Injectivity sets for the Radon transform over circles and complete systems of radial functions

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Abstract. A necessary and sufficient characterization is given that specifies which sets of sums of translations of radial functions are dense in the set of continuous functions in the plane. This problem is shown to be equivalent to inversion for the Radon transform on circles centered on restricted subsets of the plane. The proofs rest on the geometry of zero sets for harmonic polynomials and the microlocal analysis of this circular Radon transform. A characterization of nodal sets for the heat and wave equation in the plane are consequences of our theorems, and questions of Pinkus and Ehrenpreis are answered.

§1. Formulation of the problem and the main results.

In this article, we characterize all systems of translations of radial functions that are complete in the space of continuous functions in $\mathbb{R}^2$. This is done by solving a dual problem, proving injectivity (on the domain of compactly supported functions) for a Radon transform integrating over restricted sets of circles. This injectivity problem is solved by first understanding the zero sets of harmonic polynomials and then using microlocal analysis.

Our result answers a question in approximation theory [LP]. The same question in dual form is posed in the book of Ehrenpreis [E], so we also answer this question in the plane (see §9.2).

The problem of inverting the spherical Radon transform on restricted sets of spheres goes back to Courant and Hilbert [CH], John [J], and Delsarte [DL]. John, Delsarte, Zalcman, Berenstein and Zalcman [Z1, Z2, BZ1, BZ2] and others [A, ABCP, BG, F, Q2] consider the case of spheres with arbitrary center but radius restricted to a small set (see [Z2] for a lovely introduction).

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In this article, we consider the Radon transform on circles in the plane with arbitrary radius but with center restricted to lie on a set S. From the point of view of integral geometry this case is natural since centers play role of directions in the classical Radon transform. We completely characterize the sets S in the plane such that the circular transform with centers restricted to S is noninjective on compactly supported functions (Theorem B in §1.2). Courant and Hilbert prove that if S is a line, then the kernel of this transform consists of all the odd functions about that line. The line is the building block for our sets of noninjectivity: Coxeter systems of lines (1.1).

1.1 Systems of radial functions. We shall use the following notation, most of which is standard:

$C(\mathbb{R}^n)$ - the space of all continuous real-valued functions endowed with the topology of the uniform convergence on compact sets,

$C_c(\mathbb{R}^n)$ - subspace of all compactly supported functions in $C(\mathbb{R}^n)$,

$C^{\#}(\mathbb{R}^n)$ - subspace of radial functions in $C(\mathbb{R}^n)$, *i.e.,* all functions which depend only on the distance to the origin,

$\mathrm{M}(n)$ - the groups of rigid motions of $\mathbb{R}^n$.

Let S be a set in $\mathbb{R}^n$. Denote by $\mathcal{L}(S)$ the linear subspace in $C(\mathbb{R}^n)$:

$$\mathcal{L}(S) = \text{span}\ \{f_a \mid a \in S, \quad f \in C^{\#}(\mathbb{R}^n)\},$$

where f_a is the shifted function, $f_a(x) = f(x-a)$. So, $\mathcal{L}(S)$ consists of sums of continuous functions, each of which is a function of the distance to a certain fixed point in S.

Our first goal is solution of:

Problem 1. Describe all sets S for which the subspace $\mathcal{L}(S)$ is dense in $C(\mathbb{R}^n)$.

The problem is to characterize the systems of shifted radial functions that are enough to approximate any continuous function by their sums. To our knowledge, this problem was formulated by A. Pinkus (see [LP]).

Another way to phrase the problem is: describe all sets S such that the space of all continuous functions can be decomposed into a closed direct sum of "spherical waves" (by analogy with plane waves) with centers on S:

$$C(\mathbb{R}^n) = cl \bigoplus_{a \in S} \tau_a(C^{\#}(\mathbb{R}^n)).$$

FIGURE 1. $\omega(\Sigma_N) \cup F$.

Here we have let $\tau_a : f \mapsto f_a$ denote the shift operator. The sum of vector spaces above consists of finite sums of vectors, and we used the notation for direct sum since the intersection of two shifted spaces of radial functions consists only of constants.

This paper contains the complete solution of this problem for the case $n = 2$. It turns out that $\mathcal{L}(S)$ is dense in $C(\mathbb{R}^2)$ for all S except very special sets related to a nice geometric object. Before formulating the result, we will describe this object. For any $N \in \mathbb{N}$ we denote by Σ_N **the Coxeter system** of N lines $L_0, \ldots, L_{N-1}$ in the plane where:

$$L_k = \{te^{\pi ik/N} \mid -\infty < t < \infty\}. \tag{1.1}$$

Each of these lines passes through the origin and through a $2N^{th}$ root of unity.

Theorem A. *The following condition is necessary and sufficient for $\mathcal{L}(S)$ to be dense in $C(\mathbb{R}^2)$:*

(*) *the set S is not contained in any set of the form $\omega(\Sigma_N) \cup F$, where $\omega \in \mathrm{M}(2)$ and F is a finite set.*

Several people have done important work on this problem. Lin and Pinkus originally (and independently) conjectured this theorem, and subsequently, they proved that $\mathcal{L}(S)$ is dense if S is non-algebraic. Pinkus proved density for some algebraic curves such as parabolas. Pinkus and Lin solved the case when S is a union of hyperplanes in $\mathbb{R}^n$. Kuchment (see §8.2) showed the relation of this problem to the membrane equation and proved denseness for closed curves using this. Zobin and Lin first observed the connection to harmonic polynomials.

1.2 Spherical Radon transforms. By duality arguments (Theorem 6.2), the denseness of $\mathcal{L}(S)$ in $C(S)$ is equivalent to the injectivity of the Radon transform over spheres

$$Rf(x,r) = \int_{S(x,r)} f dA, \qquad f \in C_c(\mathbb{R}^n). \tag{1.2}$$

Here $x \in \mathbb{R}^n$, $r \in \mathbb{R}_+ = (0,\infty)$, $S(x,r)$ denotes the sphere centered at x and of radius r, and dA is the normalized area measure on $S(x,r)$. Of course, this transform can be defined on domain $C(\mathbb{R}^n)$, but we consider only compactly supported functions in this article. Let us say precisely what we mean by the injectivity of R.

Definition 1.1. The transform R is said to be *injective on a set S* (S is a *set of injectivity*) if for any $f \in C_c(\mathbb{R}^n)$ the condition

$$Rf(x,r) = 0 \quad \text{for all} \quad r \in \mathbb{R}_+ \quad \text{and any} \quad x \in S$$

implies $f \equiv 0$.

The problem for the spherical transform equivalent to Problem 1 is:

Problem 2. Describe all sets of injectivity for the Radon transform R on domain $C_c(\mathbb{R}^n)$

This problem is given in the book of L. Ehrenpreis [E], and our next theorem is the solution in the plane:

Theorem B. *The condition (*) in Theorem A is necessary and sufficient for S to be a set of injectivity for the Radon transform over circles.*

As with other Radon transforms (*e.g.*, [BG, Z2]), proving injectivity of R for functions not of compact support is a difficult problem (see *e.g.*, [Q3]).

As is shown in §6.1, the denseness of $\mathcal{L}(S)$ in $C(\mathbb{R}^n)$ is equivalent to the injectivity of R on S. Therefore, Theorem A is equivalent to Theorem B, and both are true for $n = 2$. We first prove Theorem B and derive Theorem A as a consequence.

The proof of Theorem B is given in §5. It consists of several steps. First we characterize sets of noninjectivity in algebraic terms (§2) and analyze geometric properties of these sets (§3). One key part of the proof is the support theorem, 4.1, which is obtained by the tools of microlocal analysis. Proposition 3.2 provides the geometric conditions needed to apply the support theorem.

In §6, Theorem A is proved and, also, the closure of the space $\mathcal{L}(S)$ is described in cases when $\mathcal{L}(S)$ is not dense. In §7 necessary conditions for $\mathcal{L}(S)$ to be dense in $C(\mathbb{R}^n)$ are given for arbitrary n. §8 is devoted to interpretations and applications of our results. In §9 some open questions are formulated. An announcement of these results appeared in [AQ].

§2. Algebraic characterization of sets of noninjectivity.

2.1 Sets of noninjectivity in $\mathbb{R}^n$. With each $f \in C_c(\mathbb{R}^n)$ we associate the set

$$S[f] = \{x \in \mathbb{R}^n \mid Rf(x,r) = 0 \quad \forall \ r \in \mathbb{R}_+\}. \tag{2.1}$$

In certain cases, for instance, when f has nonzero integral over the whole space, $S[f] = \emptyset$. Much of Sections 2, 3, and 5 is devoted to understanding the geometry of $S[f]$, and Theorem B' in §5 is a complete characterization of $S[f]$ for $n = 2$.

We also associate with each $f \in C_c(\mathbb{R}^n)$ an infinite family of polynomials

$$Q_k = Q_k[f] = r^{2k} * f, \qquad r^2 = x_1^2 + \cdots + x_n^2.$$

Each function

$$Q_k(x) = Q_k[f](x) = \int_{\mathbb{R}^n} \|x - \xi\|^{2k} f(\xi)\, d\xi \tag{2.2}$$

is a polynomial of degree $\deg Q_k \le 2k$.

For any polynomial Q with real coefficients, we denote by $V[Q]$ the real algebraic variety

$$V[Q] = \{x \in \mathbb{R}^n \mid Q(x) = 0\}. \tag{2.3}$$

Lemma 2.1. $S[f] = \bigcap_{k=0}^{\infty} V[Q_k]$.

Proof. The condition $Rf(x,r) = 0$ for all $r \in \mathbb{R}_+$ is equivalent to

$$\int_{\mathbb{R}^n} \alpha(\|x - \xi\|^2) f(\xi) d\xi = 0 \tag{2.4}$$

for any $\alpha \in C_c([0,\infty))$. Then the lemma follows from Weierstrass' theorem about the denseness of polynomials. □

Proposition 2.2. *Let $f \in C_c(\mathbb{R}^n)$. Then, $f \equiv 0$ if and only if $Q_k[f] \equiv 0$ for all $k = 0, 1, \ldots$. If f is not identically zero, and $P = Q_{k_{\min}}[f]$ is the nontrivial polynomial of minimal degree in (2.2), then P is harmonic.*

When $f \neq 0$, we will denote this minimal degree, harmonic polynomial, P, by $P[f]$.

