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LOCAL SOBOLEV ESTIMATES OF A FUNCTION BY MEANS OF ITS RADON TRANSFORM

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This article is dedicated to our mentor, colleague, and friend, Jan Boman, on the occasion of his seventy-fifth birthday.

Abstract. In this article, we will define local and microlocal Sobolev seminorms and prove local and microlocal inverse continuity estimates for the Radon hyperplane transform in these seminorms. The relation between the Sobolev wavefront set of a function f and of its Radon transform is well-known [18]. However, Sobolev wavefront is qualitative and therefore the relation in [18] is qualitative. Our results will make the relation between singularities of a function and those of its Radon transform quantitative. This could be important for practical applications, such as tomography, in which the data are smooth but can have large derivatives.

1. **Introduction.** The Radon transform that integrates over lines in the plane is the fundamental transform in X-ray tomography, and there are many good inversion methods if all X-ray CT data are given [15, 16]. Inversion is fairly stable; it is like taking 1/2 derivative of the data. However, many tomography problems, use *limited data*. That is, integrals over some lines are missing. In certain cases, one can recover the function from this limited data, but inversion is most often highly unstable. In other cases, the limited data transform is not even invertible. These observations motivate the questions how does one measure this instability and what one can determine from this limited data, if inversion is not possible. We will provide quantitative answers in this article.

Qualitative measures of instability already exist. Quinto used microlocal analysis to develop a qualitative relationship between singularities of a function f and of its Radon transform Rf [18]. This qualitative relationship is developed using the concept of Sobolev wavefront set, which we will discuss in Section 2. Roughly speaking, f has a singularity of order α at a certain point (and direction) if and only if Rf has a singularity of order $\alpha + 1/2$ at a corresponding location. However, the

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authors know of no general quantitative estimates giving the relationship between the strength (in seminorms) of the corresponding singularities. Global quantitative estimates (global continuity and inverse continuity of R) are well-known and will be discussed at the start of Section 3.

In this article, we will develop seminorms to measure this instability; the seminorms will measure directional strengths of singularities. We will measure *global* instability using Sobolev norms, (2), a measurement of derivatives of f in L^2, that make sense even when s is not a natural number. In Section 2 we will define local versions as well as microlocal versions.

Then, we will address what can be stably determined from Radon data using limited data. As noted above, classical microlocal analysis provides a qualitative correspondence, which is defined using Sobolev wavefront sets [18], (see Definition 2.4). After developing the seminorms, we will prove theorems that give quantitative bounds between the singularities of f and its Radon transform using these seminorms.

Authors have used functional analytic ideas to measure this relationship between singularities of a function and its Radon transform. Ramm and Zaslavsky developed results for functions that are piecewise smooth with discontinuities on smooth curves [21]. Candès and Donoho developed curvelets and ridgelets and explored their properties under the Radon transform [3]. Candès and Denament [2] used ridgelets and curvelets to define wavefront sets, and it is possible these can be used to develop quantitative estimates. Some years ago, Guillemin [8, 9] developed a theory of Fourier integral operators and wavefront sets based on Radon transforms, and as noted above, Quinto [18] first applied these results to singularity detection in tomography, but these results are not quantitative. Since then, many other authors have used microlocal analysis to understand limited data problems. Greenleaf and Uhlmann have developed beautiful results for admissible line complexes [5, 6, 7], an integral geometric framework that applies to important limited data problems. Then Finch *et al.* [4] analyzed the microlocal analysis for a specific admissible line complex, cone beam CT, and Katsevich [13, 12], Anastasio, *et al.* [1] and Ye *et al.* [25] used microlocal analysis to motivate better reconstruction methods for local cone beam CT. Quinto *et al.* developed a local reconstruction algorithm and for slant hole SPECT analyzed its microlocal properties [19] and electron microscopy [20].

We should point out that our results are a first step in making microlocal results quantitatively applicable. Real data is finite and so it can be considered to be approximated by a smooth function. The data can have large derivatives that would be in all Sobolev spaces, therefore not distinguishable by qualitative microlocal results. However, quantitative microlocal estimates could detect large derivatives, even if the function were smooth.

In section 2 we will define the qualitative notion of Sobolev wavefront set and define local seminorms that are related to this idea but are quantitative. In Section 3, we will present our main results, and in Section 4 we will give properties of these seminorms that will provide perspective on them and the results. Finally, in Section 5, we will present our proofs.

2. **Notation.**

We define our seminorms in this section, and we will compare them to the standard microlocal concept of singularity, Sobolev wavefront set. This will help understand the process getting from the qualitative idea of wavefront set to our more quantitative ideas. We will go successively from global ideas to local ideas to microlocal ideas.

Our seminorms are defined in terms of the Fourier transform. For a function f defined in $\mathbb{R}^n$, we define the Fourier transform of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}dx \tag{1}$$

and for any real number s we define the Sobolev s-norm of f by

$$\|f\|_s = \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi\right)^{1/2}. \tag{2}$$

The key to our results is that one can understand smoothness of a function from integrability properties of its Fourier transform. The simplest such result is the well-known result that if $\hat{f}$ is in L^1, then f is continuous.

We denote the Sobolev space of all distributions f such that $\hat{f}$ is a function and $\|f\|_s < \infty$ by $H^s(\mathbb{R}^n)$. If $s \in \mathbb{N}$ then a distribution in $H^s(\mathbb{R}^n)$ is a function for which distributional derivatives up to order s are in $L^2(\mathbb{R}^n)$ [24].

Sobolev norms are global, and one often wants to know local behavior of f. The easiest way to localize is to multiply by a cutoff function. Let $x_0 \in \mathbb{R}^n$ and let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then φ *is a cutoff function near* x_0 if $\varphi(x_0) \neq 0$.

