MORERA THEOREMS VIA MICROLOCAL ANALYSIS

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Abstract. We prove Morera theorems for curves in the plane using microlocal analysis. The key is that microlocal smoothness of functions is reflected by smoothness of their Morera integrals on curves–their Radon transforms. Parallel support theorems for the associated Radon transforms follow from our arguments by a simple correspondence.

1. Introduction

The classical Morera Theorem states that, if $\int_C f\,dz = 0$ for all closed curves in a region, then f is holomorphic in that region. More general Morera theorems specify subclasses of curves which can be used to determine holomorphy (see [Ag, BG1, BG2, Gl1, Gl2, Gl3, Gl4, Za1, Za3]). In the present paper we show how to use arguments from microlocal analysis to prove Morera theorems for circles passing through the origin (§2.1), for circles of arbitrary radius and arbitrary center (§2.2), and for translates of a fixed closed convex curve (§2.3).

The theorems we prove have somewhat different character from known Morera theorems. First, we assume that as a function of the curve, $\int_C f\,dz$ is constant; in known theorems one assumes that the integral is zero. The sets of curves and the domains are more general than in known theorems. In general, we assume that f is holomorphic on a small set and then infer that f is holomorphic on a larger set. This holomorphy assumption is not present in known Morera theorems, so we provide counterexamples to our conclusions when we drop holomorphy assumptions. Finally, our theorems are valid for distributions because the associated Radon transforms are defined for distributions and the proofs in §3 are valid for distributions.

In our proofs, we start with a distribution f that is holomorphic on a small set V and we use the microlocal analysis of an associated Radon transform to show f is holomorphic on a much larger set, Ω. The outline of our proofs is as follows. We assume that integrals of $f\,dz$ over a class of curves do not depend on the curve.

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We also assume that f is holomorphic on V. Using the microlocal properties of the Radon transform that integrates over these curves, we show f is real analytic in certain directions in the microlocal sense, that is, the analytic wave front set, $\mathrm{WF_A}(f)$, is sufficiently small. Since $\frac{\partial}{\partial \bar z}$ is a real analytic elliptic partial differential operator, $\mathrm{WF_A}(\frac{\partial f}{\partial \bar z}) = \mathrm{WF_A}(f)$. So, the distribution $\frac{\partial f}{\partial \bar z}$ is real analytic in these same directions. Finally, we use a theorem of Hörmander, Kawai, and Kashiwara to prove propagation of real analyticity of f: since $\frac{\partial f}{\partial \bar z} = 0$ on V and certain directions are not in $\mathrm{WF_A}(\frac{\partial f}{\partial \bar z})$, we show $\frac{\partial f}{\partial \bar z}$ is zero on the larger set Ω. So, f is holomorphic on Ω. These ideas can be used on other spaces such as Riemann surfaces or complex manifolds and preliminary results have recently been proven.

This outline is very close to microlocal proofs of support theorems for Radon transforms in, *e.g.,* [BQ, Gl5, Qu3]. This correspondence in proofs is seen in the Morera Theorems 2.1.2 and 2.2.1 and corresponding support Theorems 3.1.2 and 3.2.2. Loosely, whenever a support theorem for curves in $\mathbb{C}$ is proven using microlocal analysis as in [Qu3], a parallel Morera theorem can be proven in the same way and vice versa. Both theorems will apply to distributions and arbitrary nowhere-zero real analytic measures.

A similar correspondence using Green's theorem exists between some classical Morera theorems and support theorems for Pompeiu transforms for regions in the plane [Za3]. A related correspondence exists between mean-value integrals and differential equations (*e.g.,* harmonicity) [Za2]. There is even a classical correspondence between Morera theorems and support theorems for Radon transforms. This last correspondence requires enough 'degrees of freedom' in the curves so one can use Green's theorem and a differentiation to convert from an integral $\int_C f dz$ on the curve C to an integral $\int_C \frac{\partial f}{\partial \bar z} ds$. (We do provide a non-microlocal proof in §3.2 for such a case. We believe this result is new.) The classical correspondences do not allow one to prove our theorems using currently known support theorems.

2. The Morera theorems

The theorems in this section are stated for continuous functions, but they are valid for distributions.

2.1. Morera Theorems on circles passing through the origin.

Let $a \in \mathbb{C}$. We denote the circle centered at $a/2$ which passes through the origin by $\Gamma(a)$. The segment between the origin and a is a diameter of $\Gamma(a)$. The set of circles passing through the origin has dimension two and thus the associated Morera problem is not dimensionally overdetermined.