Proof. Because of Lemma 2.1 and (2.1), the condition $Q_k[f] \equiv 0$ for all $k = 0, 1, \ldots$ is equivalent to the vanishing of all integrals of f over all spheres in $\mathbb{R}^n$, that is, to $f \equiv 0$.

The second statement in the Lemma follows from the relation

$$\Delta Q_k = 2k(2k+n-2)Q_{k-1},$$

where Δ is the Laplace operator. $\square$

Lemma 2.1 and Proposition 2.2 imply that, if R is not injective on S, then S is the zero set of a harmonic polynomial. Therefore, we get a sufficient condition for injectivity:

Corollary 2.3. *Any set of uniqueness for harmonic polynomials, $S \subset \mathbb{R}^n$, is a set of injectivity for the transform R.*

Lin and Zobin independently proved this theorem. Our proof above is valid for the transform R evaluated on rapidly decreasing functions since the polynomials $Q_k[f]$ are well defined for such functions.

It is interesting to note that, because of the Mean Value Property for harmonic functions, the condition in Corollary 2.3 is necessary for injectivity of the transform R in the space of polynomials.

On the other hand, Corollary 2.3 becomes false when one replaces compactly supported functions by bounded functions or even functions vanishing at infinity. An example is the spherical function ϕ in $\mathbb{R}^n = \mathrm{M}(n)/\mathrm{SO}(n)$:

$$\phi(x) = J_k(\|x\|)\|x\|^{-k},$$

where $k = \frac{n-2}{2}$ and J_k is the Bessel function. In this case

$$S = \{x \in \mathbb{R}^n \mid \|x\| = \lambda \neq 0,\ J_k(\lambda) = 0\}$$

satisfies the condition of Corollary 2.3, ϕ is not identically zero and $R\phi(a,r) = 0$ for all $a \in S$ and $r \in \mathbb{R}_+$. The last identity follows from the general integral equation for spherical functions (*cf.* [H2, Prop. 4.2.4]). Thus the set S, which is the union of spheres, is a set of noninjectivity in any function space which contains ϕ.

2.2 Sets of noninjectivity in $\mathbb{R}^2$. Now we focus on the case $n = 2$. We let $C(a,r)$ be the circle of radius r centered at the point a. When it will not cause confusion, we will let (x,y) denote the coordinates in $\mathbb{R}^2$ and $z = x + iy \in \mathbb{C}$.

Let $f \in C_c(\mathbb{R}^2)$ and $Rf(a,\cdot) \equiv 0$ for all $a \in S \subset \mathbb{R}^2$. Let Q_k, $P[f]$, and $S[f]$ be as in §2.1. It is clear that $S \subset S[f]$. We will investigate the set $S[f]$ in more detail.

Proposition 2.4. *Let $f \in C_c(\mathbb{R}^2)$, $f \neq 0$, and assume $S[f]$ is an infinite set. There exists a non-constant polynomial $\Psi = \Psi[f]$ and a finite set F, such that*

(i) *$S[f] = V[\Psi] \cup F$, where F is a finite set.*
(ii) *$V[\Psi] = S_1 \cup \cdots \cup S_m$, where each S_j is a real-analytic topologically connected curve in $\mathbb{R}^2$.*
(iii) *Ψ divides $P[f]$.*

Proof. According to Lemma 2.1, $S[f]$ is the set of common zeros of all polynomials Q_k, including $P = P[f] = Q_{k_{\min}}$, so $V[P] \supset S[f]$.

Let us decompose P into a product of irreducible (over $\mathbb{R}$) polynomials:

$$P = P_1 \cdots P_\ell.$$

Choose an arbitrary nonnegative integer k and consider the polynomial Q_k. Because of the Bezout Theorem for real algebraic curves (*cf.* [W,Theorem 5.4]), the number of points of intersection

$$\# V[Q_k] \cap V[P_i] \leq \deg Q_k \cdot \deg P_i,$$

unless the polynomials Q_k and P_i have a common polynomial divisor. In this case, P_i divides Q_k, as P_i is irreducible.

Let $P_{i_1}, \ldots, P_{i_m}$ be all of the irreducible factors of P such that for any $\alpha = 1, \ldots, m$ $V[P_{i_\alpha}] \cap V[Q_k]$ is infinite for all $k = 0, 1, \ldots$. Then we obtain

$$S[f] = V[P_{i_1}] \cup \cdots \cup V[P_{i_m}] \cup F,$$

where F is a finite set. The polynomial $\Psi = P_{i_1} \cdots P_{i_m}$, which is just the greatest common divisor of all the Q_k, satisfies (i) and (iii) by construction.

Let us now verify the property (ii). Suppose $x_0 \in \mathbb{R}^2$ is a singular point of the real algebraic curve $V[\Psi]$, *i.e.* grad $\Psi(x_0) = 0$. Using a translation, we can assume that $x_0 = 0$. Let $\Psi = \Psi_k +$ (summands of higher degree) be the decomposition into homogeneous polynomials. Since Ψ is a divisor of a nonzero harmonic polynomial, then $\Psi_k = \ell_1 \cdots \ell_k$, where ℓ_j are linear functions defining k lines $\ell_j = 0$ with angles between them that are rational multiples of π (*cf.* [FNS]). It is not hard to show, by passing to polar coordinates (r, θ), dividing by r^k and using the Implicit Function Theorem, that each line $\ell_j = 0$ is the tangent line to some smooth curve $\theta = \theta(r)$ in a neighborhood of $x_0 = 0$. Thus, the variety $V[\Psi]$ in a neighborhood of each of its singular points is a union of k nonsingular (smooth) curves which intersect transversally at this point. Self intersections are impossible since they would imply $P \equiv 0$ by the Maximum Principle. From this we conclude that globally $V[\Psi]$ is union of a finite number of smooth curves as is stated in (ii). Note that since we have used only that Ψ is divisor of P, the algebraic variety $V[P]$ has similar structure. □

Let us call nonsingular (connected) curves which make up a corresponding algebraic curve, *nonsingular components*.

§3. Asymptotic analysis of $V[\Psi]$.

Throughout this entire section we will assume $f \in C_c(\mathbb{R}^2)$ is a nonzero function. We will assume $S = S[f]$ is an infinite set and we will let $P = P[f]$, and $\Psi = \Psi[f]$.

3.1 Asymptotic analysis of $V[P]$. According to Proposition 2.4, S is contained in the zero set of a nonzero harmonic polynomial P. This places certain restrictions on S (*cf.* [FNS]) and we want to use these restrictions to get information about geometric properties of the set $S[f]$.

The polynomial P can be represented as

$$P(z) = \operatorname{Im}\left(c_N z^N + c_{N-1} z^{N-1} + \cdots + c_0\right), \quad z = x + iy.$$

By using a rotation and translation in the plane, we can assume $c_N > 0$ and $c_{N-1} = 0$.

Let $P_k = \operatorname{Im} c_k z^k$ for $k = 0, \ldots, N$, then

$$P = P_N + P_{N-2} + \cdots + P_0$$

is the decomposition of P into a sum of homogeneous harmonic polynomials.

Since $c_N > 0$, the leading homogeneous term P_N vanishes on each line $\mathbb{R}\cdot e^{ik\pi/N}$, $k = 0, 1, \ldots, N-1$, and so P_N can be decomposed into a product of linear factors:

$$P_N(x, y) = \text{const} \prod_{k=0}^{N-1} (a_k x + b_k y), \quad \text{where} \quad a_k = \sin k\frac{\pi}{N},\ b_k = -\cos k\frac{\pi}{N} \qquad (3.1)$$

.

Denote by L_k the line $L_k = \{(x, y) \mid a_k x + b_k y = 0\}$ and by $L_k^{\pm}$ two half-lines $L_k^{\pm} = \{te^{ik\pi/N},\ t \in R_{\pm}\}$.

The following properties of zero sets of harmonic polynomials can be easily observed:

(1) each ray $L_k^{\pm}$ is an asymptote for some nonsingular component of the algebraic curve $V[P]$;

(2) each nonsingular component of $V[P]$ has two asymptotes, each of which is one of the $2N$ rays $L_0^\pm, \dots, L_{N-1}^\pm$;

(3) no ray $L_k^\pm$ can be the asymptote for two different nonsingular components of $V[P]$.

Let us comment on these statements. All of the nonsingular components of $V[P]$ must be unbounded curves. Indeed, if one of them were bounded, then it would be a closed curve because all algebraic curves are topologically closed sets; we would get a contradiction with the harmonicity of P, according to the Maximum Principle.

Writing the equation $P = 0$ in polar coordinates, dividing by r^N, and letting $r \to \infty$ shows that (2) holds. (The normalization $c_{n-1} = 0$ guarantees that the asymptotes coincide with two of the rays in (2). In general, they might be parallel.) (1) can be obtained by using the Implicit Function Theorem (we will apply this argument to $V[\Psi]$ in more detail in the proof of Lemma 3.1). Finally, (3) follows from the simplicity of the zeros, $k\frac{\pi}{N}$, of the spherical harmonic $P_N(\cos\theta, \sin\theta)$.

3.2 Asymptotic analysis of $V[\Psi]$. We will use the fact that the algebraic curve $V[\Psi]$ is contained in $V[P]$ so that it inherits some asymptotic properties of the larger curve. Recall that $V[\Psi] = S_1 \cup \cdots \cup S_m$ where S_j is a smooth connected curve.