Definition 2.1. Let f be a distribution. We say f *is locally in* H^s *near* x_0 if there is a cutoff function, φ, near x_0 such that $\varphi f \in H^s(\mathbb{R}^n)$.

The only problem with Definition 2.1 is that it provides qualitative information about f but not quantitative information. Now, we provide a way to *quantify* local Sobolev smoothness that is related to $\|f\|_s$ when f is a function defined only in Ω.

Definition 2.2 ([24, Section 4.2.1])**.** Let Ω be an open subset of $\mathbb{R}^n$ and let $s \in \mathbb{R}$. We define *the Sobolev space* $H^s(\Omega)$ to be the set of distributions on Ω that are restrictions to Ω of distributions in $H^s(\mathbb{R}^n)$. For $f \in H^s(\Omega)$, we define

$$\|f\|_{\Omega,s} = \inf\{\|\tilde{f}\|_s : \tilde{f} \in H^s(\mathbb{R}^n), \tilde{f}|_\Omega = f\}. \tag{3}$$

Now, we consider directions in which f is in H^s, and this leads to the following definition.

Definition 2.3. Let V be a measurable subset of $\mathbb{R}^n$ and f be a distribution with locally square-integrable Fourier transform. We define

$$\|f\|_{V,s} = \left(\int_V |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi\right)^{1/2}. \tag{4}$$

Note that $\|f\|_{V,s}$ is just a seminorm, since the Fourier transform of f could vanish everywhere in V even if f is not identically 0. In our applications, V will be an open cone. Such directional seminorms are the basis of the definition of Sobolev wavefront set.

Definition 2.4 ([17, p. 259])**.** Let f be a distribution and $s \in \mathbb{R}$. Let $x_0 \in \mathbb{R}^n$ and $\xi_0 \in \mathbb{R}^n \setminus 0$. We say f *is in* H^s *near* x_0 *in direction* ξ_0 if there is a cutoff function, φ near x_0 and an open, conic neighborhood V of ξ_0 such that $\|\varphi f\|_{s,V} < \infty$.

On the other hand, we say $(x_0, \xi_0) \in \mathrm{WF}^s(f)$ if f is not in H^s near x_0 in direction ξ_0.

This definition combines the local idea, Definition 2.1 with the microlocal idea, Definition 2.3, and it provides a qualitative way to decide local Sobolev smoothness in directions. However, Sobolev wavefront set is not quantitative. We now define quantitative seminorms that provide this quantitative measure.

Definition 2.5. Let s and s' be real numbers with $s > s'$. Let Ω be an open subset of $\mathbb{R}^n$ and let V be an open cone in $\mathbb{R}^n$. Let f be a distribution with locally square integrable Fourier transform. We define the seminorm

$$\|f\|_{\Omega,V,s,s'} = \inf\{(\|\tilde{f}\|_{V,s}^2 + \|\tilde{f}\|_{s'}^2)^{1/2} : \tilde{f} \in H^{s'}(\mathbb{R}^n), \tilde{f}|_\Omega = f\}. \tag{5}$$

In Section 4.1 we show that, without the s' norm, (5) would be trivial.

We should point out that Sobolev wavefront sets and inhomogeneous Sobolev spaces, such as those defined by the seminorm (5), have a rich history. The first definition in [22] is essentially the Sobolev space defined by (5), and the presence of s' helps the authors prove propagation of singularities for hyperbolic pseudodifferential operators [22, 23]. Similar results are valid for Fourier integral operators.

To state the main results, we need some more notation. For an integrable function f defined on $\mathbb{R}^n$ and decaying sufficiently fast, let Rf denote the Radon transform of f, defined on $S^{n-1} \times \mathbb{R}$ by the relation

$$Rf(\omega, p) = \int_{\omega\cdot x = p} f(x)dx \tag{6}$$

where dx now denotes $(n-1)$-dimensional Lebesgue measure on the hyperplane $\{x : \omega \cdot x = p\}$. Let R^* denote the adjoint of the Radon transform, mapping functions on $S^{n-1} \times \mathbb{R}$ to functions on $\mathbb{R}^n$:

$$R^*u(x) = \int_{S^{n-1}} u(\omega, \omega \cdot x)d\omega. \tag{7}$$

For an integrable function u on $S^{n-1} \times \mathbb{R}^n$, define the Fourier transform of u with respect to the second variable

$$\hat{u}(\omega, \sigma) = \int_{\mathbb{R}} u(\omega, p)e^{-ip\sigma}dp \tag{8}$$

and accordingly, if $\hat{u}$ is locally integrable, define the Sobolev norm

$$\|u\|_s = \left(\int_{S^{n-1}\times\mathbb{R}} |\hat{u}(\omega, \sigma)|^2(1 + \sigma^2)^s d\omega d\sigma\right)^{1/2}. \tag{9}$$

These last two definitions naturally extend to $\mathcal{S}'(S^{n-1} \times \mathbb{R})$.

Let $H^{0,s}(S^{n-1} \times \mathbb{R})$ denote the space of distributions u on $S^{n-1} \times \mathbb{R}$ with $\|u\|_s < \infty$. (The zero indicates that we are using L^2 norm in the ω variable.) For an open subset $\Lambda \subset S^{n-1} \times \mathbb{R}$, the local Sobolev norm $\|u\|_{\Lambda,s}$ is defined analogously to Definition 2.2 and $H^{0,s}(\Lambda)$ denotes the spaces of restrictions of $H^{0,s}(S^{n-1} \times \mathbb{R})$ to Λ. (We will not need a definition of directional Sobolev seminorms on $S^{n-1} \times \mathbb{R}$.)