Theorem 2.1.1. *Let $\mathcal{A}$ be an open connected subset of $\mathbb{C}$ which contains the origin. Let $\Omega = \cup_{a\in\mathcal{A}}\Gamma(a)$ and let f be a continuous function on Ω. Assume that $\int_{z\in\Gamma(a)} f(z)\, dz = 0$ for $a \in \mathcal{A}$. If for each k there is a neighborhood $U_k \subset \mathbb{C}$ of the origin such that $f|U_k$ is of class C^k, then f is holomorphic in Ω.*

Theorem 2.1.1 does not hold if one replaces the smoothness assumption by a weaker smoothness assumption that for *some* k there is a neighborhood $U_k \subset \mathbb{C}$ of the origin such that $f|U_k$ is of class C^k [Gl1]. Note also that although $\Gamma(0)$ is only a point, the Morera integral is defined and zero at $a = 0$.

Under the hypotheses of Theorem 2.1.1, f is holomorphic in a small disk centered at zero [Gl1, Theorem 1]. Now Theorem 2.1.1 follows from our next theorem.

Theorem 2.1.2. *Let $\mathcal{A}$ be an open connected subset of $\mathbb{C}$. Let $\Omega = \cup_{a\in\mathcal{A}}\Gamma(a)$ and let f be a continuous function on Ω. Assume that $a \mapsto \int_{z\in\Gamma(a)} f(z)\,dz$ is constant on $\mathcal{A}$ and suppose that f is holomorphic in a neighborhood of $\Gamma(a_0)$ for some $a_0 \in \mathcal{A}$. Then f is holomorphic on Ω.*

Note that the assumption that f is holomorphic in a neighborhood of $\Gamma(a_0)$ implies that f is defined and holomorphic in a neighborhood of the origin.

The conclusion of Theorem 2.1.2 is false in general if one drops the assumption about f being holomorphic. To illustrate this, let $f(z) = z^3/\overline{z} \quad (z \in \mathbb{C}\setminus\{0\})$, $f(0) = 0$. Then it is easy to see that f is continuous and that $\int_\Gamma f\,dz = 0$ for every circle Γ through the origin yet f is nowhere holomorphic. Similar examples also show that one cannot drop the smoothness assumption in Theorem 2.1.1.

2.2. Morera Theorems on arbitrary circles.

Now, we consider arbitrary circles, not just those containing the origin. The associated Morera problem is dimensionally overdetermined. We let $C(y,r)$ denote the circle centered at $y \in \mathbb{C}$ and of radius $r \in (0,\infty)$.

Theorem 2.2.1. *Let $\mathcal{A}$ be an open connected subset of $\mathbb{C}\times(0,\infty)$. Let $\Omega = \cup_{(y,r)\in\mathcal{A}}C(y,r)$ and let f be a continuous function on Ω. Assume that $(y,r) \mapsto \int_{C(y,r)} f\,dz$ is constant on $\mathcal{A}$ and let f be holomorphic in a neighborhood of $C(y_0,r_0)$ for some $(y_0,r_0)\in\mathcal{A}$. Then f is holomorphic on Ω.*

In some cases local two-circle theorems can be applied under weaker hypotheses. Let D be a disk and assume integrals of $f\,dz$ are zero on all circles of two well chosen radii contained in D. Then f is holomorphic on D [BG1, BG2]. The local two-circle theorems are stronger than our theorem in the sense that one assumes vanishing of integrals for circles of two well chosen radii, rather than our three-dimensional set of circles. However, our theorem does not follow from their results, since our sets, $\mathcal{A}$ and Ω, are more general than those permitted in two-circle theorems.

Our assumption that f is holomorphic somewhere is not in the classical theorems. For example, Carleman [SZ, p. 179, Ri pp. 315, 333] proved that f is holomorphic in a region, B, as long as $\int_{C(y,r)} f\,dz = 0$ for (y,r) in a neighborhood, $\mathcal{A}$, of $B\times\{0\}$. The proof consists of an easy convolution argument and an application of Green's theorem. However, the function $f(z) = z^3/\overline{z}$ has zero integrals over all circles surrounding the origin, but f is not holomorphic (see [Gl2, Theorem 2]). This shows that some holomorphy assumption is necessary for our theorems.

2.3. Morera Theorems on translates of curves.

Let γ be a regular, simple, closed real analytic curve in the plane. Such a curve divides the plane into two regions; we say γ is convex if the bounded region inside γ is convex. We say that γ is *flat to order one* at a point $w = (w_1, w_2) \in \gamma$ if the tangent line to γ at w does not have higher than first order contact with γ at w. If the curve is given locally by $x_2 = f(x_1)$ then the condition is, of course, equivalent to $f''(w_1) \neq 0$.