Lemma 3.1. *Let $f \neq 0$ and assume $S[f]$ is an infinite set. There is a collection of rays*

$$L_{i_1}^\pm, \dots, L_{i_M}^\pm, \tag{3.2}$$

where $M = \deg \Psi$, such that

(i) *each curve S_j in Proposition 2.4 has two asymptotes among the rays in (3.2);*

(ii) *each ray in (3.2) is an asymptote for some curve S_j;*

(iii) *no ray in (3.2) serves as an asymptote for two different curves S_i, S_j.*

Proof. Each S_j is unbounded and, since $S_j \subset V[\Psi] \subset V[P]$, we obtain (i) from the property (2) of $V[P]$.

Now we need to select the rays in (3.2) which are really asymptotes for $V[\Psi]$. For this purpose, let us represent Ψ as a sum of homogeneous polynomials:

$$\Psi = \Psi_M + \Psi_{M-1} + \cdots + \Psi_0, \qquad M = \deg \Psi.$$

Since the polynomial Ψ divides P, then its leading part Ψ_M divides the leading part P_N of P. Therefore, Ψ_M is a product of some of the linear factors of P_N in (3.1):

$$\Psi_M(x,y) = \text{const} \cdot \prod_{\alpha=1}^{M} (a_{k_\alpha} x + b_{k_\alpha} y)\,.$$

Let us rewrite the equation $\Psi = 0$ (the equation that determines the set $V[\Psi]$) in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, and divide by r^M;

$$\Psi_M(\cos\ \theta, \sin\ \theta) + \frac{1}{r}\ \Psi_{M-1}(\cos\ \theta, \sin\ \theta) + \cdots + \frac{1}{r^M}\ \Psi_0(\cos\ \theta, \sin\ \theta) = 0. \quad (3.3)$$

When we introduce the small parameter $\varepsilon = \frac{1}{r}$ and denote the left hand side in (3.3) by $F(\varepsilon, \theta)$, we obtain

$$F(\varepsilon, \theta) = 0. \quad (3.4)$$

Fix some index k_α and let $\theta^\circ = k_\alpha \frac{\pi}{N}$. Then $F(0, \theta^\circ) = 0$ and

$$\frac{\partial F}{\partial \theta}\ (0, \theta^\circ) = \text{const} \cdot \sin\ (\theta - \theta_{k_0}) \ldots \cos\ (\theta - \theta_{k_\alpha}) \ldots \sin\ (\theta - \theta_{k_{N-1}})\ \Big|_{\theta = k_\alpha \pi / N} \neq\ 0.$$

Using the Implicit Function Theorem, we obtain that Eq. (3.4), uniquely determines some real-analytic curve in a neighborhood of the point $\varepsilon = 0,\ \theta = \theta^\circ$, or, equivalently, in a neighborhood of $r = \infty,\ \theta = \theta^\circ$. Let $\theta = \theta(r),\ r > r_0$ be the solution of (3.4) which defines this curve. The asymptotic behavior of $\theta(r)$ for $r \to \infty$ follows from (3.3):

$$\theta(r) = \theta^\circ + C \cdot \frac{1}{r}\ + o(\frac{1}{r}), \quad C = \text{const}\ \cdot \Psi_{M-1}(\cos\ \theta^\circ, \sin\ \theta^\circ).$$

In Cartesian coordinates we have

$$x = r\cos\ \theta(r) = r\cos\ \theta^\circ + o(1),$$
$$y = r\sin\ \theta(r)\ = r\sin\ \theta^\circ + C \cdot \cos\ \theta^\circ + o\left(\frac{1}{r}\right).$$

Therefore, the curve $\theta = \theta(r), r > r_0$, has an asymptote which is parallel to $L^+_{k_\alpha} = \{\theta = \theta^\circ\}$. But this curve is a subset of $V[\Psi]$ and no half-line parallel to and in the same direction as $L^+_{k_\alpha}$ is an asymptote to $V[\Psi]$, except the ray $L^+_{k_\alpha}$ itself. The case of $L^-_{k_\alpha}$ can be treated analogously. Thus, (ii) is proved. (iii) trivially follows from the analogous property (3) of $V[P]$. Finally, we have a 1 - 1 correspondence between m pairs of asymptotes of the curves S_j's and certain M pairs of rays $L^\pm_{k_\alpha}$. Therefore, $m = M = \deg\Psi$, and the Lemma is proved completely. $\square$

At this point we will be able to infer the most useful information about $V[\Psi]$ for our purposes:

Proposition 3.2. *Let $f \neq 0$ and assume $S[f]$ is an infinite set. Only the two following cases are possible:*

(a) *there exists $t \in \mathbb{R}^2$ such that the shifted polynomial $\Psi^t(x) = \Psi(x+t)$ is homogeneous;*

(b) *at least two nonsingular components S_i, S_j of $V[\Psi]$ are disjoint.*

Proof. Suppose that condition (a) is not fulfilled, and each two connected components of $V[\Psi]$ intersect. Observe that since the polynomial P is harmonic, not identically zero, and vanishes on $V[\Psi]$, then no three curves S_i, S_j, S_k intersect pairwise in three different points because, in this case, P vanishes on a closed contour (a curved triangle). Since P is harmonic, the Maximum Principle would imply that $P \equiv 0$. For the same reason, no two curves S_i and S_j intersect at two different points and no curve S_i has points of self intersections.

Then the only remaining possibility is that all the curves S_j intersect at some point $t \in \mathbb{R}^2$. In order to show that, in this case, $\Psi^t(x) = \Psi(x+t))$ is a homogeneous polynomial, let us decompose it into a sum of homogeneous summands:

$$\Psi^t = \Psi^t_{\underline{M}} + \cdots + \Psi^t_M,$$

where $\underline{M}$ is the minimal degree of the nonzero summands. Note that $\underline{M} > 0$ as $\Psi^t(0) = 0$.

We know that Ψ divides P, therefore, the minor homogeneous term, $\Psi^t_{\underline{M}}$, divides that of the shifted polynomial P^t, which is also harmonic, and for this reason

$$\Psi^t_{\underline{M}} = \ell_1 \cdots \ell_{\underline{M}}$$

where the ℓ_i are some of the irreducible factors of the minor homogeneous part $P_{\underline{N}}$ of P. These factors are linear since $P_{\underline{N}}$ is harmonic.

Let $S^t_j = -t + S_j$ be the shifted curve. All the curves S^t_j intersect at the origin and, clearly the lines $\ell_i = 0$ are exactly the family of tangents to these curves at the origin. No two curves S^t_i, S^t_j are tangent to each other at the origin because it contradicts the simplicity of zeros of the spherical harmonic $P_{\underline{N}}(\cos\ \theta, \sin\ \theta)$ (see also [FNS] about this and other properties of zeros of harmonic polynomials). By Lemma 3.1 the number of connected components S_j is equal to M, therefore, the number $\underline{M}$ of tangent lines $\ell_i = 0$ is the same.

Thus we arrive at the identity $\underline{M} = M$ and, therefore, $\Psi^t = \Psi^t_M$ is homogeneous. This completes the proof. $\square$

§4. Microlocal Fourier analysis. Support Theorem.

Guillemin [GS, pp. 336-337, 364-365] first used the microlocal techniques of Fourier analysis (Fourier integral operators and wavefront sets) to understand Radon transforms, and others (*e.g.* [BQ, Q2, Q3]) have used them to prove support theorems for Radon transforms on hyperplanes, line complexes, groups, and manifolds. Recall that the set of distributions (continuous linear functionals on $C_c^\infty(\mathbb{R}^n)$) is denoted $\mathcal{D}'(\mathbb{R}^n)$ and the set of distributions of compact support is denoted $\mathcal{E}'(\mathbb{R}^n)$.

Theorem 4.1 (Support Theorem). *Let S be a regular real-analytic curve (possibly disconnected). Assume that S contains two points, a and b, $a \neq b$, such that the segment $\overline{ab}$ is perpendicular to the tangent lines L_a and L_b at the point a and b respectively. Then the Radon transform, R, is injective on S.*

This theorem is true if $f \in \mathcal{E}'(\mathbb{R}^2)$ and if an arbitrary nowhere zero real-analytic weight is added in (1.2) because the proof, using microlocal analysis, is valid in this setting [Q3]. Other support theorems for f not of compact support are given in [*ibid.*].

4.1 The microlocal analysis. We begin by introducing the microlocal terminology and then we prove the microlocal regularity theorem, Lemma 4.3.

For $x \in \mathbb{R}^2$, the cotangent space $T_x^*\mathbb{R}^2$ is the set of all linear functionals on the tangent space $T_x\mathbb{R}^2$. So, if $x = (x_1, x_2)$, a basis of $T_x^*\mathbb{R}^2$ is formed by the differentials $\mathbf{dx}_1$ and $\mathbf{dx}_2$. We write $(x;\xi) \in T^*\mathbb{R}^2$ when $\xi \in T_x^*\mathbb{R}^2$. If C is a smooth regular curve in $\mathbb{R}^2$, then the conormal bundle of C, N^*C, is the set of all covectors $(x;\xi) \in T^*\mathbb{R}^2$ that are conormal to the tangent space of C (*i.e.* $x \in C$ and the linear functional ξ is zero on the tangent space $T_xC \subset T_x\mathbb{R}^2$).

The analytic wavefront set of a distribution $f \in \mathcal{D}'(\mathbb{R}^2)$ is a conic subset, $\mathrm{WF_A}(f)$, of the cotangent bundle $T^*\mathbb{R}^2$ consisting of "directions" in which f is not real-analytic. This is defined either in terms of the very rapid decrease of localized Fourier transforms of f [T, Definition 1.1, p. 243] or in terms of exponential decrease of the FBI (Fourier-Bros-Iagolnitzer) transform [Hö, Theorem 9.6.3]. For example, if f is the characteristic function of a disk, D, then $\mathrm{WF_A}(f)$ is the conormal space of the boundary of D, $N^*\partial D$.

