REAL ANALYTIC RADON TRANSFORMS
ON RANK ONE SYMMETRIC SPACES

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ABSTRACT. Using microlocal techniques, we prove support theorems for Radon transforms with real analytic measures on horocycles in rank one symmetric spaces. We generalize Helgason's support theorem to this case and prove a new local support theorem.

1. Introduction.

Radon transforms on symmetric spaces have a rich theory because of the extra structure that the groups provide. In this article, we will use this structure plus the theory of analytic Fourier integral operators to prove support theorems for real analytic Radon transforms on horocycles in symmetric spaces. This study is based on the pioneering work of Helgason (*e.g.,* [He 1970, 1973]) on group Radon transforms and Guillemin [G 1975, GS, G 1985] on Radon transforms as Fourier integral operators. Since then several authors have studied Radon transforms as Fourier integral operators [GU, BQ 1991]. The microlocal analysis is easiest for rank one, the case considered here.

Let X be a real analytic manifold and let the manifold Y parameterize certain subsets of X. Let μ be a weight function on each of these subsets. Let R_μ be a Radon transform that integrates functions on X over the sets parameterized by Y in the weight μ. A typical support theorem for R_μ is as follows: given appropriate functions f on X and an appropriate subset $\mathcal{A}$ of Y, if $Rf(\xi) = 0$ for each submanifold ξ in $\mathcal{A}$, then f is zero on all points in the union of the submanifolds in $\mathcal{A}$. Cormack [Co], Solmon [So], and others have proven support theorems for transforms integrating over various curves and surfaces and with non-standard measures (*e.g., ,* [Fi], [O], [Q 1983]). However there are examples depending on the function class (*e.g.,* [ShK]) or measure [Bo] for which support theorems do not hold. Support theorems are useful in partial differential equations [He 1973] and tomography [Q 1988].

The support theorems in this article are for the Radon transform on horocycles on rank one symmetric space with arbitrary nowhere zero real analytic measures. Under the assumption that the measures are group invariant, Helgason [He 1970] proved theorem 4.1 for symmetric spaces of all ranks using purely Lie algebraic techniques. Theorem 4.3 is new, even in the classical case.

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In our proofs, the theory of real analytic Fourier integral operators is used to deduce analytic smoothness of a distribution f from support restrictions on $R_\mu f$ (theorem 3.1, see also [G 1975]). Then a theorem of Hörmander Kawai, and Kashiwara [Hö 1983, Ka] about analytic singularities and support is used to deduce support restrictions on f from analytic smoothness of f. The Radon transform, R_μ, and its dual, R_μ^*, are defined using the group double fibration in §2; the microlocal analysis is given in §3; and the support theorems are proven in §4.

2. The point-horocycle double fibration.

Let G be a real semisimple non-compact Lie group. Let $G = KAN$ be its Iwasawa decomposition and let M be the centralizer of A in K. The point-horocycle *double fibration* is:

$$Z = G/M \quad \xrightarrow{p_Y} \quad Y = G/MN$$
$$\Big\downarrow p_X \tag{2.1}$$
$$X = G/K$$

where p_X and p_Y are the natural maps [He 1984].

The double fibration provides manifolds of integration for a Radon transform. For each $\xi \in Y$, there corresponds a unique closed horocycle in X,

$$\hat{\xi} = p_X p_Y^{-1}(\{\xi\}) \text{ with measure } dm_\xi(x)$$

and for each $x \in X$, there corresponds a unique compact subset of Y,

$$\check{x} = p_Y p_X^{-1}(\{x\}) \text{ with measure } dm_x(\xi).$$

These measures are defined canonically in terms of the appropriate Haar measures on the groups. Because the maps in (2.1) are fibrations, all $\hat{\xi}$ are diffeomorphic (to $MN/M \simeq N$), and all $\check{x}$ are diffeomorphic (to K/M) [He 1963, 1984]. The set Z is called the *incidence relation* because Z can be imbedded in $X \times Y$ as $Z = \{(x, \xi) \in X \times Y | x \in \hat{\xi}\}$, and under this identification, the maps p_X and p_Y are projections.