Theorem 2.3.1. *Let γ be a regular, simple, closed, convex curve parameterized in polar coordinates by $r = r(\theta)$ where $r : [0,2\pi] \to (0,\infty)$ is periodic and real analytic. Assume γ is flat to order one at all points on γ. Let $\mathcal{A}$ be an open connected subset of $\mathbb{C}$. Let D be the convex hull of γ and let Ω be the union of all translates $y+D$ for $y\in\mathcal{A}$. Assume f is continuous on Ω, and assume that $y \mapsto \int_{y+C} f\,dz$ is constant* *for $y \in \mathcal{A}$. Assume that, for some $y_0 \in \mathcal{A}$, f is holomorphic in a neighborhood of the set $y_0 + D$. Then f is holomorphic on Ω.*

The convexity hypothesis in Theorem 2.3.1 is given to make the picture clear; any smooth closed curve that is flat to order one at all points is also strictly convex by a simple Mean Value Theorem argument.

Strong holomorphy assumptions are necessary here. The case where γ is a circle has been studied, and negative results are known if the assumptions are weakened. The example in [Qu2] shows that the conclusion of Theorem 2.3.1 is false for circles if one weakens the hypotheses about holomorphy of f to become: *for some $y_0 \in \mathcal{A}$, f is holomorphic in the neighborhood of the curve $y_0+\gamma$ (rather than in a neighborhood of $y_0 + D$).* That example is given for integration in arc length measure, but it can be easily adapted to the measure dz, since the function in that example is radial (see also [Jo, p. 115]).

Counterexamples to the conclusion of this theorem exist when γ is not convex. One counterexample is as follows. Let the curve γ defined in polar coordinates by $r = \sqrt{2} - \cos\theta$. This curve looks a little like the boundary of a very fat letter 'C.' Let f be the function that is $+1$ on $[1/2, 1] \times [0, 1]$ and -1 on $[1/2, 1] \times [-1, 0]$. We let $\mathcal{A}$ be a small neighborhood of $(0, 0)$. Whenever a translate of C (with center in $\mathcal{A}$) meets the top half of supp f, it meets the bottom half and the integrals cancel. One can make f smooth as long as the smoothed function is constant on small vertical segments near $(1/2, \pm 1/2)$, the two 'points' of the 'C.' Then, the integral on the top part will cancel the integral on the bottom part, as long as the center is sufficiently close to $(0, 0)$.

3. Proofs

3.0. Preliminaries.

The theorems in §2 will now be proven for distributions. This simplifies some details in the proofs; for example, $\frac{\partial f}{\partial \bar{z}}$ does not have to be a smooth function. In each case, we will first explain why the associated Radon transform is a real analytic Fourier integral operator. Then we use the microlocal analysis of this operator to prove the theorem.

Definition 3.0.1. Let $f \in \mathcal{D}'(\mathbb{R}^2)$ and let C be a smooth curve. Let $x \in C \cap \operatorname{supp} f$. We say that *supp f is on one side of C at x* if there is an open neighborhood U of x such $U \setminus C$ has two connected components on one of which f vanishes identically.

The analytic wave front set, $\mathrm{WF_A}(f)$, of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is defined in [Tr] and [Hö2]. Loosely speaking, if $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ then $(x, \xi) \in \mathrm{WF_A}(f)$ if the Fourier transform (localized about x) of f does not decrease rapidly enough in directions near ξ. If $C \subset \mathbb{R}^2$ is a regular, simple curve, then we let N^*C denote the conormal bundle of C *i.e.,* the set of all $(x, \xi) \in T^*\mathbb{R}^2$ such that $x \in C$ and the linear functional ξ is zero on the tangent space T_xC. Then, $N^*C \setminus 0$ will denote this conormal bundle with zero section removed. We will need the following theorem of Hörmander, Kawai, and Kashiwara [Hö2, Theorem 8.5.6].

Lemma 3.0.2. *Let $h \in \mathcal{D}'(\mathbb{R}^2)$ and let C be a smooth curve. Let $x \in \operatorname{supp} h \cap C$ and assume that* supp h *is on one side of C at x. If $(x, \xi) \in N^*C \setminus 0$, then $(x, \xi) \in \mathrm{WF_A}(h)$.*

This Lemma is a refinement of the obvious statement that, if $x \in \operatorname{supp} h \cap C$ and supp h is on one side of C at x, then h is not real analytic near x.

3.1. The Radon transform on circles passing through the origin.
Let $a \in \mathbb{C}$ and define the Radon transform $S_o f(a) = \int_{z\in\Gamma(a)} f(z)\,dz$. To prove the Morera theorem, we first need a microlocal regularity theorem for S_o.

Proposition 3.1.1. *Let $a_0 \in \mathbb{C}\setminus\{0\}$ and let $W \subset \mathbb{C}$ be an open set containing the circle $\Gamma(a_0)$. Let $f \in \mathcal{D}'(W)$ and assume $S_o f(a_0)$ is real analytic in a neighborhood of a_0. Furthermore, assume f is real analytic in a neighborhood of zero. Then* $\mathrm{WF_A}(f) \cap N^*(\Gamma(a_0)) = \emptyset$.