We will parameterize S in order to do the microlocal calculations. To this end, let A be an open subset of $\mathbb{R}$ and let $\gamma : A \to \mathbb{R}^2$ parameterize the regular real-analytic curve S. For $f \in \mathcal{E}'(\mathbb{R}^2)$ and $(t, r) \in A \times (0, \infty)$, we change notation a little to reflect this parameterization:

$$Rf(t,r) := Rf(\gamma(t),r) \text{ and } c(t,r) := C(\gamma(t),r). \tag{4.1}$$

Rf is just the spherical mean of f over the circle $c(t,r)$, the circle centered at $\gamma(t)$ and of radius r.

Definition 4.2. Points x and x' in $c(t,r)$ are said to be $c(t,r)$*-mirror* if and only if they are reflections about the diameter of $c(t,r)$ that is tangent to γ at $\gamma(t)$.

The fundamental microlocal result is the following regularity theorem for the Radon transform, R. The hypotheses include an assumption on the vanishing of f at certain points.

Lemma 4.3. *Let $f \in C(\mathbb{R}^2)$. Assume Rf is zero in an open neighborhood of $(t,r) \in A \times (0,\infty)$. Let $(x;\xi) \in N^*c(t,r) \setminus 0$, and assume that f is zero in a neighborhood of the $c(t,r)$-mirror point to x. Then $(x;\xi) \notin \mathrm{WF_A}(f)$.*

In general, Radon transforms detect singularities ($\mathrm{WF_A}(f)$) conormal to the curve being integrated over. This lemma implies that singularities at $(x;\xi) \in N^*c(t,r)$ will be detected by "data" $Rf(t,r)$ when f is zero (or real-analytic) near the mirror point to x.

If $x \in c(t,r)$ is on the diameter of $c(t,r)$ tangent to γ, then x is its own mirror (that is: *self-mirror*) and Lemma 4.3 gives no conclusion about x. In other cases, if f is zero in a neighborhood of the $c(t,r)$-mirror point to x, then the theorem provides information about $\mathrm{WF_A}(f)$ above x.

Proof. The proof is essentially the same as the proof of Proposition 4.3 in [Q3]. The incidence relation for R is defined to be $Z = \{(x,t,r) \in \mathbb{R}^2 \times A \times (0,\infty) \bigm| x \in c(t,r)\}$ [H2]. The appropriate microlocal diagram [GS, pp. 364-365] (see also [Q1]) is:

$$\begin{array}{ccc} \Gamma = N^*(Z) \setminus 0 & \xrightarrow{\pi_2} & T^*(A \times (0,\infty)) \setminus 0 \\ \big\downarrow{\scriptstyle \pi_1} & & \\ T^*(\mathbb{R}^2) \setminus 0 & & \end{array} \tag{4.2}$$

where the maps π_1 and π_2 are projections from $\Gamma \subset T^*(\mathbb{R}^2 \times A \times \mathbb{R}_+)$ onto the indicated factors.

We must show the map π_2 is close enough to being an injective immersion (the Bolker Assumption, [GS, pp. 364-365, Q1] that the calculus of Fourier integral operators can be used to prove the lemma. Specifically, the goal is to prove that π_2 in (4.2) satisfies:

(4.3) *covectors* $(x, t, r; \xi, \eta) \in \Gamma$ *and* $(x', t, r; \xi', \eta) \in \Gamma$ *have the same image under* π_2 *only if* x *and* x' *are* $c(t, r)$*-mirror.* π_2 *is a local diffeomorphism except above points* (x, t, r) *where* $x \in c(t, r)$ *is its own mirror.*

To this end, we first calculate N^*Z in good coordinates. Points $(x, t, r) \in Z$ are determined by the equation $|x - \gamma(t)|^2 - r^2 = 0$, and the differential of this equation gives a basis of the fibers of N^*Z. Coordinates for $N^*Z \setminus 0$ are:

$$\begin{aligned} [0, 2\pi] \times A \times (0, \infty) \times (\mathbb{R} \setminus 0) &\to N^*Z \setminus 0 \\ (\theta, t, r, a) &\to \big(x, t, r; a([r\overline{\theta}]\mathbf{dx} - [(r\overline{\theta}) \cdot \gamma'(t)]\mathbf{dt} - r\mathbf{dr})\big) \\ \text{where} \quad x &= r\overline{\theta} + \gamma(t). \end{aligned} \tag{4.4}$$

Here, $(w_1, w_2)\mathbf{dx} = w_1\mathbf{dx}_1 + w_2\mathbf{dx}_2$ is the covector in $T^*\mathbb{R}^2$ corresponding to $(w_1, w_2) \in \mathbb{R}^2$.

Eq. (4.4) shows that π_1, and π_2 do not map to the zero section so, R is a Fourier integral operator associated to the Lagrangian manifold, Γ [T, Theorem 2.1, p. 316]. This explains why R can be evaluated on distributions. R is real-analytic elliptic since the measure of integration for R, dA, is real-analytic and nowhere zero.

The map π_2 is equivalent to the corresponding map in coordinates (4.4):

$$(\theta, t, r, a) \xrightarrow{\tilde{\pi}_2} \big(t, r; -a([(r\overline{\theta} \cdot \gamma'(t))]\mathbf{dt} - r\mathbf{dr})\big). \tag{4.5}$$

Therefore, π_2 determines only $\overline{\theta} \cdot \gamma'(t)$ so $x = \gamma(t) + r\overline{\theta}$ is known only up to its $c(t, r)$-mirror. This shows the first claim of (4.3). The calculation that $\tilde{\pi}_2$ is a local diffeomorphism except at self-mirror points is left to the reader.

Now, assume f is as in the hypotheses of Lemma 4.3. R has been shown to be an analytic elliptic Fourier integral operator associated with Γ. The calculus of such operators implies the conclusion of Lemma 4.3. Here is the idea: let $(x; \xi) \in N^*(c(t, r)) \setminus 0$ and assume f is zero near the $c(t, r)$-mirror point to x. By (4.3), only singularities at the $c(t, r)$-mirror point, x', to x can mask singularities of f above x. But, $\mathrm{WF_A}(f)$ is empty above x' as $f = 0$ near x'. Therefore, singularities at x' cannot mask singularities at x. Since Rf is zero near (t, r), $(x; \xi) \notin \mathrm{WF_A}(f)$. (*The precise argument is:, we make a* C^∞ *partition of unity,* $1 = \psi_p + \psi_x + \psi_0$*, with the following conditions:* $\psi_p = 1$ *near* x' *and* ψ_p *is sufficiently localized around* x' *so that* $\psi_p f = 0$*;* $\psi_x = 1$ *near* x *and supp* ψ_x *is sufficiently* *localized around x so that Rg satisfies the Bolker Assumption (the restricted π_2 is an injective immersion) locally above (x, t, r) for functions supported in a neighborhood of supp ψ_x. Therefore, $\psi_0 = 0$ near x and x', so by (4.3) and the calculus of real-analytic Fourier integral operators* [SKK, Ka], *$R\psi_0 f$ is real-analytic in directions $(t, r; \eta)$ when $(x, t, r; \xi, -\eta) \in \Gamma$. Therefore, as $Rf = R\psi_p f = 0$ near (t, r) and $(t, r; \eta) \notin \mathrm{WF_A}(R\psi_0 f)$, $(t, r; \eta) \notin \mathrm{WF_A}(R\psi_x f)$. Since the operator R satisfies the Bolker assumption above (x, t, r) for functions supported in a neighborhood of supp ϕ_x, $(x; \xi) \notin \mathrm{WF_A}(f)$.)* $\square$

4.2 Proof of Support Theorem. Let a and b be points satisfying the condition in the theorem, and let L_a and L_b be their respective tangent lines to S. The trick is to eat away at supp f using Lemma 4.3 and Lemma 4.4 (below) by successively using circles centered at a and then circles centered at b. Recall that the circle centered at a and of radius $r \in \mathbb{R}_+$ is $C(a, r)$ and similarly for $C(b, r)$. The following lemma is a special case of [Hö] Theorem 8.5.6.

Lemma 4.4. *Let $f \in \mathcal{D}'(\mathbb{R}^2)$ and let C be a circle. Let $x \in$ supp $f \cap C$ and assume* supp f *is enclosed by C. If $(x; \xi) \in N^*C \setminus 0$, then $(x; \xi) \in \mathrm{WF_A}(f)$.*

The lemma states that, if f is zero on one side of C and x is in supp $f \cap C$, then f is not analytic in this conormal direction, $(x; \xi)$, to C. This is a refinement of the well known fact that if f is zero outside of C and $x \in C\cap$ supp f, then f is not real-analytic near x. Lemma 4.4 says that for such x, not only is f not real-analytic at x, but also f is not real-analytic in the conormal direction $(x; \xi)$.

FIGURE 2. *$C(a, r_0)$, $C(b, r_1)$ and tangent lines to S.*

Our final reasoning is illustrated by Figure 2. Assume the tangent lines, L_a and L_b are horizontal and a is below b. Let $r_0 > 0$ be the smallest radius, r, such that $C(a,r)$ encloses supp f. Let r_1 be half of the length of the segment on L_b that is between the points $L_b \cap C(a,r_0)$. Note that $r_1 < r_0$. (If $L_b \cap C(a,r_0) = \emptyset$, then the argument continues in the next paragraph.) If $r > r_1$ and $C(b,r)$ meets supp f and $C(b,r)$ encloses supp f, then $C(b,r)$ meets supp f only at points below L_b. Therefore, there are no mirror points (or self-mirror points) on $C(b,r)$ that meet supp f. So, if $x \in C(b,r) \cap$ supp f, and $(x;\xi) \in N^*C(b,r)$, then by Lemma 4.3, $(x;\xi) \notin \mathrm{WF_A}(f)$. However, Lemma 4.4 implies $(x;\xi)$ must be in $\mathrm{WF_A}(f)$. This contradiction shows that $x \notin$ supp f, so we have eaten away at supp f in order to conclude that supp f is inside $C(b,r_1)$. Using the same argument, we now find an $r_2 < r_1$ such that supp f is inside $C(a,r_2)$. We can continue, alternately eating away at supp f using circles centered at a and then circles centered at b. Note that the r_j decrease faster as j increases because the circles $C(a,r_j)$ and $C(b,r_k)$ are getting smaller.