These manifolds and measures define a Radon transform and its dual in the following way. Let $\mu(x, \xi)$ be a nowhere zero real analytic function for $(x, \xi) \in Z$. Let $f \in C_c(X)$, then the Radon transform of f is defined by

$$R_\mu f(\xi) = \int_{x \in \hat{\xi}} f(x)\mu(x, \xi) dm_\xi(x). \tag{2.2a}$$

This is the integral of f over the horocycle $\hat{\xi} \subset X$ with respect to the analytic measure μdm_ξ. The dual Radon transform of $g \in C(Y)$ is

$$R_\mu^* g(x) = \int_{\xi \in \check{x}} g(\xi)\mu(x, \xi) dm_x(\xi), \tag{2.2b}$$

the integral of g over the compact set $\check{x} \subset Y$ of all horocycles through x with respect to the analytic measure μdm_x. As in the classical case [He 1963] R_μ and R_μ^* are formally dual under the G invariant measures on X and Y. The Radon transform and its dual can be understood as push forwards of pull backs of measures on X, Y, and Z [GS, p.365].

We include the weight μ in the definition of the Radon transform, (2.2), because canonical measures do not occur in general. Employing non-standard measures can help focus on properties intrinsic to the Radon transform rather than on specific symmetry relations that are valid only for special cases. Also, our proofs, involving microlocal analysis, do not require canonical measures.

3. The Radon transform as an analytic Fourier integral operator.

The key to the support theorems is the way that R_μ detects singularities, theorem 3.1. Recall that X is a real analytic manifold and so the analytic wave front set, $WF_A(f)$, of a distribution $f \in \mathcal{E}'(X)$ is defined using local coordinates and the definition in $\mathbb{R}^n$ [Treves].

Theorem 3.1. *Let G be a semisimple Lie group of real rank one. Let R_μ be a Radon transform on horocycles with a weight function μ that is real analytic and never zero. Let $f \in \mathcal{E}'(X)$ and let ξ_o be a given horocycle. Assume $R_\mu f = 0$ for all horocycles in an open neighborhood of ξ_o. Let $N^* \hat{\xi}_o$ be the conormal bundle of $\hat{\xi}_o$ in T^*X. Then $WF_A(f) \cap N^* \hat{\xi}_o = \emptyset$.*

The essential calculations in this proof were done by Victor Guillemin [G 1975] in unpublished work. Guillemin proves general conditions on an arbitrary group fibration for $R_\mu^* R_\mu$ to be an elliptic pseudodifferential operator. For rank greater than one, $R_\mu^* R_\mu$ is *not* elliptic and theorem 3.1 is not true. We prove a weaker statement for rank greater than one in [GQ] (see end of §3).

Proof of Theorem 3.1. Because Guillemin's proof [G 1975] was never published, the relevant details are given here. The key to theorem 3.1 is an understanding of the microlocal analysis of the operators R_μ and R_μ^*. This is gotten from the diagram on the cotangent level corresponding to the double fibration (2.1):

$$\Gamma = N^* Z \setminus 0 \xrightarrow{\pi_Y} T^*(Y) \setminus 0$$

$$\downarrow \pi_X \qquad\qquad\qquad\qquad (3.1)$$

$$T^*(X) \setminus 0$$

where π_X and π_Y are the natural projections.

Without loss of generality, we can assume ξ_o is the identity coset in Y. The set $p_Y^{-1}(\xi_o) = MN/M \simeq N$ is the orbit of MN in $Z = G/M$ (see [He 1963], theorem 3.4). Let Γ_o be the subset of Γ lying above $p_Y^{-1}(\xi_o)$. We first show

Lemma 3.2. $\pi_X : \Gamma_o \to N^* \hat{\xi}_o \setminus 0$ *is a diffeomorphism.*