Proof. We first show that S_o is equivalent to a dual Radon transform, R^*, for which the Bolker Assumption [GS, pp. 364-365], [Qu1, equation (9)] is known to hold. This assumption will be described below; the microlocal analysis of transforms satisfying the Bolker Assumption is especially easy. The proof becomes more complicated because of the degeneracies near $x = 0$ or $a = 0$ (for example $\Gamma(0) = \{0\}$ is a point, not a curve).

Let X, Y, and Z be manifolds. If $A \subset T^*X \times T^*Y$ and $B \subset T^*Y \times T^*Z$ are manifolds, then we define

$$\begin{aligned} A' &= \{(x,y;\xi,-\eta) \bigm| (x,y;\xi,\eta) \in A\}, \\ A^t &= \{(y,x;\eta,\xi) \bigm| (x,y;\xi,\eta) \in A\} \end{aligned} \tag{3.1a}$$

if $A \subset T^*X \times T^*Y$ and $B \subset T^*Y$ then

$$A \circ B = \{(x,\xi) \in T^*X \bigm| \exists (y,\eta) \in B \text{ such that } (x,y;\xi,\eta) \in A\} \tag{3.1b}$$

Let $\Lambda \subset T^*X \setminus 0 \times T^*Y \setminus 0$ be a Lagrangian manifold and let S be a Fourier integral operator (FIO) associated to Λ. If the projection from Λ to T^*Y is an injective immersion, then we say S (or Λ) satisfies the *Bolker Assumption* [GS, pp. 364-365], [Qu1, equation (9)]. If $\dim X = \dim Y$, then Λ is a local canonical graph [Hö1, Definition 4.1.5, Qu1]. The microlocal analysis of FIO associated to local canonical graphs is especially easy. In particular, if S is real analytic elliptic and f is a distribution, then

$$\mathrm{WF_A} f = \Lambda' \circ \mathrm{WF_A} Sf. \tag{3.2}$$

The inclusion "$\supset$" follows immediately from the real analytic equivalent of ([Tr, Theorem II 8.5.4]), and the other one from ellipticity. (Essentially, one constructs a parametrix T, a FIO associated to Λ^t, and shows that $T \circ S$ is an analytic elliptic pseudodifferential operator. This is similar to the argument on the bottom of p. 337 below (14) in [Qu1]. See also [Hö1], Theorem 4.2.2 and discussion at the bottom of p. 180 for how to compose FIO.)

Let $\Xi = ([0,2\pi] \times \mathbb{R})/[(\theta,p) \sim (\theta+\pi,-p)]$; Ξ represents the set of lines in the plane. Let $P : \Xi \to \mathbb{R}^2$ be defined by $P(\theta,p) = (p\cos\theta, p\sin\theta)$. Then, as in [CQ, Lemma on p. 578],

$$\begin{aligned} &S_o f(a) = \\ &1/2 \int_{\theta=0}^{2\pi} f \circ P(\theta, x\cdot\overline{\theta})\big[(a\cdot\theta^{\perp})(\cos\theta + i\sin\theta) + (a\cdot\overline{\theta})(-\sin\theta + i\cos\theta)\big]d\theta \\ &\qquad = R^* f \circ P(a) \end{aligned} \tag{3.3}$$

where $\overline{\theta} = (\cos\theta, \sin\theta)$ and $\theta^\perp = (-\sin\theta, \cos\theta)$ and the inner product is the *real* inner product on $\mathbb{R}^2$. The last term in (3.3) is the dual transform R^* to a generalized Radon transform on lines in the plane [Qu1, §3]. The incidence relation [He] for R^* is the set $Z = \{([\theta,p],x) \in \Xi \times \mathbb{R}^2 \bigm| x\cdot\overline{\theta} = p\}$.

R^* is a Fourier integral operator (FIO) associated with the Lagrangian manifold $\Lambda = N^*Z \setminus 0$ [GS, Q1, §3]. The relation $x\cdot\overline{\theta} - p = 0$ defines Z and its differential gives a basis of the fibers of Λ. Therefore, [Q1, bottom of p. 339]

$$\Lambda = \{\big([\theta,p],x;\alpha(x\cdot\theta^\perp)\mathbf{d}\boldsymbol{\theta}, -\alpha\mathbf{dp}, \alpha\overline{\theta}\cdot\mathbf{dx}\big) \bigm| ([\theta,p],x)\in Z, \alpha\in\mathbb{R}\setminus 0\}. \tag{3.4}$$

Here, $x = (x_1, x_2)$ and $\overline{\theta}\cdot\mathbf{dx} = \cos\theta\mathbf{dx}_1 + \sin\theta\mathbf{dx}_2$.

Since the weight in brackets in (3.3) is nowhere zero, the symbol of R^* is nowhere zero, R^* is elliptic [GS, Q1 Theorem 2.1] (see [Tr] for the definition of ellipticity). Since the weight and Lagrangian manifold, Λ, are real analytic, R^* is a real analytic elliptic Fourier integral operator associated with Λ [Ka, SKK].