3. **Results.** In this section, we will provide our local and microlocal Sobolev inverse continuity results for the Radon transform. It is well known that R is an elliptic Fourier integral operator of order $-(n-1)/2$ and its inverse is of order $(n-1)/2$ [9]. Global Sobolev continuity and inverse continuity are also well known. If f has compact support then there is a constant C depending only on n and the size of the support of f such that $(1/C)\|f\|_s \leq \|Rf\|_{0,s+(n-1)/2} \leq C\|f\|_s$ [15, 10, 11, 14]. As expected, this factor of $(n-1)/2$ will appear in our microlocal estimates.

Definition 3.1. If Ω is an open subset of $\mathbb{R}^n$, let Ω' denote the set of all (ω, p) corresponding to hyperplanes intersecting Ω. If $\epsilon > 0$, let Ω'_ϵ denote the set of (ω, p) corresponding to hyperplanes passing within a distance ϵ from Ω.

Note that if f is a distribution of compact support then $\widehat{Rf}$ is a locally integrable function by the Projection Slice Theorem (e.g., [16]).

Theorem 3.2. *Suppose n is an odd integer, $n > 1$, s is a real number, and Ω is a bounded subset of $\mathbb{R}^n$. Let f be a distribution of compact support. Then*

$$\|f\|_{\Omega,s} \leq \frac{1}{\sqrt{2}}\|Rf\|_{\Omega',s+(n-1)/2}. \tag{10}$$

Theorem 3.3. *Suppose n is an even positive integer, s and s' are real numbers, ϵ is a positive real number, and Ω is a bounded subset of $\mathbb{R}^n$. Let f be a distribution of compact support. Then there are constants C_n and $C'_{n,s,s',\epsilon}$ such that*

$$\|f\|_{\Omega,s} \leq C_n\|Rf\|_{\Omega'_\epsilon,s+(n-1)/2} + C'_{n,s,s',\epsilon}\|f\|_{s'}. \tag{11}$$

In the following two theorems we assume that V is a symmetric open cone. By this we mean that V is an open cone in $\mathbb{R}^n$ such that $\xi \in V$ if and only if $-\xi \in V$.

Theorem 3.4. *Suppose n is odd, $n > 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$ and let V be a symmetric open cone in $\mathbb{R}^n$. Let $s > s'$ be real numbers. Let f be a distribution of compact support. Then*

$$\|f\|_{\Omega,V,s,s'} \leq \|Rf\|_{\Omega'\cap(V\times\mathbb{R}),s+(n-1)/2} + \|f\|_{s'}. \tag{12}$$

Theorem 3.5. *Suppose n is even. Let Ω be a bounded open subset of $\mathbb{R}^n$ and let V be a symmetric open cone in $\mathbb{R}^n$. Let $s > s'$ be real numbers and $\epsilon > 0$. Let f be a distribution of compact support. Then there are constants C_n and $C'_{n,s,s',\epsilon}$ such that*

$$\|f\|_{\Omega,V,s,s'} \leq C_n\|Rf\|_{\Omega'_\epsilon\cap(V\times\mathbb{R}),s+(n-1)/2} + C'_{n,s,s',\epsilon}\|f\|_{s'}. \tag{13}$$

These theorems make quantitative the microlocal Sobolev results in [18] in the following way. To see if a singularity at (x_0, ξ_0) is large, one chooses a small neighborhood, Ω of x_0 and a small open cone V containing ξ_0. Then, by the results of Theorem 3.4 or 3.5, one can bound the seminorm $\|f\|_{\Omega,V,s,s'}$ in terms of local bounds of Rf in hyperplanes meeting (or, for even dimensions, near) Ω and in directions in V. Thus, our results bound this microlocal seminorm of f by a local seminorm of Rf (plus, for even dimensions, a lower Sobolev norm of f).

4. Properties of local Sobolev norms.

We now examine in more detail the Sobolev seminorms of Section 2.

4.1. The role of s' in the definition of $\|f\|_{\Omega,V,s,s'}$.

A more natural and apparently plausible definition for a local and directional Sobolev norm would be

$$\|f\|_{\Omega,V,s} = \inf\{\|\tilde{f}\|_{V,s} : \tilde{f}|_\Omega = f\}.$$

The problem with this definition is that for many Ω and V, $\|f\|_{\Omega,V,s}$ would be 0 for all f. For example, if Ω is bounded and the complement of V has non-empty interior, then $\|f\|_{\Omega,V,s} = 0$ for all $f \in H^s(\Omega)$. Here is a proof of that statement.

Note first that an equivalent statement is that

$$F(\Omega, V) = \{\hat{g}|_V : g \in H^s(\mathbb{R}^n), g|_\Omega = 0\}$$

is dense in the weighted L^2 space $L^2(V,(1+|\xi|^2)^s)$. Suppose now that $\phi \in L^2(V,(1+|\xi|^2)^{-s})$ is orthogonal to every function in $F(\Omega,V)$. We will show that $\phi=0$. We are assuming that

$$\int_V \psi(\xi)\overline{\phi(\xi)}d\xi = 0$$

for every $\psi \in F(\Omega,V)$. There is an $h \in H^{-s}(\mathbb{R}^n)$ with $\hat{h}(\xi)=\phi(\xi)$ for every $\xi \in V$ and $\hat{h}(\xi)=0$ for $\xi \notin V$, and it follows from Parseval's identity that

$$\int_{\mathbb{R}^n} g(x)\overline{h(x)}dx = 0$$

for every $g \in H^s(\mathbb{R}^n)$ vanishing on Ω. This implies that $\operatorname{supp} h \subset \overline{\Omega}$ and since Ω is bounded it follows that $\hat{h}$ is real analytic. Since $\hat{h}$ vanishes in the complement of V, which is assumed to have non-empty interior, it follows that $\hat{h}$ vanishes identically, which implies that $\phi=0$. This shows that $F(\Omega,V)$ is indeed dense in $L^2(V,(1+|\xi|^2)^s)$, and the statement is proved.