Proof. The argument is valid for any double fibration. First, note for $(x, \xi_o) \in Z$ that the sequence

$$0 \longrightarrow T_x \hat{\xi}_o \xrightarrow{i} T_x X \xrightarrow{j} \frac{T_{(x,\xi_o)}(X \times Y)}{T_{(x,\xi_o)} Z} \longrightarrow 0 \qquad (3.2)$$

is exact where i is the inclusion and for $v \in T_x X$, $j(v)$ is the equivalence class of $(v, 0)$ in the quotient space in (3.2). Clearly $\operatorname{im} i = \ker j$ as $v \in T_x \hat{\xi}_o$ **iff** $(v, 0) \in T_{(x,\xi_o)} Z$. The map j is onto because $p_Y : Z \to Y$ is a submersion: *let $(v, w) \in T_{(x,\xi_o)}(X \times Y)$, then there is a $\bar{v} \in T_x X$ such that $(\bar{v}, w) \in T_{(x,\xi_o)} Z$; thus, (v, w) is in the same equivalence class as $(v - \bar{v}, 0)$, and so j is onto.* Taking the dual diagram of (3.2) shows that π_X, the dual map to j in (3.2), is a diffeomorphism on each fiber of Γ_o. Now, because Z is imbedded in $X \times Y$, $p_X : p_Y^{-1}(\xi_o) \to \hat{\xi}_o$ is a diffeomorphism on the base, so the lemma is proved. ∎

Theorem 3.1 follows from the properties of the map π_Y in (3.1). Lemma 3.4 will imply that π_Y is an injective immersion if G has real rank one. This is the *Bolker Assumption* [GS; see also Q

1980, p. 335], and it implies that Γ is a local canonical graph. Because of the assumptions on the measure μ, it also implies that R_μ is an elliptic Fourier integral operator associated with Γ [GS, see also Q 1980 p. 337 above (15)]. Although this result was proven in the C^∞ category, it is true in the real analytic category [Bj, SKK, Ka]. The calculus of such operators proves that, if $R_\mu f = 0$ for horocycles near ξ_o, then $WF_A f \cap [\pi_X \circ \pi_Y^{-1} T^*_{\xi_o} Y \setminus 0] = \emptyset$. However, by the definition of Γ_o and lemma 3.2, $\pi_X \circ \pi_Y^{-1} T^*_{\xi_o} Y \setminus 0 = N^* \hat{\xi}_o \setminus 0$. This proves theorem 3.1 assuming the Bolker Assumption.

The composition rules of Fourier integral operators also imply that $R^*_\mu R_\mu$ is an analytic elliptic pseudodifferential operator [GS, see also Q 1980 theorem 2.1].

A translation argument on Γ above the fibers of $Z \to Y$ shows that the Bolker Assumption is equivalent to $\pi_Y : \Gamma_o \to T^*_{\xi_o} Y$ being an injective immersion. Let

$$\overline{\rho} = \pi_Y \circ \pi_X^{-1} : N^* \hat{\xi}_o \setminus 0 \to T^*_{\xi_o} Y \setminus 0. \tag{3.3}$$

By lemma 3.2, the map $\pi_Y : \Gamma_o \to T^*_{\xi_o} Y$ is an injective immersion **iff** the map $\overline{\rho}$ is. To finish the proof, we will prove that $\overline{\rho}$ is an injective immersion (lemma 3.4).

To this end, we will first express $\overline{\rho}$ in terms of the Lie algebras (lemma 3.3). Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}$ be the Lie algebras of G, K, A, M and N respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, and let B denote the Killing form. Recall that $B|_\mathfrak{k}$ is negative definite; $B|_\mathfrak{p}$ is positive definite; and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an orthogonal decomposition with respect to B [He 1962, Prop. III 7.4]. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using B. If $\mathfrak{h} \subset \mathfrak{g}$, then let

$$\mathfrak{h}^\perp = \{S \in \mathfrak{g} | B(S, \mathfrak{h}) = 0\}.$$

Under the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$, $\mathfrak{h}^\perp$ is the "conormal space" of $\mathfrak{h}$. Thus,

$$\mathfrak{k}^\perp = \mathfrak{p} \quad \text{and} \quad (\mathfrak{m} + \mathfrak{n})^\perp = \mathfrak{a} + \mathfrak{n}. \tag{3.4}$$

The second equality in (3.4) is true for the following reasons. As $\mathfrak{k} \perp \mathfrak{p}$, $\mathfrak{m} \perp \mathfrak{a}$. Recall that $\mathfrak{n}$ is the direct sum of the positive root spaces [Wa 7.3.5]; $(\mathfrak{m} + \mathfrak{a})$ is the "zero root space" [Wa 7.3.4]; and two root spaces are orthogonal under the Killing form if the sum of their roots is not zero (see [He 1962, III 4.2]). These facts imply $(\mathfrak{m} + \mathfrak{n}) \perp (\mathfrak{a} + \mathfrak{n})$, and now a dimension argument using [Wa 7.3.5] completes the proof.