Let π_2 be the projection from Λ to the second factor, $T^*\mathbb{R}^2$; $\pi_2\big([\theta,p],x;\alpha(x\cdot\theta^\perp)\mathbf{d}\boldsymbol{\theta}, -\alpha\mathbf{dp}, \alpha\overline{\theta}\cdot\mathbf{dx}\big) = (x, \alpha\overline{\theta}\cdot\mathbf{dx})$. It is easy to show, using (3.4) and the definition of Ξ, that π_2 is an injective immersion. Therefore, the Bolker Assumption holds. So, by (3.2), if $\tilde{f}$ is a distribution on Ξ, then

$$\mathrm{WF_A}(\tilde{f}) = \Lambda' \circ \mathrm{WF_A}(R^*\tilde{f}). \tag{3.5}$$

Now, we go back to (3.3) and apply (3.5) to the composition $\tilde{f} = f\circ P$ where f is a function on $\mathbb{R}^2$. As P is a diffeomorphism away from $p = 0$ (corresponding to $x = 0$), and f is a real analytic function near $x = 0$, $\tilde{f}$ is a distribution on Ξ (essentially the same arguments work if f is a distribution). By (3.3) and the fact $S_o f$ is real analytic near a_0, $\mathrm{WF_A}(R^*\tilde{f})$ does not meet $T^*_{a_0}\mathbb{R}^2$. Now, by (3.5),

$$\mathrm{WF_A}(\tilde{f}) \cap \big(\Lambda' \circ T^*_{a_0}\mathbb{R}^2\big) = \emptyset \tag{3.6}$$

By explicitly calculating this second set in (3.6) one sees

$$\mathrm{WF_A}(\tilde{f}) \cap N^*\tilde{\Gamma}(a_0) = \emptyset \text{ where } \tilde{\Gamma}(a_0) = \{[\theta,p]\in\Xi \bigm| a_0\cdot\overline{\theta} = p\} \tag{3.7}$$

Because P is a diffeomorphism away from $x = 0$, the distribution $\tilde{f}$ differs from the push forward distribution $(P^{-1})_* f$ by a nonzero real analytic (Jacobian) factor away from $p = 0$. Therefore, $\mathrm{WF_A}(f\circ P) = (P^{-1})_+\mathrm{WF_A}(f)$, at least away from $p = 0$ [Tr, I Ch. 5, Theorem 3.5 and p. 53 for the definition of $(P^{-1})_+$]. Since f is real analytic near $x = 0$, both of these wave front sets are empty near $x = 0$ ($p = 0$). Using this fact, (3.7), and the fact that $N^*\tilde{\Gamma}(a_0) = (P^{-1})_+\big(N^*\Gamma(a_0)\big)$, we see that $(P^{-1})_+(\mathrm{WF_A}(f)\cap N^*\Gamma(a_0)) = \emptyset$. This last equation implies that $\mathrm{WF_A}(f)\cap N^*\Gamma(a_0) = \emptyset$. □

Proof of Theorem 2.1.2. Let f be a distribution. Without loss of generality, we can assume $0 \notin \mathcal{A}$ (*if* $0\in\mathcal{A}$, *then* $\mathcal{A}\setminus\{0\}$ *is connected and open; if* $a_0 = 0$, *then* $\exists a_0' \in \mathcal{A}\setminus\{0\}$ *such that* f *is holomorphic in a neighborhood of* $\Gamma(a_0')$). Since $\mathrm{WF_A}(f) = \mathrm{WF_A}(\frac{\partial f}{\partial\overline{z}})$, Proposition 3.1.1 gives

$$\mathrm{WF_A}(\partial f/\partial\overline{z}) \cap N^*(\Gamma(a)) = \emptyset\ \forall a\in\mathcal{A}. \tag{3.8}$$

Assume f is not holomorphic on Ω. Then, there is an $a_2 \in \mathcal{A}$ such that $\Gamma(a_2)$ meets supp $\frac{\partial f}{\partial \bar{z}}$. Since $\mathcal{A}$ is connected and $a_0 \in \mathcal{A}$ there is a path in $\mathcal{A}$ from a_0 to a_2. Therefore, there is an $a_1 \in \mathcal{A}$ such that supp $\frac{\partial f}{\partial \bar{z}}$ is on one side of $\Gamma(a_1)$ at some $x \in \Gamma(a_1)$ $\big(a_1$ can be chosen as the first point in this path whose circle, $\Gamma(a_1)$, meets supp $\frac{\partial f}{\partial \bar{z}}\big)$. If $x \in$ supp $\frac{\partial f}{\partial \bar{z}} \cap \Gamma(a_1)$, and $(x,\eta) \in N^*\Gamma(a_1) \setminus 0$, then Lemma 3.0.2 implies that $(x,\eta) \in \mathrm{WF_A}(\frac{\partial f}{\partial \bar{z}})$. However, (3.8) implies that $(x,\eta) \notin \mathrm{WF_A}(\frac{\partial f}{\partial \bar{z}})$. This contradiction shows that $x \notin$ supp $\frac{\partial f}{\partial \bar{z}}$ and completes the proof of the theorem. □

The same type of reasoning gives a new support theorem. Let

$$S_o^\mu f(a) = \int_{z\in\Gamma(a)} f(z)\mu(z,a)ds \tag{3.9}$$

where ds is the arc length measure on the circle and the weight μ is continuous.