This process stops when we have an $r_m < \mathrm{dist}(a,b)$ such that supp f is enclosed by $C(a,r_m)$ or $C(b,r_m)$. Assume supp f is enclosed by $C(a,r_m)$. Then as in the last paragraph, we can use circles centered at b and Proposition 4.3 and Lemma 4.4 to eat completely away at supp f and show that $f \equiv 0$. □

§5. Proof of Theorem B.

5.1 Sufficiency. One needs to prove that for any $f \in C_c(\mathbb{R}^2)$ the set $S[f]$, introduced in §2.1, either coincides with the whole plane $\mathbb{R}^2$ (which is equivalent to $f \equiv 0$) or is contained in some set $\omega(\Sigma_N) \cup F$, where Σ_N is the Coxeter system of N lines, $\omega \in \mathrm{M}(2)$ and F is a finite set.

Suppose $f \not\equiv 0$. Then, by Corollary 2.3, $S[f] \neq \mathbb{R}^2$. If $S[f]$ is a finite set, we are done. Assume $S[f]$ is infinite and let $P = P[f]$ and $\Psi = \Psi[f]$ be as in Proposition 2.4. Then, by Proposition 3.2 two cases (a) and (b) are possible:

(a) For some $t \in \mathbb{R}^n$ the shifted polynomial $\Psi^t(x) = \Psi(x+t)$, is homogeneous.

In this case, according to Proposition 2.4 (iii), Ψ^t divides the leading homogeneous part P^t_N of the shifted polynomial P^t and so the zero set $V[\Psi^t]$ is contained in $V[P^t_N]$, which is Σ_N. It remains to mention that $V[\Psi] = V[\Psi^t] + t$ and to remember that in 3.1 we have used a rotation and translation to normalize the polynomial P. Therefore, $V[\Psi] \subset \omega(\Sigma_N)$ for some $\omega \in \mathrm{M}(2)$, and Proposition 2.4 yields $S[f] \subset \omega(\Sigma_N) \cup F$.

(b) There exist two disjoint nonsingular components of $V[\Psi]$, say, S_1 and S_2.

In this case we claim we can use Theorem 4.1. Indeed, the distance between S_1 and S_2 cannot be attained at infinity, since the curves S_1 and S_2 have two different and not parallel asymptotes according to Lemma 3.1. Therefore, there exist points $a \in S_1$, and $b \in S_2$ for which

$$d = \operatorname{dist}(S_1, S_2) = \operatorname{dist}(a, b) > 0.$$

We claim that a and b satisfy the condition of Theorem 4.1. This is true for the following reasons. As d is the minimal distance from a to points of S_2, the circle centered at the point a and of radius d is tangent to S_2 at the point b (the circle cannot meet the curve S_2 transversally because d is minimum, S_2 is regular and a is not an end point of S_2). Therefore, the segment $\overline{ab}$ is perpendicular to the tangent to S_2 at b. Similarly, $\overline{ab}$ is perpendicular to the tangent to S_1 at the point a. Thus, the curve $S = S_1 \cup S_2$ satisfies the conditions of Theorem 4.1 and $f \equiv 0$.

This shows that (b) is impossible since we have assumed from the beginning that $f \neq 0$. The sufficiency part of Theorem B is proved. $\square$

5.2 Necessity. To show that the condition (*) in Theorem B is necessary, we have to construct, for any set $\Sigma_N \cup F$ where F is finite, a nonzero function $f \in C_c(\mathbb{R}^2)$ such that $Rf(a, \cdot) \equiv 0$ for all $a \in \Sigma_N \cup F$. (The motion ω is obviously unessential.)

Lemma 5.1. *For any function $f \in C_c(\mathbb{R}^2)$ of the form*

$$f(x) = \sum_{j=1}^{\ell} f_j(r) \ \sin \ jN\theta, \quad x = re^{i\theta}, \tag{5.1}$$

the Radon transform $Rf(a, \cdot) \equiv 0$ for all $a \in \Sigma_N$.

Proof. It is easy to see that f in (5.1) is odd with respect to the reflection w_k about the line $L_k = \left\{te^{ik\frac{\pi}{N}} \;\middle|\; t \in \mathbb{R}\right\} \subset \Sigma_N, k = 0, 1, \ldots, N-1$. Therefore, if $a \in L_k$, then

$$\int_{C(a,r)} f dA = \int_{C(a,r)} (f \circ w_k) dA = -\int_{C(a,r)} f dA$$

and we obtain $Rf(a, r) = 0$. The Lemma is proved. $\square$

Now we have to satisfy the additional finite number of conditions $Rf(a, r) = 0, \ \forall a \in F$. Let $F = \{a_1, \ldots, a_q\}$ and write $a_s = r_s e^{i\theta_s}$ for $s = 1, \ldots, q$. We may assume $\forall s, \ a_s \notin \Sigma_N$. The condition we need to solve is:

$$\int_0^{2\pi} f\left(a_s + ze^{i\theta}\right) d\theta = 0 \quad \text{for} \quad s = 1, \ldots, q \quad \text{and} \quad z = x + iy. \tag{5.2}$$

Now apply the Fourier transform in (x, y):

$$\int_0^{2\pi} e^{i\langle a_s, \lambda e^{i\varphi}\rangle} \hat{f}(\lambda e^{i\varphi}) d\varphi = 0, \quad \text{for all} \quad \lambda \in \mathbb{C}, \quad \text{and} \quad s = 1, \dots, q \tag{5.3}$$

where $\langle a_s, \lambda e^{i\varphi}\rangle$ is the *real* inner product of these points in $\mathbb{R}^2$.

We are looking for a solution of the system of integral equations (5.3) in the class of functions of the form (5.1). For these functions the Fourier transform in polar coordinates (ρ, φ) is:

$$\hat{f}(\rho, \varphi) = \sum_{k=1}^{\ell} \hat{f}_k(\rho) \ \sin kN\varphi, \tag{5.4}$$

where $\hat{f}_k(\rho) = \int\limits_0^{\infty} f_k(r) \ J_{kN}(r\rho) \ rdr$ is the Fourier-Bessel transform.

If one substitutes the decomposition (5.4) in (5.3), uses [GR, 3.915.2], and simplifies, one gets q linear equations for ℓ functions. If we let $\ell = q + 1$, we get:

$$\begin{gathered} \sum_{k=1}^{q+1} M_{s,k}(\rho) \ \hat{f}_k(\rho) = 0 \text{ for each } s = 1, \dots, q \\ \text{where } M_{s,k}(\rho) = i^{kN} \sin(kN\theta_s) J_{kN}(\rho r_s) \end{gathered} \tag{5.5}$$

The matrix of this system will be denoted $M(\rho) = \left[M_{s,k}(\rho)\right]_{s,k=1}^{q,q+1}$.

Let $\bar{q} = \max\{ \text{ rank } M(\rho) \mid \rho \in \mathbb{R}_+\}$ and let $\rho_0 \in \mathbb{R}_+$ be a point at which the maximum is attained. We have assumed no point a_s lies on any line in Σ_N, so $M(\rho)$ is not identically the zero matrix. Hence, this maximal rank is greater than zero. This implies the existence of a neighborhood W of ρ_0 such that rank $M(\rho) = \overline{q}$ is constant on W and some $\bar{q} \times \bar{q}$ minor of the matrix $M(\rho)$ does not vanish in W. Without loss of generality we can assume that it is the principal $\bar{q} \times \bar{q}$ minor, which we will denote by $\Delta(\rho)$.

Now we consider the truncated system (5.5) taking only the first $\bar{q}$ equations. We can set $\hat{f}_{\overline{q}+1} = \cdots = \hat{f}_q = 0$ and then solve for $\hat{f}_{q+1}$ in the truncated system, getting:

$$\tilde{M}(\rho)\hat{F}(\rho) = -\hat{F}_{q+1}(\rho), \tag{5.6}$$

where we have denoted $\tilde{M}(\rho) = \left[M_{s,k}(\rho)\right]_{s,k=1}^{\overline{q},\overline{q}}$, $\hat{F} = (\hat{f}_1, ..., \hat{f}_{\bar{q}})^T$ and $\hat{F}_{q+1} = (M_{1,q+1} \cdot \hat{f}_{q+1}, ..., M_{\bar{q},q+1} \cdot \hat{f}_{q+1})^T$.

We solve the system (5.6) as follows. Let $\hat{f}_{q+1}(\rho) = \Delta(\rho)\hat{u}(\rho)$, where $u(r)$ is an arbitrary fixed smooth nonzero radial function of compact support that satisfies $\Delta(\rho) \cdot \hat{u}(\rho) \not\equiv 0$ in W.

Then, we can find the other functions $\hat{f}_1, \dots, \hat{f}_{\bar{q}}$ from (5.6) using Cramer's rule: $\hat{f}_k(\rho) = -\Delta_k(\rho)$, where $\Delta_k(\rho)$ is the determinant which is obtained by replacing the k^{th} column in $\Delta(\rho)$ by the column

$$\left(M_{1,q+1}(\rho)\hat{u}(\rho), \dots, M_{\overline{q},q+1}(\rho)\hat{u}(\rho)\right)^T.$$

The functions $\hat{f}_1, ..., \hat{f}_{q+1}$ give a solution of truncated homogeneous system (5.5) (the first $\bar{q}$ equations). For $\rho \in W$ this is also a solution of the whole system (5.5), since the last $q - \bar{q}$ equations are linear combinations of the first $\bar{q}$. Because all functions under consideration are real-analytic, the solution of (5.5) is valid for all ρ.