4.2. **Properties of the local and directional Sobolev norms.** Seeing how the attempted definition of $\|f\|_{\Omega,V,s}$ fails, it is natural to ask if our definitions of $\|f\|_{\Omega,s}$ and $\|f\|_{\Omega,V,s,s'}$ perform any better. In this section we shall give a few results which indicate that in fact they do.

Lemma 4.1. *The mapping $f \mapsto \|f\|_{\Omega,s}$ is a norm on $H^s(\Omega)$. In particular, $\|f\|_{\Omega,s}=0$ implies that $f=0$ as a distribution in Ω.*

Proof. This is a well-known result [24], but for completeness we present a simple argument. All the properties of a norm are straightforward to check, except that $\|f\|_{\Omega,s}=0$ implies that $f=0$ almost everywhere in Ω. Suppose that $\|f\|_{\Omega,s}=0$. Then there is a sequence of distributions $f_n \in H^s(\mathbb{R}^n)$ such that $f_n|_\Omega = f$ and $\|f_n\|_s \to 0$. If $s \geq 0$ it follows immediately that $f_n \to 0$ in $L^2(\mathbb{R}^n)$ and hence $f=0$. In case $s<0$ let ϕ be any compactly supported smooth function defined in $\mathbb{R}^n$. Since $\hat{\phi}$ decays rapidly at ∞ it follows then that $\phi * f_n \to 0$ in $L^2(\mathbb{R}^n)$, and hence $\phi * f = 0$ where it is defined, that is at a distance away from the boundary of Ω depending on the support of ϕ. By letting ϕ approximate the Dirac delta distribution, it follows that $f=0$ as a distribution in Ω. □

Since evidently $\|f\|_{\Omega,V,s,s'} \geq \|f\|_{\Omega,s'}$ we have the following consequence.

Corollary 4.2. *If $\|f\|_{\Omega,V,s,s'}=0$, then $f=0$ as a distribution in Ω.*

This is of course interesting, but one would like to say a bit more.

Lemma 4.3. *If K is a compact subset of Ω, there is a constant C depending only on Ω, K and s such that*

$$\|f\|_{\Omega,s} \leq \|f\|_s \leq C\|f\|_{\Omega,s} \tag{14}$$

for every $f \in H^s(\mathbb{R}^n)$ with $\operatorname{supp} f \subset K$.

Lemma 4.4. *If K is a compact subset of Ω, and U and V are open cones with $\overline{U} \subset V$, then there is a constant C depending only on Ω, K, U, V, s, and s' such that*

$$\|f\|^2_{\Omega,U,s,s'} \leq \|f\|^2_{U,s} + \|f\|^2_{s'} \leq C\|f\|^2_{\Omega,V,s,s'} \tag{15}$$

for every $f \in H^s(\mathbb{R}^n)$ with $\operatorname{supp} f \subset K$.

If $w(\xi)$ is any non-negative function defined on $\mathbb{R}^n$ we define

$$\|f\|_w = \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 w(\xi) d\xi \right)^{1/2}. \tag{16}$$

The proofs of Lemma 4.3 and Lemma 4.4 follow easily from the following inequality.

Lemma 4.5. *Let f, ϕ, w_1, and w_2 be functions defined on $\mathbb{R}^n$, w_1 and w_2 positive. Let*

$$C = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{\phi}(\xi)| \sup_{\eta \in \mathbb{R}^n} \left(\frac{w_2(\xi+\eta)}{w_1(\eta)} \right)^{1/2} d\xi \tag{17}$$

and assume that C is finite. Then

$$\|\phi f\|_{w_2} \leq C \|f\|_{w_1}. \tag{18}$$

Proof. We will use the fact that $\widehat{\phi f} = (2\pi)^{-n} \hat{\phi} * \hat{f}$. By using this and the convexity of the functional $f \mapsto \|f\|_{w_2}$ it follows that

$$\begin{aligned} \|\phi f\|_{w_2} &= \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} |\hat{\phi} * \hat{f}(\eta)|^2 w_2(\eta) d\eta \right)^{1/2} \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{\phi}(\xi)| \left(\int_{\mathbb{R}^n} |\hat{f}(\eta-\xi)|^2 w_2(\eta) d\eta \right)^{1/2} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{\phi}(\xi)| \left(\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 w_2(\xi+\eta) d\eta \right)^{1/2} d\xi. \end{aligned}$$

Now let

$$v(\xi) = \sup_{\eta \in \mathbb{R}^n} \frac{w_2(\xi+\eta)}{w_1(\eta)}.$$

Then it follows that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\hat{\phi}(\xi)| \left(\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 w_2(\xi+\eta) d\eta \right)^{1/2} d\xi \\ &\qquad \leq \int_{\mathbb{R}^n} |\hat{\phi}(\xi)| v(\xi)^{1/2} \left(\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 w_1(\eta) d\eta \right)^{1/2} d\xi \\ &\qquad = \int_{\mathbb{R}^n} |\hat{\phi}(\xi)| v(\xi)^{1/2} d\xi \|f\|_{w_1}. \end{aligned}$$

This proves the lemma. □

Proof of Lemma 4.3 and Lemma 4.4. The first inequality in both lemmas follows simply by taking $\tilde{f} = f$ in the definition of the norms. To prove the second inequality, let ϕ be a smooth function, equal to 1 in K and vanishing in the complement of Ω. If $\tilde{f}$ is an arbitrary function, equal to f in Ω, we then have $\phi \tilde{f} = f$.