Theorem 3.1.2. *Let $\mathcal{A}$ be an open connected subset of $\mathbb{C}$. Let $\Omega = \cup_{a\in\mathcal{A}}\Gamma(a)$ and let f be a continuous function on Ω. Assume the weight, μ, in (3.9) is nowhere zero and real analytic. Assume that $S_o^\mu f(a) = 0$ for all $a \in \mathcal{A}$. If f is zero in a neighborhood of $\Gamma(a_0)$ for some $a_0 \in \mathcal{A}$, then $f = 0$ on Ω.*

The analogous theorem is true for the transform on spheres through the origin in $\mathbb{R}^n$ and the proof is essentially the same. Furthermore, a support theorem corresponding to Morera Theorem 2.1.1 is true for S_o^1 (*i.e.* with weight $\mu = 1$). Let $\mathcal{A}$ be a connected open set containing zero and let $\Omega = \cup_{a\in\mathcal{A}}\Gamma(a)$. Let $f \in C(\Omega)$ be C^∞ in a neighborhood of zero. The main result in [CQ] shows that, under these conditions, f is zero in a neighborhood of 0. Now, Theorem 3.1.2 shows that f is zero on Ω.

Proof of Theorem 3.1.2. Proposition 3.1.1 holds for any Radon transform on circles $\Gamma(a)$ that has nowhere zero real analytic weight because the proof is valid for such weights (S_o^μ defines a dual Radon transform in the same way as (3.3)). So, since $S_o^\mu f(a)$ is real analytic (it is zero) for $a \in \mathcal{A}$, $\mathrm{WF_A}(f) \cap N^*(\Gamma(a)) = \emptyset\ \forall a \in \mathcal{A}$. We assume f is not identically zero on Ω. Now, we use the argument in the last paragraph of the proof of Theorem 2.1.2 applied to f instead of $\frac{\partial f}{\partial \bar{z}}$ to prove that f is zero in Ω. □

3.2. Radon transforms on arbitrary circles.

Recall that $C(y,r)$ is the circle centered at $y \in \mathbb{C}$ of radius $r > 0$. For $f \in C(\mathbb{C})$,

$$Sf(y,r) = \int_{z\in C(y,r)} f(z)\, dz \tag{3.10}$$

is the Radon transform integrating $f\, dz$ over $C(y,r)$.

Proposition 3.2.1. *Let $(y,r) \in \mathbb{C} \times (0,\infty)$ and let W be an open set containing the circle $C(y,r)$. Let $f \in \mathcal{D}'(W)$ and assume Sf is real analytic in a neighborhood of (y,r). Then $\mathrm{WF_A}(f) \cap N^*(C(y,r)) = \emptyset$.*

Proof. We prove that the Radon transform S is a Fourier integral operator that satisfies the Bolker Assumption. To do this, we will calculate the Lagrangian manifold, Λ associated to this operator. Then, the conclusion of Proposition 3.2.1 follows from the theory of Fourier integral operators.

The incidence relation for this Radon transform [He] is the set $Z = \{(x, y, r) \in \mathbb{R}^2 \times \big(\mathbb{R}^2 \times (0,\infty)\big) \;\big|\; (x-y)\cdot(x-y) - r^2 = 0\}$. Here again $\cdot$ is the *real* inner product on $\mathbb{R}^2$. Let Λ be the conormal bundle of Z in $T^*\big(\mathbb{R}^2 \times (\mathbb{R}^2 \times (0,\infty))\big)$ with the zero section removed. Let π_2 be the projection on the second factor, $\pi_2 : \Lambda \to T^*(\mathbb{R}^2 \times (0,\infty))$. We have to show that π_2 is an injective immersion and for this we must explicitly calculate Λ. As Z is defined by the equation,

$$(x-y)\cdot(x-y) - r^2 = 0, \tag{3.11}$$

the differential of (3.11), $2(x-y)(\mathbf{dx} - \mathbf{dy}) - 2r\mathbf{dr}$, gives a basis for the fibers of Λ. Here, $x\mathbf{dx} = x_1\mathbf{dx}_1 + x_2\mathbf{dx}_2$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Therefore,

$$\begin{aligned}\Lambda = \{&(x, y, r; \alpha(x-y)\mathbf{dx}, \alpha(y-x)\mathbf{dy}, -\alpha r\mathbf{dr}) \\ &\big|\; (x, y, r) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (0,\infty), \alpha \neq 0, (x-y)\cdot(x-y) - r^2 = 0\}.\end{aligned} \tag{3.12}$$