Lemma 3.3. *Let $G = KAN$ be the Iwasawa decomposition of a semisimple Lie group of any real rank and let M be the centralizer of A in K. Let ξ_o be the horocycle MN. Then the conormal space $N^* \hat{\xi}_o \setminus 0 = \{(gK, Ad(g)S) \mid g \in MN, \ S \in \mathfrak{a} \setminus 0\}$ and the cotangent space $T^*_{\xi_o} Y = \mathfrak{a} + \mathfrak{n}$. The map $\overline{\rho} = \pi_Y \circ \pi_X^{-1}$ is projection on the second factor:*

$$\begin{aligned} \overline{\rho} : N^* \hat{\xi}_o \setminus 0 &\to T^*_{\xi_o} Y \setminus 0 \\ (gK, \ Ad \ (g)S) &\to Ad(g)S. \end{aligned} \tag{3.5}$$

Proof. $T_{\xi_o} Y$ is identified with $\mathfrak{g}/(\mathfrak{m} + \mathfrak{n})$ and so $T^*_{\xi_o} Y$ can be identified with $(\mathfrak{m} + \mathfrak{n})^\perp = \mathfrak{a} + \mathfrak{n}$ by (3.4).

Let $g \in MN$, then the isotropy group of gK in X is gKg^{-1}. Using this, we see $T_{gK}X = \mathfrak{g}/(Ad(g)\mathfrak{k})$. The manifold $\hat{\xi}_o$ is canonically diffeomorphic to $MN/M \simeq N$ and so

$$T_{gK}\hat{\xi}_o = \frac{\mathfrak{m} + \mathfrak{n}}{Ad(g)\mathfrak{m}} = \frac{Ad(g)(\mathfrak{m} + \mathfrak{n}) + Ad(g)\mathfrak{k}}{Ad(g)\mathfrak{k}}.$$

Note in the last equality that $Ad(g)(\mathfrak{m} + \mathfrak{n}) = \mathfrak{m} + \mathfrak{n}$ because $g \in MN$. Using (3.4) one sees that the conormal space

$$N^*_{gK}\hat{\xi}_o = Ad(g)\big[(\mathfrak{m} + \mathfrak{n})^{\perp} \cap \mathfrak{k}^{\perp}\big] = Ad(g)\big[(\mathfrak{a} + \mathfrak{n}) \cap \mathfrak{p}\big]. \tag{3.6}$$

This simplification is valid because of the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form and that $B(Ad(g^{-1})S, T) = B(S, Ad(g)T)$ for all $S \in \mathfrak{g}$, $T \in \mathfrak{g}$, and $g \in G$.

Let $\bar{\mathfrak{n}}$ be the sum of the negative root spaces. Now, because $\mathfrak{a} \subset \mathfrak{p}$, and $\mathfrak{n} \cap \bar{\mathfrak{n}} = \{0\}$, and because the Cartan involution maps $\mathfrak{p}$ onto itself and $\mathfrak{n}$ onto $\bar{\mathfrak{n}}$ [Wa 7.3.5], $(\mathfrak{a} + \mathfrak{n}) \cap \mathfrak{p} = \mathfrak{a}$ so (3.6) becomes $N^*_{gK}\hat{\xi}_o = Ad(g)\mathfrak{a}$. This proves that $N^*\hat{\xi}_o$ is as given in lemma 3.3. Using this expression for $N^*\hat{\xi}_o$, lemma 3.2, and that the maps are all natural, one sees that $\bar{\rho}$ is just inclusion in the second factor. $\blacksquare$

The final step in the proof of theorem 3.1 is the confirmation that the Bolker Assumption holds.