Now define $\hat{f}$ according to (5.4). It is easy to see from the construction that the function $\hat{f}$ is in $L^2(\mathbb{R}^2)$ and has analytic extension to $\mathbb{C}^2$ as a function of exponential growth. Because of the Paley-Wiener Theorem, the inverse Fourier transform, f, belongs to $C_c(\mathbb{R}^2)$ and $Rf(a_s, \cdot) \equiv 0$ for $s = 1, \dots, q$, by construction. Also, $f \not\equiv 0$ since $\hat{f}_{q+1} \not\equiv 0$ in W. The condition $Rf(a, \cdot) \equiv 0$ for $a \in \Sigma_N$ is satisfied because of Lemma 5.1.

The necessity part of Theorem B is completely proved. □

Remark. The proof of sufficiency for Theorem B above shows that $S[f] = V[\Psi] \cup F$ (up to a rigid motion of the plane) where Ψ is a homogeneous polynomial of degree M that divides a homogeneous harmonic polynomial P_N. Therefore, $V[\Psi]$ consists of lines which are a subset of the lines from the Coxeter system $V[P_N] = \Sigma_N$. Since the Radon transform Rf vanishes on $V[\Psi]$, f must be odd with respect to reflections around any line in $V[\Psi]$ (Lemma 6.3 below). This implies that the system of lines $V[\Psi]$ must be invariant under the Coxeter group of reflections generated by $V[\Psi]$, *i.e.* $V[\Psi]$ itself is a Coxeter system, $V[\Psi] = \Sigma_M$ for some $M \leq N$. This provides a complete characterization of $S[f]$:

Theorem B'. *If $f \in C_c(\mathbb{R}^2)$ is not identically zero, then $S[f] = \omega(\Sigma_M) \cup F$, where $\omega \in \mathrm{M}(2)$, F is finite, and Σ_M is a Coxeter system of lines for some $M = 0, 1, \dots$*

§6. Complete systems of radial functions. Proof of Theorem A.

In §6.1, we prove the equivalence of Theorem A and Theorem B (Theorem 6.2). Therefore Theorem A is true in $\mathbb{R}^2$. In §6.2, we examine the case when $S \subset \Sigma_N$.

6.1 Proof of Theorem A. Recall the notation from §1:

$$\mathcal{L}(S) = \{f_a \mid a \in S,\ f \in C^{\#}(\mathbb{R}^n)\}, \quad f_a(x) = f(x-a).$$

Let $C'(\mathbb{R}^n)$ be the dual space to $C(\mathbb{R}^n)$, the set of all regular Borel measures of compact support, equipped with the weak topology. Denote by $\mathcal{L}^{\perp}(S) \subset C'(\mathbb{R}^n)$ the annihilator of $\mathcal{L}(S)$:

$$\mathcal{L}^{\perp}(S) = \{\mu \in C'(\mathbb{R}^n) \mid \langle g, \mu \rangle = 0 \quad \text{for all} \quad g \in \mathcal{L}(S)\},$$

where $\langle g, \mu \rangle = \int_{\mathbb{R}^n} g d\mu$.

By the Hahn-Banach Theorem, the space $\mathcal{L}(S)$ is dense in $C(\mathbb{R}^n)$ if and only if $\mathcal{L}^{\perp}(S) = 0$. Note also that according to the definition of the space $\mathcal{L}(S)$, its annihilator $\mathcal{L}^{\perp}(S)$ can be described as:

$$\mathcal{L}^{\perp}(S) = \{\mu \in C'(\mathbb{R}^n) \mid \alpha * \mu \big|_S \equiv 0 \quad \text{for any} \quad \alpha \in C^{\#}(\mathbb{R}^n)\}. \tag{6.1}$$

Identify $C_c(\mathbb{R}^n)$ with a subspace of $C'(\mathbb{R}^n)$ by associating the measure fdx to the function $f \in C_c(\mathbb{R}^n)$.

Lemma 6.1. *$\mathcal{L}^{\perp}(S) \cap C_c(\mathbb{R}^n)$ is dense in $\mathcal{L}^{\perp}(S)$.*

Proof. For any $\alpha \in C_c^{\#}(\mathbb{R}^n)$ and $\mu \in \mathcal{L}^{\perp}(S)$, the convolution $\alpha * \mu$ belongs to $\mathcal{L}^{\perp}(S) \cap C_c(\mathbb{R}^n)$ (by (6.1)). These convolutions approximate μ. □

It will be convenient at this point to introduce the kernel of the transform R on S :

$$\ker_S R = \{f \in C_c(\mathbb{R}^n) \mid Rf(a, \cdot) \equiv 0 \quad \text{for all} \quad a \in S\}.$$

Since the vanishing of the integrals of $f \in C_c(\mathbb{R}^n)$ over all circles $C(a, r)$, $a \in S$, is equivalent to the orthogonality of the measure fdx to any function $\alpha_a(x) = \alpha(x-a)$, $\alpha \in C^{\#}(\mathbb{R}^n)$, we easily conclude that

$$\ker_S R = \mathcal{L}^{\perp}(S) \cap C_c(\mathbb{R}^n) \tag{6.2}$$

and, therefore, from Lemma 6.1 follows:

Theorem 6.2. *$\mathcal{L}^{\perp}(S) = 0$ if and only if $\ker_S R = 0$ (i.e., R is injective on S).*

This important relationship was independently observed by E.A. Gorin.

Proof of Theorem A. Now the proof of Theorem A becomes just a rewording of Theorem B into a different language. Indeed, the denseness of $\mathcal{L}(S)$ in $C(\mathbb{R}^n)$ is equivalent to $\mathcal{L}^{\perp}(S) = 0$, which, in turn, is equivalent (by Theorem 6.2) to the injectivity of the transform R on S.

Now, because Theorem B is true for $n = 2$, the condition (*) of Theorem A is a necessary and sufficient condition for $\mathcal{L}(S)$ to be dense in $C(\mathbb{R}^n)$. □

6.2 The closure of $\mathcal{L}(\Sigma_n)$ and $\ker_{\Sigma_N} R$. We know that $\mathcal{L}(S)$ is dense in $C(\mathbb{R}^2)$ if S is not a rigid motion of some $\Sigma_N \cup F$. The natural question arises: what is the closure of $\mathcal{L}(\Sigma_N)$? In other words, which continuous functions can be approximated by linear combinations of radial functions with centers on Σ_N? This is answered in Theorem 6.5. We characterize $\ker_{\Sigma_N}$ in Proposition 6.4.

Lemma 6.3. *Let L be a line in $\mathbb{R}^2$ and $f \in C(\mathbb{R}^2)$. Then $Rf(a,r) = \int\limits_{C(a,r)} f dA = 0$ for all $a \in L, r \in \mathbb{R}_+$ if and only if f is odd with respect to reflection w about L.*

Proof. The "if" part has already been verified in the proof of Lemma 5.1. Denote

$$f^+ = \frac{1}{2}(f + f \circ w), \; f^- = \frac{1}{2}(f - f \circ w).$$

Then, f^+, f^- are respectively w-even and w-odd and

$$f = f^+ + f^-.$$

Then, $Rf^-(a,\cdot) \equiv 0$ for all $a \in L$ and for this reason $Rf^+(a,\cdot) \equiv 0,\; a \in L$.

Courant and Hilbert [CH, p. 699 *ff.*] proved many years ago that if a function f is even with respect to the reflection about a line L, and integrals over all circles centered at L vanish, then $f \equiv 0$. Therefore, $f^+ = 0$ and $f = f^-$. □

Lemma 6.3 shows that if all circular means $Rf(a,\cdot) \equiv 0$, $a \in S$, for $f \in C_c(\mathbb{R}^2)$, then f is odd with respect to reflection about any line $L_k \in \Sigma_N$.

We let W_N be the set of reflections about lines in Σ_N. Lemma 6.1 and (6.2) give:

Proposition 6.4. *$\ker_{\Sigma_N} R$ consists of all W_N-odd functions in $C_c(\mathbb{R}^2)$, i.e., of all $f \in C_c(\mathbb{R}^2)$ with the property $f \circ w_k = -f$ for any reflection $w_k \in W_N$, and this space is dense in $\mathcal{L}^\perp(\Sigma_N)$.*

By duality, $\mathcal{L}(\Sigma_N)$ consists of all functions $g \in C(\mathbb{R}^n)$ for which $\langle g, f\rangle = 0$, where $f \in C_c(\mathbb{R}^2)$ is W_N-odd.

It remains for us to concretely describe $\ker_{\Sigma_N} R$ and $\mathcal{L}(\Sigma_N)$. The appropriate tool is the Fourier series.

Theorem 6.5.

(1) $\ker_{\Sigma_N} R$ *consist of all $f \in C_c(\mathbb{R}^2)$ with "sparse" trigonometric Fourier series:*

$$f(r,\theta) = \sum_{m=1}^{\infty} b_m(r) \sin mN\theta$$

(2) *Correspondingly, the space* closure $\mathcal{L}(\Sigma_N)$ *consists of all* $g \in C(\mathbb{R}^2)$ *with trigonometric Fourier series:*

$$g(r,\theta) = \sum_{m=0}^{\infty} a_m(r)\cos\ m\theta + \sum_{m\in\mathbb{N}\setminus N\mathbb{N}} b_m(r)\sin\ m\theta.$$

Proof. First we prove (1). Let $f \in \ker_{\Sigma_N} R$ and expand f in a trigonometric Fourier series. Now use Proposition 6.4 ($f(r, \frac{k\pi}{N} - \theta) = -f(r, \frac{k\pi}{N} + \theta)\ \forall\ k \in \mathbb{N}$) and the independence of the trigonometric Fourier system to show the Fourier series of f is of the given form. The other direction is essentially covered by Lemma 5.1.