It is straight forward to show that π_2, the projection onto the second factor, $T^*(\mathbb{R}^2 \times (0,\infty))$, satisfies the Bolker Assumption. So, just as in the proof of Proposition 3.1.1, S is a real analytic elliptic FIO associated to Λ. Therefore, if Sf is real analytic near (y_0, r_0), then $\mathrm{WF}_\mathrm{A}(f) \cap \left[\Lambda' \circ \left(T^*_{(y_0,r_0)}\big(\mathbb{R}^2 \times (0,\infty)\big)\right)\right] = \emptyset$. Since the expression in brackets above is equal to $N^*C(y, r) \setminus 0$, this is exactly the conclusion of Proposition 3.2.1. $\square$

Note that this transform does not satisfy the Bolker assumption for $r = 0$ because the cotangent coordinate in (3.12) is zero when $r = 0$.

Microlocal proof of Theorem 2.2.1. Since $Sf(y, r)$ is real analytic for $(y, r) \in \mathcal{A}$, we use Proposition 3.2.1 to conclude $\mathrm{WF}_\mathrm{A}(f) \cap N^*C(y, r) = \emptyset \;\forall (y, r) \in \mathcal{A}$. Since $\frac{\partial}{\partial \bar z}$ is a real analytic elliptic differential operator, $\mathrm{WF}_\mathrm{A}(f) = \mathrm{WF}_\mathrm{A}(\frac{\partial f}{\partial \bar z})$, so

$$\mathrm{WF}_\mathrm{A}(\partial f/\partial \bar z) \cap N^*C(y, r) = \emptyset \quad \forall (y, r) \in \mathcal{A}. \tag{3.13}$$

Since f is holomorphic in a neighborhood, V, of $C(y_0, r_0)$, $\frac{\partial f}{\partial \bar z} = 0$ in V.

Assume f is not holomorphic on Ω. Then, there is a $(y_2, r_2) \in \mathcal{A}$ such that $C(y_2, r_2)$ meets supp $\frac{\partial f}{\partial \bar z}$. Since $\mathcal{A}$ is connected and $(y_0, r_0) \in \mathcal{A}$ there is a path in $\mathcal{A}$ between (y_0, r_0) and (y_2, r_2). Therefore, there is a $(y_1, r_1) \in \mathcal{A}$ such that $\frac{\partial f}{\partial \bar z}$ is on one side of $C(y_1, r_1)$ at some point on $C(y_1, r_1)$ $\big((y_1, r_1)$ can be chosen as the first point in this path whose circle, $C(y_1, r_1)$, meets supp $\frac{\partial f}{\partial \bar z}\big)$. If $x \in$ supp $\frac{\partial f}{\partial \bar z} \cap C(y_1, r_1)$, and $(x, \eta) \in N^*C(y_1, r_1) \setminus 0$, then Lemma 3.0.2 implies that $(x, \eta) \in \mathrm{WF}_\mathrm{A}(\frac{\partial f}{\partial \bar z})$. However, (3.13) implies that $(x, \eta) \notin \mathrm{WF}_\mathrm{A}(\frac{\partial f}{\partial \bar z})$. This contradiction shows that $x \notin$ supp $\frac{\partial f}{\partial \bar z}$, and it proves the theorem. $\square$

Proof outline of Theorem 2.2.1 without microlocal analysis. Because $C(y, r)$ is parameterized by the independent variables, y and r, one can use Green's Theorem to reduce this Morera theorem to a support theorem for the classical Radon transform on circles. Then, arguments similar to those in [John p. 115] can be used to prove this support theorem. No such trick works for circles through the origin and convex curves because there are not enough degrees of freedom. This second proof is included for completeness.

One can prove the existence of $(y_1, r_1) \in \mathcal{A}$ and positive constants ϵ_1 and ϵ_2 such that

(3.14a) f is not identically zero on $C(y_1, r_1)$,
(3.14b) $U = \{(z, s) \mid |z - y_1| < \epsilon_1, |s - r_1| < \epsilon_2\} \subset \mathcal{A}$,
(3.14c) There is an $R \in (r_1 - \epsilon_2, r_1 + \epsilon_2)$ such that $C(y, R)$ does not meet supp $\frac{\partial f}{\partial \bar{z}}$ whenever $|y - y_1| < \epsilon_1$.

The justification is similar to the geometric arguments of the microlocal proof above that come up with $C(y_1, r_1)$ but now we use thin annuli made up of circles in $\mathcal{A}$ and letting y_1 be the center of the first one that just touches supp$\frac{\partial f}{\partial \bar{z}}$.

A convolution argument allows us to assume f is a smooth function.