To prove Lemma 4.3, take a function $\tilde{f}$ with $\|\tilde{f}\|_s$ close to $\|f\|_{\Omega,s}$ (say $\|\tilde{f}\|_s < 2\|f\|_{\Omega,s}$). Let $w_1(\xi) = w_2(\xi) = (1+|\xi|^2)^s$ and apply Lemma 4.5. Note that

$$\begin{aligned} \frac{1+|\xi+\eta|^2}{1+|\eta|^2} &\leq \frac{1+(|\xi|+|\eta|)^2}{1+|\eta|^2} = 1 + \frac{2|\xi||\eta| + |\xi|^2}{1+|\eta|^2} \\ &\leq 1 + |\xi| + |\xi|^2 \leq (1+|\xi|)^2. \end{aligned}$$

Replacing η by $\eta - \xi$ and ξ by $-\xi$ this implies that

$$\frac{1+|\eta|^2}{1+|\xi+\eta|^2} \leq (1+|\xi|)^2.$$

From one of these inequalities (since s can be either positive or negative) it follows that

$$\left(\frac{w_2(\xi+\eta)}{w_1(\eta)}\right)^{1/2} \le (1+|\xi|)^{|s|}$$

Since $\hat{\phi}(\xi)$ decays more rapidly at ∞ than any power of $|\xi|$, the constant C in Lemma 4.5 is finite, and it follows that

$$\|f\|_s = \|\phi\tilde{f}\|_{w_2} \le C\|\tilde{f}\|_{w_1} \le 2C\|f\|_{\Omega,s}.$$

This completes the proof of Lemma 4.3.

To prove Lemma 4.4 we use a similar argument. Take $\tilde{f}$ with $\|\tilde{f}\|^2_{V,s} + \|\tilde{f}\|^2_{s'} < 2\|f\|^2_{\Omega,V,s,s'}$ and define

$$w_1(\xi) = \begin{cases} (1+|\xi|^2)^s + (1+|\xi|^2)^{s'} & \xi \in V \\ (1+|\xi|^2)^{s'} & \xi \notin V \end{cases}$$

and

$$w_2(\xi) = \begin{cases} (1+|\xi|^2)^s + (1+|\xi|^2)^{s'} & \xi \in U \\ (1+|\xi|^2)^{s'} & \xi \notin U \end{cases}.$$

Again we want to apply Lemma 4.5, and must check that the constant is finite. Consider three different cases. First, if $\eta \notin V$ and $\xi+\eta \notin U$, then the same argument as in the proof of Lemma 4.3 shows that

$$\left(\frac{w_2(\xi+\eta)}{w_1(\eta)}\right)^{1/2} \le (1+|\xi|)^{|s'|}.$$

Second, if $\eta \in V$ and $\xi+\eta$ is arbitrary, then

$$\begin{aligned}\left(\frac{w_2(\xi+\eta)}{w_1(\eta)}\right)^{1/2} &\le \left(\frac{(1+|\xi+\eta|^2)^s + (1+|\xi+\eta|^2)^{s'}}{(1+|\eta|^2)^s + (1+|\eta|^2)^{s'}}\right)^{1/2} \\ &\le \max\left(\left(\frac{(1+|\xi+\eta|^2)^s}{(1+|\eta|^2)^s}\right)^{1/2}, \left(\frac{(1+|\xi+\eta|^2)^{s'}}{(1+|\eta|^2)^{s'}}\right)^{1/2}\right) \\ &\le (1+|\xi|)^{\max(|s|,|s'|)}.\end{aligned}$$

Finally, if $\eta \notin V$ and $\xi+\eta \in U$, then since $\overline{U} \subset V$, there is a constant C_1 depending only on U and V such that $|\eta| \le C_1|\xi|$. It follows that

$$\begin{aligned}\left(\frac{w_2(\xi+\eta)}{w_1(\eta)}\right)^{1/2} &= \left(\frac{(1+|\xi+\eta|^2)^s}{(1+|\eta|^2)^{s'}} + \frac{(1+|\xi+\eta|^2)^{s'}}{(1+|\eta|^2)^{s'}}\right)^{1/2} \\ &\le \left(\frac{(1+|\xi+\eta|^2)^s}{(1+|\eta|^2)^s}\frac{(1+|\eta|^2)^s}{(1+|\eta|^2)^{s'}} + \frac{(1+|\xi+\eta|^2)^{s'}}{(1+|\eta|^2)^{s'}}\right)^{1/2} \\ &\le \left((1+|\xi|)^{2|s|}(1+C_1^2|\xi|^2)^{|s-s'|} + (1+|\xi|)^{2|s'|}\right)^{1/2}.\end{aligned}$$

In any case, since $\hat{\phi}(\xi)$ decays more rapidly at ∞ than any power of $|\xi|$, it follows that the constant C in Lemma 4.5 is finite. Just as in the proof of Lemma 4.3, we have

$$\|f\|^2_{U,s} + \|f\|^2_{s'} = \|\phi\tilde{f}\|^2_{w_2} \le C^2\|\tilde{f}\|^2_{w_1} \le 2C^2\|f\|_{\Omega,V,s,s'}.$$

The proof is complete. □

It is natural to ask if the constants C in Lemma 4.3 and Lemma 4.4 can be made independent of the subset $K \subset \Omega$. If s is a non-negative integer, it is easy to see that the constant in Lemma 4.3 can always be taken equal to 1, since $\|f\|_s$ can be computed in terms of L^2 norms of f and its derivatives. For arbitrary s, the answer is however negative, as the following example shows.

We give an example on the real line, and it can easily be generalized to higher dimensions. Let Ω be an open interval on the real line, and let χ be the characteristic function of Ω. The example is based on two observations: First $\|\chi\|_s < \infty$ for all $s < 1/2$ and

$$\|\chi\|_s \to \infty \quad \text{as} \quad s \to 1/2. \tag{19}$$

Second, if $s < 1/2$ and $f \in H^s(\mathbb{R})$, then a simple calculation using properties of the Fourier transform shows

$$\|f(x/\epsilon)\|_s \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{20}$$

while for $f \in H^{1/2}(\mathbb{R})$, $\|f(x/\epsilon)\|_{1/2}$ is a increasing function of ϵ (which generally tends to a nonzero limit as $\epsilon \to 0$.)