Lemma 3.4. *If G has real rank one, then the map $\bar{\rho}$ in (3.3) is an injective immersion.*

Proof. Using lemma 3.3, we first prove $\bar{\rho}$ is injective. Let $g \in MN$ be such that $Ad(g)S \in \mathfrak{a}$ for some $S \in \mathfrak{a} \setminus 0$. As $\mathfrak{a}$ is one dimensional, g normalizes $\mathfrak{a}$. Let $g = mn$ for some $m \in M$, $n \in N$. Because m centralizes $\mathfrak{a}$ and g normalizes $\mathfrak{a}$, n normalizes $\mathfrak{a}$. We now prove $n = e$. Assume $n \neq e$. Because $\dim \mathfrak{a} = 1$, all restricted roots are 1 - 1 maps from $\mathfrak{a}$ to $\mathbb{R}$. Therefore the following map is 1 - 1 from N to $\mathfrak{n}$ [He 1962: Lemma 4.6, p. 231]; $\phi(n_1) = Ad(n_1)S - S$. As $\phi(e) = 0$, $\phi(n) = Ad(n)S - S \in \mathfrak{n} \setminus 0$. Because $\mathfrak{n} \cap \mathfrak{a} = \{0\}$, $Ad(n)S = S + \phi(n) \notin \mathfrak{a}$. Therefore, $n = e$, $g = m \in M$, and therefore $\bar{\rho}$ is injective.

To show $\bar{\rho}$ is an immersion at $(eM, S_o) \in N^*\hat{\xi}_o \setminus 0$, we construct an arbitrary path through (eM, S_o) and take its derivative. Let $g(t)$ be a path in MN with $g(0) = e$ and $\frac{dg}{dt}(0) = V_o \in \mathfrak{m} + \mathfrak{n}$, and let $S(t)$ be a path in $\mathfrak{a} \setminus 0$ with $S(0) = S_o$ and $\frac{dS}{dt}(0) = T_o \in \mathfrak{a}$. Then $v(t) = \big(g(t)K, Ad(g(t))S(t)\big)$ is a path in $N^*\xi_o \setminus 0$ with derivative $\frac{dv}{dt}(0) = (V_o, ad(V_o)S_o + T_o)$. Assume $d\bar{\rho}v(t) = ad(V_o)S_o + T_o$ is zero. Therefore, $ad(V_o)S_o = -T_o \in \mathfrak{a}$ and, since $\mathfrak{a}$ is one dimensional and $S_o \neq 0$, V_o normalizes $\mathfrak{a}$. Because G has a finite Weyl group [Wa, 7.5.6], the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$ is $\mathfrak{m}$. Because $\mathfrak{n}$ is the sum of the positive root spaces [He 1962], $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$. This shows that the normalizer of $\mathfrak{a}$ in $\mathfrak{g}$ is $\mathfrak{m} + \mathfrak{a}$ and so the normalizer of $\mathfrak{a}$ in $\mathfrak{m} + \mathfrak{n}$ is $\mathfrak{m}$. This shows $V_o \in \mathfrak{m}$ and so $\frac{dv}{dt}(0) = 0$ and so $\bar{\rho}$ is an immersion. $\blacksquare$

When the real rank of G is greater than one, the Bolker Assumption does not hold. [GQ] has a more restrictive theorem in this case under the other hypotheses of Theorem 3.1:

$$WF_A(f) \cap N^o = \emptyset \quad \text{where} \quad N^o = \{(gK, Ad(g)S) \mid g \in MN, \ S \in \mathfrak{a} \setminus 0 \text{ is regular }\}. \tag{3.7}$$

$S \in \mathfrak{a} \setminus 0$ is regular [He 1962] if $\alpha(S) \neq 0$ for all restricted roots α.[1]

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[1] If $\mathfrak{a}$ has dimension one, then all restricted roots are one-to-one maps and so all $S \in \mathfrak{a} \setminus 0$ are regular.

4. The support theorems.

Theorem 4.1 is the generalization of Helgason's support theorem to non-invariant measures.