We now reduce to an integral of the function $\frac{\partial f}{\partial \bar{z}}$ over $C(y, r)$ in measure ds. To do this, use Green's Theorem to show for $(y, r) \in U$,

$$\int_{A(y,r,R)} \frac{\partial f}{\partial \bar{z}} dA = \pm(Sf(y, r) - Sf(y, R)) = 0 \tag{3.15}$$

where $A(y, r, R)$ is the annulus between the two circles $C(y, r)$ and $C(y, R)$. The sign, $\pm$, in (3.15) depends on whether r is larger or smaller than R. Now, differentiate the area integral with respect to r to show that the arc length integral

$$0 = \int_{C(y,r)} \frac{\partial f}{\partial \bar{z}} ds \quad \forall (y, r) \in U. \tag{3.16}$$

We now use a perturbation argument parallel to that in equation (6.18) on p. 115 of [Jo]. (The only changes are that the integrals here are in $\mathbb{R}^2$ and that the last integral in (6.18) [Jo] must be taken in our proof over the annulus $A(y, r, R)$. This argument uses the Divergence Theorem (for the area integral at the end of (6.18)) and the assumption 3.14c.) The perturbation argument shows that the integral of $\frac{\partial f}{\partial \bar{z}}$ times any polynomial on $C(y, r)$ is zero for all $(y, r) \in U$. This shows $\frac{\partial f}{\partial \bar{z}} = 0$ on all $C(y, r)$ for $(y, r) \in U$. This contradicts 3.14a and finishes the proof. $\square$

Using the correspondence between microlocal proofs of Morera theorems and support theorems, we get the following new support theorem. Define

$$S^\mu f(y, r) = \int_{z \in C(y,r)} f(z)\mu(z, y, r) ds, \tag{3.17}$$

where $\mu(z, y, r)$ is a continuous weight. Of course, the non-microlocal proof gives a support theorem only for the classical measure, $\mu = 1$.

Theorem 3.2.2. *Let $\mathcal{A}$ be an open connected subset of $\mathbb{C} \times (0, \infty)$. Let $\Omega = \cup_{(y,r)\in\mathcal{A}} C(y, r)$, and let f be a continuous function on Ω. Assume the weight, μ, in (3.17) is nowhere zero and real analytic. Assume $S^\mu f(y, r) = 0$ for all $(y, r) \in \mathcal{A}$. If f is zero in a neighborhood of $C(y_0, r_0)$ for some $(y_0, r_0) \in \mathcal{A}$, then $f(z) = 0$ for all $z \in \Omega$.*

Proof of Theorem 3.2.2. Proposition 3.2.1 holds for any Radon transform on circles $C(y, r)$ that has nowhere zero real analytic weight. So, since $S^\mu f(y, r)$ is real analytic on $\mathcal{A}$ (it is zero) $\mathrm{WF_A}(f) \cap N^*(C(y, r)) = \emptyset \ \forall (y, r) \in \mathcal{A}$. We assume f is not identically zero on Ω. Now, we use the argument in the last paragraph of the microlocal proof of Theorem 2.2.1 applied to f instead of $\frac{\partial f}{\partial \bar{z}}$ to prove that f is zero in Ω. $\square$

3.3. The Radon transform on translates of a curve.

The equivalent support theorem to Theorem 2.3.1 is the following:

Theorem 3.3.1 [Qu3, Theorem 1.2]**.** *Let γ be a smooth closed convex curve parameterized in polar coordinates by $r = r(\theta)$ where $r : [0, 2\pi] \to (0, \infty)$ is real analytic. Assume γ is flat to order one at all points on γ. Let $\mathcal{A} \subset \mathbb{R}^2$ be open and connected. Let R_μ be the Radon transform on translates of γ with nowhere zero real analytic weight μ. Let D be the convex hull of γ and let $\Omega = \cup_{y\in\mathcal{A}}(y + D)$. Assume $f \in \mathcal{D}'(\Omega)$ and assume $R_\mu f(y) = 0$ for all $y \in \mathcal{A}$. If, for some $y_0 \in \mathcal{A}$, the set $y_0 + D$ is disjoint from* supp f*, then f is zero on Ω.*

In the last two sections, we have seen that each Morera theorem (*e.g.,* 2.2.1) corresponds exactly to a support theorem (*e.g.,* 3.2.2), and the microlocal proofs correspond, too. For example, in the Morera theorem proofs, we show certain directions are not in $\mathrm{WF}_{\mathrm{A}}(\frac{\partial f}{\partial \bar{z}})$ and then use Lemma 3.0.2 to show $\frac{\partial f}{\partial \bar{z}}$ is zero; in the support theorem proofs, we show certain directions are not in $\mathrm{WF}_{\mathrm{A}}(f)$ and then use Lemma 3.0.2 to show f is zero. The microlocal arguments for the Support Theorem 3.3.1 are given in detail in [Qu3], so the proof of the corresponding Morera theorem will be left to the reader.

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