Now let ϕ be a smooth and compactly supported function, equal to 1 on Ω. Since $\phi \in H^s(\mathbb{R})$ for every s, we can according to (19), for any arbitrary constant C, find an s so close to $1/2$ that $\|\chi\|_s > C\|\phi\|_{1/2}$. Next according to (20), we can modify the function χ near the endpoints of the interval Ω, to obtain a smooth function f with $\operatorname{supp} f \subset \Omega$ and with $\|f\|_s \geq \|\chi\|_s/2$. Similarly, we can modify ϕ near the endpoints of Ω to obtain a function $\tilde{f}$, equal to f in Ω and with $\|\tilde{f}\|_{1/2} \leq 3\|\phi\|_{1/2}$. Now it follows that

$$\|f\|_{1/2} \geq \|f\|_s \geq \frac{C}{6}\|\tilde{f}\|_{1/2} \geq \frac{C}{6}\|f\|_{\Omega,1/2}. \tag{21}$$

Since C could be chosen arbitrarily, this shows that when $s = 1/2$, the constant in Lemma 4.3 must depend on K.

5. **Proofs of main results.**

The proofs are straightforward once we have the following lemmas.

Definition 5.1. Let k be a distribution in $\mathbb{R}^n$ and u a distribution on $S^{n-1} \times \mathbb{R}$. If $\hat{k}$ is a locally integrable function on $\mathbb{R}^n$ and $\hat{u}$ is a locally integrable function on $S^{n-1} \times \mathbb{R}$, we define $T_k u$ to be the distribution on $\mathbb{R}^n$ whose Fourier transform is a locally integrable function satisfying

$$\widehat{T_k u}(\sigma\omega) = \frac{(2\pi)^{n-1}}{|\sigma|^{n-1}} \hat{k}(\sigma\omega)\big(\hat{u}(\omega,\sigma) + \hat{u}(-\omega,-\sigma)\big) \tag{22}$$

provided that this expression defines a locally integrable function on $\mathbb{R}^n$.

Lemma 5.2. *If k is a distribution in $\mathbb{R}^n$ such that*

$$|\hat{k}(\xi)| \leq \frac{C}{\sqrt{2}(2\pi)^{n-1}} |\xi|^{(n-1)/2}(1+|\xi|^2)^{l/2} \tag{23}$$

and $u \in H^{0,s+l}(S^{n-1} \times \mathbb{R})$ then $T_k u \in H^s(\mathbb{R}^n)$ and

$$\|T_k u\|_s \leq C\|u\|_{s+l}. \tag{24}$$

Lemma 5.3. *If k be a distribution in $\mathbb{R}^n$ such that*

$$|\hat{k}(\xi)| \leq \frac{C}{2(2\pi)^{n-1}} |\xi|^{n-1}(1+|\xi|^2)^{l/2} \tag{25}$$

and $f \in H^{s+l}(\mathbb{R}^n)$ is such that Rf is defined, then $T_kRf \in H^s(\mathbb{R}^n)$ and

$$\|T_kRf\|_s \leq C\|f\|_{s+l}. \tag{26}$$

Lemma 5.4. *If k and u satisfy the hypothesis of Lemma 5.2, and k has compact support, then $T_ku = k * R^*u$. In particular, if $\operatorname{supp} k$ is contained in an ϵ-neighborhood of the origin, then the restriction of T_ku to an open set Ω depends only on the restriction of u to Ω'_ϵ.*

Proof of Lemma 5.2. We have to show that the expression (22) defines a locally integrable function on $\mathbb{R}^n$ and that the required estimate on the Sobolev norm holds. This is accomplished by the following computation:

$$\begin{aligned}
&\|T_ku\|_s^2 = \int_{\mathbb{R}^n\setminus 0} |\widehat{T_ku}(\xi)|^2(1+|\xi|^2)^s d\xi \\
&= \frac{1}{2}\int_{S^{n-1}\times(\mathbb{R}\setminus 0)} |\widehat{T_ku}(\sigma\omega)|^2(1+\sigma^2)^s|\sigma|^{n-1} d\sigma d\omega \\
&= \frac{(2\pi)^{2(n-1)}}{2}\int_{S^{n-1}\times(\mathbb{R}\setminus 0)} |\hat{k}(\sigma\omega)|^2|\hat{u}(\omega\sigma)+\hat{u}(-\omega,-\sigma)|^2(1+\sigma^2)^s|\sigma|^{-(n-1)} d\sigma d\omega \\
&\leq \frac{C^2}{4}\int_{S^{n-1}\times(\mathbb{R}\setminus 0)} |\hat{u}(\omega\sigma)+\hat{u}(-\omega,-\sigma)|^2(1+\sigma^2)^{s+l} d\sigma d\omega \\
&\leq C^2\int_{S^{n-1}\times(\mathbb{R}\setminus 0)} |\hat{u}(\omega\sigma)|^2(1+\sigma^2)^{s+l} d\sigma d\omega \\
&= C^2\|u\|_{s+l}^2.
\end{aligned}$$

□

The proofs of Lemma 5.3 and Lemma 5.4 rely on the well known Projection Slice Theorem, which states that

$$\widehat{Rf}(\omega,\sigma) = \hat{f}(\sigma\omega). \tag{27}$$

Proof of Lemma 5.3. From the definition (22) of T_ku and (27) it follows that

$$\begin{aligned}
\widehat{T_kRf}(\sigma\omega) &= \frac{(2\pi)^{n-1}}{|\sigma|^{n-1}}\hat{k}(\sigma\omega)(\widehat{Rf}(\omega,\sigma)+\widehat{Rf}(-\omega,-\sigma)) \\
&= 2\frac{(2\pi)^{n-1}}{|\sigma|^{n-1}}\hat{k}(\sigma\omega)\hat{f}(\sigma\omega).
\end{aligned}$$