Since the trigonometric Fourier system is orthogonal and complete and $\ker_{\Sigma_N} R$ is dense in $\mathcal{L}^{\perp}(\Sigma_N)$ (by Lemma 6.1 and formula (6.2)), the second statement follows from the first one by orthogonality arguments. □

Remark. (1) Each system S of lines through the origin with angles between lines that are rational multiples of π, can be embedded in some Coxeter system Σ_N. Therefore, the space $\mathcal{L}(S)$, generated by radial functions with centers on S, is not dense in $C(\mathbb{R}^2)$. The closure of this space can be described by the conditions on Fourier coefficients in Theorem 6.5. On the other hand, if the angle between two lines in S is an irrational multiple of π, the $\mathcal{L}(S)$ is dense in $C(S)$. It can be seen from Theorem 6.5 that it is related to the impossibility of arranging corresponding lacunas in the Fourier series. In terms of the reflection group, the denseness of $\mathcal{L}(S)$ in this case can be explained by the infiniteness of the reflection group and the denseness of its orbits.

(2) The Coxeter system Σ_N can be defined as the system of lines through the origin; each line intersects the unit circle at a zero of the N^{th} spherical harmonic $\sin N\theta$. Thus, the condition (*) in Theorems A and B can be reworded as follows:

(6.3) *S is not contained in any set of the form $\omega(V) \cup F$, where $\omega \in \mathrm{M}(2)$, V is the zero set of some nonzero homogeneous harmonic polynomial, F is a finite set.*

We will see in the next section that the formulation (6.3) seems to be more suitable for the generalization of our results to higher dimensions, so perhaps, the whole problem can be viewed as a problem in harmonic analysis.

§7. The case $\mathbb{R}^n$ $(n > 2)$. Necessary conditions for injectivity of the Radon transform on spheres.

To motivate our theorems in $\mathbb{R}^n$, we first examine the analogous problem on the compact space S^{n-1}. E. Quinto and L. Zalcman [Z4] noted the following (see also [Sc, U]):

Theorem 7.1. *Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and let R be the spherical Radon transform on S^{n-1} :*

$$Rf(x,r) = \int_{S(x,r)} f dA, \quad f \in C(S^{n-1}),$$

where $S(x,r)$ denotes the geodesic sphere in S^{n-1} of radius r, centered at $x \in S^{n-1}$, dA is the area measure on $S(x,r)$.

Then $Rf(x,r) = 0$ for $x \in S \subset S^{n-1}$ and all $r > 0$ implies $f = 0$ if and only if S is not a subset of the zero set of some spherical harmonic $h \in \mathcal{H}^k$.

The proof uses harmonic analysis (the Funke-Hecke theorem) on S^{n-1} and is quite transparent. The reasons that the analogous problem becomes much more difficult on $\mathbb{R}^n$ than on S^{n-1} could have to do with the more difficult Fourier analysis on $\mathbb{R}^n$. It is interesting that the conditions of injectivity in Theorem 7.1 and in Theorem B are almost the same (see (6.3)). The only difference is that in the noncompact case, according to Theorem B, there is the additional freedom of applying rigid motions and adding finite sets. Moreover, this observation essentially leads to necessary condition for injectivity of the Radon transform $Rf(x,r)$ in $\mathbb{R}^n$:

Theorem 7.2. *If the transform $Rf(x,r)$ is injective on a set $S \subset \mathbb{R}^n$ for $f \in C_c(\mathbb{R}^n)$, then S is not contained in any algebraic variety $V[h_a]$,where $a \in \mathbb{R}^n$ and h is a nonzero homogeneous harmonic polynomial.*

Proof. It suffices to consider the case $a = 0$. Suppose $S \subset V[h]$, where h is a nonzero homogeneous harmonic polynomial and $V[h] = \{x \in \mathbb{R}^n \mid h(x) = 0\}$. Denote $\varphi = h\,\big|_{S^{n-1}}$, the corresponding spherical harmonic $\varphi \in \mathcal{H}^N$, $N = \deg h$. The set $V[h]$ is the conical set determined by the zero set of φ on S^{n-1}.

Let us define the measure μ in $C'(\mathbb{R}^n)$ by formula

$$\int_{\mathbb{R}^n} g d\mu = \int_{S^{n-1}} g\varphi dA, \qquad g \in C(\mathbb{R}^n),$$

where dA is the normalized area measure on S^{n-1}.

Pick $x \in \mathbb{R}^n \setminus 0$ and denote by $SO(x,n)$ the subgroup of the special orthogonal group, $SO(n)$, which preserves the point $e_x = \frac{x}{\|x\|} \in S^{n-1}$. Then $SO(x,n)$ leaves fixed the line $\{tx \mid t \in \mathbb{R}\}$, and for any radial function $\alpha \in C^{\#}(\mathbb{R}^n)$ and any $k \in SO(x,n)$, we have

$$\begin{aligned}(\alpha * \mu)(x) &= \int_{\mathbb{R}^n} \alpha(x-\xi)d\mu(\xi) = \int_{S^{n-1}} \alpha(x-\xi)\varphi(\xi)dA(\xi) \qquad (7.1)\\ &= \int_{S^{n-1}} \alpha(x-k\xi)\varphi(k\xi)dA(\xi) = \int_{S^{n-1}} \alpha(x-\xi)\varphi(k\xi)dA(\xi).\end{aligned}$$

We have used the $SO(x,n)$-invariance of α and the fact that $k^{-1}x = x$. Integrating over $k \in SO(x,n)$ yields

$$(\alpha * \mu)\ (x) = \int_{S^{n-1}} \alpha(x-\xi)\ \tilde{\varphi}(\xi)\ dA(\xi),$$

where $\tilde{\varphi}(\xi) = \int\limits_{O(x,n)} \varphi(k\xi)dk$.

The $SO(x,n)$-invariant function $\tilde{\varphi} \in \mathcal{H}^N$, coincides, up to a constant factor, with the zonal spherical function $Y_x \in \mathcal{H}^N$: $\tilde{\varphi}(\xi) = cY_x(\xi)$.

Now let $x \in V[h]$, $x = \|x\|e_x$, $e_x \in S^{n-1}$. Then, $\tilde{\varphi}(e_x) = \varphi(e_x) = \frac{1}{\|x\|^N}\ h(x) = 0$. Since $Y_x(e_x) = \dim \mathcal{H}^N/\text{area } S^{n-1} \neq 0$ ([SW, Ch. 6, §2]), it follows that $c = 0$ and $\tilde{\varphi}(\xi) \equiv 0$. Then (7.1) yields:

$$(\alpha * \mu)\ (x) = 0 \qquad \text{for} \qquad x \in V[h].$$

This means $\mu \in \mathcal{L}^{\perp}(S)$. Since $\mu \neq 0$ and $\ker_S R$ is dense in $\mathcal{L}^{\perp}(S)$ (see 6.1) we conclude that $\ker_S R \neq 0$ and, therefore, R is not injective on S. □

Remark. Theorem 7.2 shows that when $n > 2$, the correct analog of Coxeter systems Σ_N must be cones $V[h]$ rather than systems of hyperplanes, as could be expected.

§8. Applications and interpretations of Theorems A and B.

8.1 Uniqueness theorem for the Darboux equation. Let us consider the ultrahyperbolic Darboux equation

$$\Delta_x u(x,y) = \Delta_y u(x,y);$$

where $u \in C^2(\mathbb{R}^2 \times \mathbb{R}^2)$, Δ is the Laplace operator in $\mathbb{R}^2$ and u depends only on $r = \|x\|$. Only the radial part of Δ_x operates in the x-variable so the equation can be written in the form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\ \frac{\partial u}{\partial r} = \Delta_y u. \tag{8.1}$$

If u is the spherical means of some function $f \in C(\mathbb{R}^2)$:

$$u(r,y) = \int_{C(y,r)} f dA,$$

which is, in our previous notation, $u(r,y) = Rf(y,r)$, then u satisfies (8.1) (*cf.* [H1, Lemma 2.14]) and $u(0,y) = f(y)$.

By a theorem due to L. Asgeirsson (*cf.* [H1, Ch. II, §6]), any solution u of (8.1) is the spherical mean of the function $f(y) = u(0,y)$. Theorem B immediately implies the following uniqueness theorem for the Darboux equation:

Proposition 8.1. *The equation (8.1) with data*

(i) *$u(0, y)$ has compact support;*
(ii) *$u(r, y) = 0$ for $(r, y) \in \mathbb{R}_+ \times S$*

has the unique solution $u = 0$ as long as S is not contained in a set $\omega(\Sigma_N) \cup F$ in Theorem B.

Thus, the solution of (8.1) with data given on a cylindrical set $\mathbb{R}_+ \times S \subset \mathbb{R}^3$ has a unique solution, unless this set is the union of a special system $\mathbb{R}_+ \times \omega(\Sigma_N)$ of planes in $\mathbb{R}^3$ and of a finite number of "vertical" lines $\mathbb{R}_+ \times \{a\},\ a \in F$.

8.2 Zero temperature sets in the 2-dimensional heat equation and nodal sets of oscillating membranes. Let $T > 0$. Let us consider the heat equation initial value problem in $\mathbb{R}^2$:

$$\begin{gathered} \frac{\partial u}{\partial t} = c^2 \Delta u, \quad u = u(x,t) \ : \ x \in \mathbb{R}^2,\ t \in [0,T] \\ u(x,0) = f(x) \end{gathered} \tag{8.2}$$

The solution of (8.2) is given by the convolution with the heat kernel:

$$u(x,t) = \frac{1}{4\pi c^2 t} \int_{\mathbb{R}^2} e^{-\frac{\|x-\xi\|^2}{4c^2 t}} f(\xi) d\xi. \tag{8.3}$$

Let $Z(f)$ be the set in the plane where the temperature is zero for all time t:

$$Z(f) = \{x \in \mathbb{R}^2 \ \big| \ u(x,t) = 0 \quad \forall t \in [0,T]\}.$$

The Taylor expansion of the kernel of the integral in (8.3) yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{c^{2k} k!}\ t^{-k} \int_{\mathbb{R}^2} \|x - \xi\|^{2k} f(\xi) d\zeta = 0, \quad \forall\ x \in Z(f),\ \forall\ t \in (0,T)$$

and, therefore, $Z(f) = \bigcap\limits_{k=0}^{\infty} V[Q_k] = S[f]$ where Q_k and $S[f]$ are defined in §2.

