general, if F is a Fourier integral operator with canonical relation C, then $\mathrm{WF}(Ff) \subset C \circ \mathrm{WF}(f)$ [13].

In the second part of the theorem we assume S is a closed curve. Therefore, the function φ in definition (3.6) is one and P_m is elliptic on all of S. Since P_m satisfies the Bolker assumption above $\Pi_R(C \setminus \Sigma)$, the composition $\mathcal{L}$ is a standard elliptic pseudodifferential operator on $\Pi_R(C \setminus \Sigma)$. Note that by the definition of N_x (5.1), added singularities are only in $\Pi_R(\Sigma)$. The final statement follows since $\mathcal{L}$ is a regular pseudodifferential operator in $\Pi_R(C \setminus \Sigma)$ and its order is one. ∎

This specific theorem also follows from Theorems 3.6 and 3.9 in [22] in which a more general Radon transform is considered.

If P_m satisfied the Bolker assumption, then $\mathcal{L}$ would map H^s_c to H^{s-1}_{loc} since P_m and its dual are of order $-1/2$, and D is a second order differential operator on Y_S. However, because $\mathcal{L}$ is in the class $I^{1,0}(\Delta, \Lambda)$, $\mathcal{L}$ takes $1/2$ more derivative. Thus we have by [8] $\mathcal{L} : H^s_c(\mathbb{R}^3) \to H^{s-3/2}_{\mathrm{loc}}(\mathbb{R}^3)$.

For the slant-hole SPECT in the degenerate case $\phi = \pi/2$ we see that $\mathcal{L} \in I^{1,0}(\Delta, \Lambda_{\Pi_R(\Sigma)})$ using the same arguments as in Remark 4.1.

6. De-emphasis of added singularities and proof of Theorem 1.2. In [20] we define the reconstruction operator $\mathcal{L}$ using the "good" operators

$$D_g = -\partial_r^2 = -\partial_{y_1}^2, \qquad \mathcal{L}_g = P_{1/m}^* D_g P_m, \tag{6.1}$$

which takes a second derivative in the $\alpha(a)$ direction that is tangent to the curve S at $\theta(a)$. The symbol of D_g in coordinates is $\sigma(D_g) = \eta_1^2$, and this symbol is zero on $\Pi_L(\Sigma)$. In [20], reconstructions from simulations for the slant-hole SPECT operator (Example 3.1) illustrate how the added singularities appear to be de-emphasized when using this operator as compared to an elliptic differential operator. Now we will prove this.

Note that $\Pi_R(\Sigma) = \{(x, \eta_s\beta(a)) \,|\, x \in \mathbb{R}^3,\ a \in I\}$, and these are the "bad" cotangent directions above which Π_L is not an injective immersion. However, D_g is not elliptic off of $\Pi_L(\Sigma)$ (although its symbol is nonzero on $\Pi_L(C) \setminus \Pi_L(\Sigma)$).

The differential operator

$$D' = \partial_{y_1}^2 + (\partial_{y_2} - c(y_3)y_1\partial_{y_3})^2 \tag{6.2}$$

in coordinates $y = (r, s, a)$ is zero on $\Pi_L(\Sigma)$ *and* is elliptic off of $\Pi_L(\Sigma)$ because the two defining equations for $\Pi_L(\Sigma)$ are $\eta_1 = 0$ and $\eta_2 - c(y_3)y_1\eta_3 = 0$. Note that for the slant-hole SPECT operator with $\phi \in (0, \pi/2)$, (4.11) is $c(a) = \cot(\phi)$, which is constant, and so D' would be easy to implement numerically in this case.

Proof of Theorem 1.2. To prove this theorem we refer to the discussion on page 459 of [8] to see that $P_{1/m}^* D' P_m \in I^{0,1}(\Delta, \Lambda_{\Pi_R(\Sigma)})$. This implies that when using D', the order of $\mathcal{L}$ on $\Delta \setminus \Lambda_{\Pi_R(\Sigma)}$ is one and the order of $\mathcal{L}$ on $\Lambda_{\Pi_R(\Sigma)} \setminus \Delta$ is zero. Because $\Pi_R(\Pi_L^{-1}(\eta_1 = 0)) = \Pi_R(\Sigma)$, the discussion in [8] also applies to D_g and $\mathcal{L}_g$. $\square$

Note that if one uses a higher order differential operator with symbol being zero on $\Pi_L(\Sigma)$ of higher order, such as $D = (D')^k$ for some $k > 1$, then the added singularities are emphasized relatively less than the ones in f at x. This is true because as D is order $2k$ and vanishes to order $2k$ on $\Pi_L(\Sigma)$, then $\mathcal{L} \in I^{k-1,k}(\Delta, \Lambda_{\Pi_R(\Sigma)})$ and so is of order $k-1$ on $\Lambda_{\Pi_R(\Sigma)} \setminus \Delta$ and order $2k-1$ on $\Delta \setminus \Lambda_{\Pi_R(\Sigma)}$. In numerical simulations, one might want to smooth before applying this operator for large k since noise would be amplified and jump singularities could become too singular.

Remark 6.1. By using a well-chosen *pseudodifferential* operator that is close to our operator in (6.1) we can create a classical pseudodifferential operator that is close to $\mathcal{L}_g$. We choose $\mathcal{D}$ a smooth second order pseudodifferential operator in y that is equal to zero on a conic neighborhood U of $\eta_1 = 0$ and equal to D_g away from that neighborhood. Then the composition $\mathcal{D}P_m$ kills the covectors in a neighborhood of $\Pi_L(\Sigma) \subset U$ and then when composed with $P_{1/m}^*$ satisfies the Bolker assumption where the operator is nonzero since Π_L c is an injective immersion off of Σ.

Therefore, $P_{1/m}^*\mathcal{D}P_m$ is a classical pseudodifferential operator and does not add singularities. It does smooth singularities that are near $\Pi_R(\Sigma)$ so it would not be elliptic everywhere on $\mathcal{V}$.

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