Hence it follows from (25) that

$$|\widehat{T_kRf}(\xi)| \leq C(1+|\xi|^2)^{l/2}|\hat{f}(\xi)|$$

and the desired conclusion follows easily from this. □

Proof of Lemma 5.4. Here we need to show that the Fourier transform of $k * R^*u$ is a locally integrable function that satisfies

$$\widehat{k * R^*u}(\sigma\omega) = \frac{(2\pi)^{n-1}}{|\sigma|^{n-1}}\hat{k}(\sigma\omega)(\hat{u}(\omega,\sigma)+\hat{u}(-\omega,-\sigma)). \tag{28}$$

To see this, let φ be an arbitrary test function on $\mathbb{R}^n$. If we define $\check{k}(x) = k(-x)$ we then have

$$\begin{aligned}
\langle \varphi, k * R^* u\rangle &= \langle \check{k} * \varphi, R^* u\rangle \\
&= \langle R(\check{k} * \varphi), u\rangle \\
&= \frac{1}{2\pi}\int_{S^{n-1}\times\mathbb{R}} \widehat{R(\check{k} * \varphi)}(\omega, -\sigma)\hat{u}(\omega,\sigma)d\omega d\sigma \\
&= \frac{1}{2\pi}\int_{S^{n-1}\times\mathbb{R}} \widehat{\check{k} * \varphi}(-\sigma\omega)\hat{u}(\omega,\sigma)d\omega d\sigma \\
&= \frac{1}{2\pi}\int_{S^{n-1}\times\mathbb{R}} \hat{k}(\sigma\omega)\hat{\varphi}(-\sigma\omega)\hat{u}(\omega,\sigma)d\omega d\sigma \\
&= \frac{1}{4\pi}\int_{S^{n-1}\times\mathbb{R}} \hat{k}(\sigma\omega)\hat{\varphi}(-\sigma\omega)(\hat{u}(\omega,\sigma)+\hat{u}(-\omega,-\sigma))d\omega d\sigma \\
&= \frac{1}{2(2\pi)^n}\int_{S^{n-1}\times\mathbb{R}} \widehat{T_k u}(\omega\sigma)\hat{\varphi}(-\sigma\omega)|\sigma|^{n-1}d\omega d\sigma \\
&= \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n} \widehat{T_k u}(\xi)\hat{\varphi}(-\xi)d\xi \\
&= \langle \varphi, T_k u\rangle .
\end{aligned}$$

□

In the following proofs, we will simplify notation by defining

$$m = (n-1)/2.$$

From Definition 5.1 and (27) it follows that

$$f = T_k Rf \quad \text{if} \quad \hat{k}(\xi) = \frac{|\xi|^{n-1}}{2(2\pi)^{n-1}} \tag{29}$$

for all $f \in H^s(\mathbb{R}^n)$ such that Rf is defined (in particular, for all $f \in H^s(\mathbb{R}^n)$ with compact support). If n is odd, the condition on k means that

$$k = \frac{1}{2}(2\pi)^{-(n-1)}(-\Delta)^m\delta_0.$$

If n is even, k is a distribution with support in all of $\mathbb{R}^n$, homogeneous of order $1-2n$.

Proof of Theorem 3.2. An inversion formula for the Radon transform can in this case be written $f = T_k Rf$ where $\operatorname{supp} k = \{0\}$ and k satisfies the hypothesis of Lemma 5.2 with $C = 1/\sqrt{2}$ and $l = m$. If $u \in H^{s+m}(S^{n-1}\times\mathbb{R})$ is equal to Rf in Ω', let $\tilde{f} = T_k u$. It follows from Lemma 5.4 that $\tilde{f} = f$ in Ω, and hence

$$\begin{aligned}
\|f\|_{\Omega,s} &\le \|\tilde{f}\|_s \\
&\le \frac{1}{\sqrt{2}}\|u\|_{s+m}.
\end{aligned}$$

By taking infimum over all u that agree with Rf on Ω', the desired inequality follows. □

Proof of Theorem 3.3. In the case when n is even, the support of the distribution k appearing in the inversion formula (29) is all of $\mathbb{R}^n$, so we can not directly apply the argument from the previous proof. However, k is homogeneous of order $1-2n$ and smooth outside the origin. Therefore we can write $k = k_1 + k_2$ where k_1

is supported in an ϵ-neighborhood of the origin and k_2 is smooth. Since k_1 is compactly supported, its Fourier transform is smooth, and by adding a suitable compactly supported smooth function to k_1 (and subtracting the same function from k_2) we can make $\hat{k}_1$ vanish at the origin to any order we wish. Specifically, we will assume that $\hat{k}_1$ vanishes to order $n-1$ at the origin. Since $\hat{k}$ also vanishes to order $n-1$, the same will be true for $\hat{k}_2$. Since $\hat{k}_2$ is rapidly decaying and $\hat{k}_1(\xi)$ does not grow faster than $|\xi|^{n-1}$, we have estimates

$$|\hat{k}_1(\xi)| \leq \frac{C_1}{\sqrt{2}(2\pi)^{n-1}}|\xi|^m(1+|\xi|^2)^{m/2} \tag{30}$$

and

$$|\hat{k}_2(\xi)| \leq \frac{C_2}{2(2\pi)^{n-1}}|\xi|^{n-1}(1+|\xi|^2)^{(s'-s)/2}. \tag{31}$$

By using the homogeneity of k, it is easy to check that C_1 can be made independent of ϵ.