Theorem 4.1. *Let G be a semisimple Lie group of real rank one. Let R_μ be a Radon transform on horocycles with a weight function μ that is real analytic and never zero (2.1-2). Let $f \in \mathcal{E}'(X)$ and let $\mathcal{A} \subset Y$ be an open connected subset. Assume $R_\mu f(\xi) = 0$ for all horocycles $\xi \in \mathcal{A}$ and assume some horocycle in $\mathcal{A}$ is disjoint from supp f. Then $\cup\{\hat{\xi} \mid \xi \in \mathcal{A}\}$ is disjoint from supp f.*

Theorem 4.1 implies a limited angle theorem, namely: *Let V be an open neighborhood of $k_o \in K$. If $R_\mu f(\xi) = 0$ for all horocycles $kaMN$ with $a \in A$ and $k \in V$, then $f \equiv 0$.* The proof uses the decomposition of $Y \cong K/M \times A$ [He 1963, Prop. 3.6] and that $\mathcal{A} = \{kaMN | k \in V, \ a \in A\}$ is open unbounded and connected in Y.

Proof. The arguments are as in [BQ 1987] for the hyperplane transform. The first key idea is theorem 3.1. Let ξ_1 be a horocycle in $\mathcal{A}$ that is disjoint from supp f and let ξ_2 be an arbitrary horocycle in $\mathcal{A}$. Assume supp f meets ξ_2. Let $\mathcal{P}$ be a path in $\mathcal{A}$ between ξ_1 and ξ_2 and let ξ_o be the first horocycle in the path that meets supp f. As $R_\mu f = 0$ for horocycles near ξ_o, the conclusion of Theorem 3.1 implies $WF_A f \cap N^* \hat{\xi}_o = \emptyset$. Furthermore, f is zero on one side of $\hat{\xi}_o$.

The second key idea is a theorem of Kawai, Kashiwara and Hörmander [Hö 1983 Theorem 8.5.6]:

Lemma 4.2. *Let $h \in \mathcal{E}'(X)$ and assume h is zero on one side of the horocycle $\hat{\xi}_o$. Let $x_o \in \hat{\xi}_o \cap$ supp f and let $(x_o, \omega_o) \in N^* \hat{\xi}_o \setminus 0$. Then $(x_o, \omega_o) \in WF_A f$.*

Hörmander's result is stated for $\mathbb{R}^n$, but it can be proved in X using real analytic local coordinates and the fact that in rank one spaces, horocycles are complete codimension one surfaces.

Applying lemma 4.2 to $x_o \in \hat{\xi}_o \cap$ supp f gives a contradiction that proves the theorem. ∎

A local support theorem similar Globevnik's theorem [Gl] for the X-ray transform in $\mathbb{R}^n$ also follows from theorem 3.1 and lemma 4.2.

Theorem 4.3. *Let G be a semisimple Lie group of real rank one. Let R_μ be a Radon transform on horocycles with a weight function μ that is real analytic and never zero (2.1-2). Let $B(r)$ be the open geodesic ball of radius r in X. Let $f \in \mathcal{E}'(X)$ and let $A_1 = B(1) \cap$ supp f. Assume A_1 is compact, and assume $R_\mu f(\xi) = 0$ for all horocycles $\hat{\xi}$ that meet $B(1)$ but not $B(1/2)$. Then supp $f \cap [B(1) \setminus Cl\ B(1/2)] = \emptyset$.*

The hypotheses of theorem 4.3 allow supp f to include points outside $B(1)$. The choice of $B(1)$ and $B(1/2)$ are somewhat arbitrary and local support theorems can be proved for other sets determined by the horocycles disjoint from them.

Proof. Let r be the smallest number such that $A_1 \subset Cl\ B(r)$, and assume the conclusion of theorem 4.3 is false. As A_1 is a compact subset of $B(1)$, $r \in (1/2, 1)$. Under these hypotheses, theorem 3.1 implies $WF_A f \cap N^* \partial B(r) = \emptyset$ because through each $x \in \partial B(r)$ there is a horocycle lying outside $B(1/2)$ in each normal direction to $\partial B(r)$ [He 1963]. Applying lemma 4.2 to $f_1 = f \chi_{A_1}$ gives the contradiction that proves the theorem.

If the real rank of G is greater than one, then (3.7) shows that some wavefront directions conormal to $\hat{\xi}_o$ are not detected by R_μ. This is analogous to the X-ray transform on line complexes in $\mathbb{R}^3$ [BQ 1991] and on symmetric spaces [GU] where some wavefront directions conormal to lines in Y are not detected by R_μ. More subtle geometric arguments are needed to prove support theorems in these cases [BQ 1991, GQ].

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