Thus, by Theorem B', we obtain the following fact which seems to be quite interesting:

Proposition 8.2. *If the initial distribution $f(x), x \in \mathbb{R}^2$, is compactly supported, then only two types of zero temperature sets $Z(f)$ are possible, either a very large set or a very special one:*

(1) *$Z(f) = \mathbb{R}^2$, which means $f = 0$,*
(2) *$Z(f)$ is a Coxeter system of lines $\omega(\Sigma_M), \omega \in \mathrm{M}(2)$ union with a number of isolated points.*

This leads to the following amusing fact (for compactly supported initial distributions). It is impossible to have temperature zero all the time on any nonlinear smooth curve unless the temperature is zero everywhere and all the time.

Similar corollaries are true for other differential equations with radial source functions, say, for the wave equation. Theorem B' implies:

Proposition 8.3. *Consider the Cauchy problem for the membrane equation*

$$\frac{\partial^2}{\partial t^2}u = a^2\Delta u,$$
$$u(x,0) = 0$$
$$u_t(x,0) = f(x),$$

where the initial velocity f is in $C_c(\mathbb{R}^2)$. Let $N[f]$ be the nodal set, that is the set of all $x \in \mathbb{R}^2$ for which $u(x,t) = 0$ for all time t.

Then $N[f]$ is a set of the form $\omega(\Sigma_M) \cup F$, for some $M = 0, 1\ldots$ and where F is a finite set, unless $f = 0$, i.e. the membrane does not move at all.

The proof follows from the Poisson-Kirchoff formula:

$$u(x,t) = \frac{1}{at}\int_{\|x-\xi\|\le at} \frac{f(\xi)d\xi}{(a^2t^2 - \|x-\xi\|^2)^{\frac{1}{2}}} = \frac{1}{at}\int_0^{at} \frac{(Rf)(x,r)r\,dr}{(a^2t^2-r^2)^{\frac{1}{2}}}$$

The last identity in this formula is obtained by a polar change of variables at the point x. We see that $u(x,t)$ is related to $Rf(x,r)$ by an (invertible) Abel integral equation and therefore zeros of u (for all time) coincide with zeros of Rf , $N[f] = S[f]$.

For instance, an oscillating membrane with compactly supported initial velocity cannot remain stationary on a small smooth curve which is not a segment of line. Earlier P. Kuchment found a proof of this statement in the special case when the curve is closed. His proof is independent of ours and is valid also for the wave equation in Euclidean spaces of arbitrary dimension with a closed surface instead of a closed curve. His proof shows the equivalence of the PDE problem to our problem. His result follows from Corollary 3.2, since closed surfaces are uniqueness sets for harmonic polynomials.

Remark. The arguments above show that the assertions of Proposition 8.3, as well as Proposition 8.2, are equivalent to Theorems A and B. These propositions can be understood also as results about smooth extendibility across sets (curves) of solutions of parabolic and hyperbolic equations. For instance, Proposition 8.2 gives the following symmetry principle: Let S be a simple smooth curve dividing the plane into two parts D^+ and D^- and let u^+ and u^- be solutions of the heat equation in the corresponding parts with zero boundary data on S (and compactly supported initial data). Then u^+ and u^- are restrictions of a solution u of the heat equation in the whole plane if and only if:

(1) S is a line and

(2) u^+ and u^- are skew-symmetric to each other under reflection in the line S.

The analogous statement is true for the wave (membrane) equation.

8.3 Riesz Potentials. In a similar way, the heat kernel in (8.3) can be replaced by any function which generates "radial monomials" r^{2k}. For instance, we can take the Riesz potentials

$$I_\lambda(x) = \int_{\mathbb{R}^2} \frac{f(\xi)d\zeta}{\|x-\xi\|^\lambda} \quad , \quad \lambda < 2 \tag{8.4}$$

and obtain the following statement in a similar way.

Proposition 8.4. *Any function $f \in C_c(\mathbb{R}^2)$ is uniquely determined by the values $I_\lambda(x)$ of its Riesz potentials on a set S, where λ is in an open interval $\lambda \in (a,b)$, unless S is the subset of some $\omega(\Sigma_N) \cup F$, $\omega \in \mathrm{M}(2)$, F is finite.*

For instance, any nonlinear arc S provides uniqueness in the above sense. The proof of this theorem rests on the fact that $\{r^{\lambda-1} \mid \lambda \in (a,b)\}$ is dense in $L^1([0,R])$ for any $R > 0$.

8.4 Charges on balls. By duality arguments, the statements of Theorems A and B can be reformulated in terms of recovering measures from charges on balls (disks).

Take $\mu \in C'(\mathbb{R}^n)$. Suppose we have a family of balls $\mathcal{B}_S = \{B(a,r) \mid a \in S, r > 0\}$. Which families $\mathcal{B}_S$ are large enough to determine μ by knowledge of the charges, $\mu(B)$, $\forall B \in \mathcal{B}_S$? Theorem B gives the full answer to this question for $n = 2$.

There are many interesting results on the problem of recovering measures from their values on balls in metric or Banach spaces and related questions (Hoffman-Jorgensen [H-J], Davies [Da], Gorin and Koldobskii [GK], Preis and Tiser [PT], Zalcman [Z3] (see also the references in [Z3]). These results show that on some infinite dimensional spaces, μ can be recovered from knowledge of charges on all balls but on others, it cannot be recovered from this information. Obviously, in the finite dimensional case it does, and Theorem B shows that only special (thin enough) families of balls (disks) of arbitrary radii fails to determine compactly supported measures in the plane.

§9. Open problems.

9.1 Generalization to higher dimensions. Theorem B states that in $\mathbb{R}^2$ the circular Radon transform, R, is injective on S if and only if S is not contained in the zero set of a nonzero homogeneous harmonic polynomial union a finite set (see (6.3)). We already know by Corollary 2.3 that if $S \subset \mathbb{R}^n$ is not contained in the zero set of a nontrivial harmonic polynomial, then R is injective on S. This plus the necessary conditions, given by Theorem 7.2, enable us to conjecture the solution Problem 1 in the general case.

Conjecture. *Let $S \subset \mathbb{R}^n$. The space $\mathcal{L}(S)$ of radial functions (defined in §1) is dense in $C(\mathbb{R}^n)$ (and S is a set of injectivity for the transform R) if and only if the set S is not included in any algebraic variety $V[h_a] \cup F$ where $a \in \mathbb{R}^n$, h is a nonzero homogeneous harmonic polynomial, and F is an algebraic variety of codimension ≥ 2.*

We think that our method of proving Theorem B also works in the multi-dimensional case and we are going to return to this problem in the future.

Remark. For $S = V[h]$, the space $\mathcal{L}(S)$ can be described (for $n = 2$, it is done in 6.2). Any $f \in C(\mathbb{R}^n)$ can be decomposed in the series corresponding to $O(n)$-irreducible subspaces:

$$f(x) = \sum_{k=0}^{\infty} f_k(r, \theta), \quad x = r\theta, \quad r \in \mathbb{R}_+, \ \theta \in S^{n-1}, \tag{9.1}$$

where $f_k(r, \cdot) \in \mathcal{H}^k(S^{n-1})$ - the space of spherical harmonics of degree k.

Then the space $\mathcal{L}(S)$ consists of all $f \in C(\mathbb{R}^n)$ such that $f_k(r, \theta) = 0$ if $h(\theta) = 0, \quad \theta \in S^{n-1}$. □

9.2 Other nonlinear Radon transforms. L. Ehrenpreis has formulated in the manuscript of his book [E, Theorem 5.5, Remark 4] the problem of injectivity for the following nonlinear Radon transform.

Let p be a nonzero homogeneous polynomial in $\mathbb{R}^n$ and

$$R_p f(x, t) = \int_{p(\xi - x) = t} f d\nu, \quad f \in C_c(\mathbb{R}^n), \tag{9.2}$$

with appropriate measure ν. This is a Radon transform on algebraic varieties $V_{x,t} = \{x \in \mathbb{R}^n \mid p(\xi - x) = t\}$; the case $\deg p = 1$ corresponds to the classical Radon transform.

The simplest nonlinear case $p(x) = x_1^2 + \cdots + x_n^2$ corresponds to the spherical Radon transform considered in this paper.

There are two natural types of families $V_{x,t}$ which involve

(1) many centers x and few radii $r = t^2$,
(2) few centers x and many radii $r = t^2$.

The problem of uniqueness for R_p in case (1) goes back to Delsarte [DL] and has been well investigated [Z2, BG, BZ1, BZ2, A, ABCP] and even on Riemannian manifolds [Q2]. However, case (2) has been much less well studied. Theorem B, of this paper, gives the solution to the problem in [E] for case (2) when $n = 2$ and $p(x_1, x_2) = x_1^2 + x_2^2$. Some other support theorems for case (2) that are valid for functions not of compact support are given in [Q3].

It is an open problem to find families of uniqueness for the transform R_p for other polynomials p of two variables.

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We had many fruitful discussions on the subject with Lawrence Zalcman, who also made us aware of his results in [Z4].

The reduction to the case of algebraic curves and then harmonic curves, which plays an important role in the sufficiency part of the proof of Theorem B, had earlier been used for a partial solution to this problem by V. Ya. Lin and N. Zobin. Lin also originally conjectured Theorem A.

The first author discussed some alternative approaches to the problem with N. Zobin, V. Ya. Lin. V. Volchkov. and P. Kuchment. Though these approaches did not lead to the final solution, the discussions were very stimulating and useful in understanding the difficulties.

We also benefited from conversations on the problem with T. Bandman, H.S. Shapiro, B. Korenblum, C. Berenstein, and E.L. Grinberg. E.A. Gorin brought to our attention the papers [Da, H-J, PT].

As this paper was being written, Allan Pinkus made us aware of his paper with Vladimir Lin [LP] in which this problem is mentioned.

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