Suppose now that $u = Rf$ in Ω'_ϵ, and let $\tilde{f} = f_1 + f_2$ where $f_1 = T_{k_1}u$ and $f_2 = T_{k_2}Rf$. In Ω it then holds that $\tilde{f} = T_{k_1}u + T_{k_2}Rf = T_{k_1}Rf + T_{k_2}Rf = T_kRf = f$, and it follows from Lemma 5.2 and Lemma 5.3 that

$$\begin{aligned}\|f\|_{\Omega,s} &\leq \|\tilde{f}\|_s \\ &\leq \|f_1\|_s + \|f_2\|_s \\ &\leq C_1\|u\|_{s+m} + C_2\|f\|_{s'}.\end{aligned}$$

By taking infimum over all u that agree with Rf on Ω'_ϵ, the estimate (11) follows. □

Proof of Theorem 3.4. Let k be the distribution from the inversion formula (29). Let $u \in H^{0,s+m}(S^{n-1}\times\mathbb{R})$ be equal to Rf on $\Omega'\cap(V\times\mathbb{R})$. We can also assume that $u = 0$ outside $(S^{n-1}\cap V)\times\mathbb{R}$, since setting u to zero outside this set will not increase its norm. Also, let $v = Rf$ outside $(S^{n-1}\cap V)\times\mathbb{R}$ and $v = 0$ in this set. Define $\tilde{f} = f_1 + f_2$ where $f_1 = T_ku$ and $f_2 = T_kv$. It follows that in Ω, $\tilde{f} = T_k(u+v) = T_kRf = f$. Note also that $\hat{f}_2(\xi) = \hat{f}(\xi)$ for ξ outside V and $\hat{f}_2(\xi) = 0$ for ξ in V.

We have the following estimates for f_1 and f_2:

$$\begin{aligned}\|f_1\|_{V,s} \leq \|f_1\|_s &\leq \frac{1}{\sqrt{2}}\|u\|_{s+m} \\ \|f_1\|_{s'} &\leq \|f_1\|_s \\ \|f_2\|_{V,s} &= 0 \\ \|f_2\|_{s'} &\leq \|f\|_{s'}.\end{aligned}$$

Putting this together, we obtain an estimate for $\|f\|_{\Omega,V,s,s'}$. By also using the fact that $\hat{f}_1$ and $\hat{f}_2$ have disjoint support, we get a slightly better estimate:

$$\begin{aligned}\|f\|^2_{\Omega,V,s,s'} &\leq \|\tilde{f}\|^2_{V,s} + \|\tilde{f}\|^2_{s'} \\ &\leq \|f_1\|^2_{V,s} + \|f_2\|^2_{V,s} + \|f_1\|^2_{s'} + \|f_2\|^2_{s'} \\ &\leq \|u\|^2_{s+m} + \|f\|^2_{s'}.\end{aligned}$$

Taking the infimum over u proves the estimate (12). □

Proof of Theorem 3.5. This proof uses a combination of ideas from the proofs of Theorem 3.3 and Theorem 3.4. Let k be the distribution in the inversion formula (29). Let $u = Rf$ in $\Omega'_\epsilon \cap (V \times \mathbb{R})$ and $u = 0$ outside $(S^{n-1} \cap V) \times \mathbb{R}$. Let k_1 and k_2 be as in the proof of Theorem 3.3 and define k_j^V and k^{V^c} by $\hat{k}_j^V(\xi) = \hat{k}_j(\xi)$ if $\xi \in V$ and 0 otherwise, while $\hat{k}^{V^c}(\xi) = \hat{k}(\xi)$ if $\xi \notin V$ and 0 otherwise.

Define $\tilde{f} = f_1 + f_2 + f_3$ where $f_1 = T_{k_1} u$, $f_2 = T_{k_2^V} Rf$, and $f_3 = T_{k^{V^c}} Rf$. In Ω it then holds that $\tilde{f} = T_{k_1} u + T_{k_2^V} Rf + T_{k^{V^c}} Rf = T_{k_1^V} Rf + T_{k_2^V} Rf + T_{k^{V^c}} Rf = T_k Rf = f$. We also obtain the following estimates:

$$\begin{aligned}
\|f_1\|_{V,s} &\le \|f_1\|_s \le C_1 \|u\|_{s+m} \\
\|f_1\|_{s'} &\le \|f_1\|_s \\
\|f_2\|_{V,s} &\le \|f_2\|_s \le C_2 \|f\|_{s'} \\
\|f_2\|_{s'} &\le \|f_2\|_s \\
\|f_3\|_{V,s} &= 0 \\
\|f_3\|_{s'} &\le \|f\|_{s'}.
\end{aligned}$$

Since the support of $\hat{f}_3$ is disjoint from the supports of $\hat{f}_1$ and $\hat{f}_2$, we obtain the estimate

$$\begin{aligned}
\|f\|^2_{\Omega,V,s,s'} &\le \|\tilde{f}\|^2_{V,s} + \|\tilde{f}\|^2_{s'} \\
&\le \|f_1 + f_2\|^2_{V,s} + \|f_3\|^2_{V,s} + \|f_1 + f_2\|^2_{s'} + \|f_3\|^2_{s'} \\
&\le 2\|f_1\|^2_{V,s} + 2\|f_2\|^2_{V,s} + \|f_3\|^2_{V,s} + 2\|f_1\|^2_{s'} + 2\|f_2\|^2_{s'} + \|f_3\|^2_{s'} \\
&\le 4C_1^2 \|u\|^2_{s+m} + (4C_2^2 + 1)\|f\|^2_{s'}
\end{aligned}$$

and taking infimum over u completes the proof. □

Remark 1. We would like to point out that the kind of estimates we obtain can also be deduced from the theory of pseudodifferential operators. However, those arguments do not provide any information about the constants. We have obtained explicit constants in Theorems 3.2 and 3.4. Moreover, the constants in Theorems 3.3 and 3.5 can be estimated by making a choice of k_1 and k_2 and computing the constants C_1 and C_2 in (30) and (31).

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