Helgason's Support Theorem and Spherical Radon Transforms

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This article is dedicated to Sigurdur Helgason on the occasion of his 80th birthday.

Abstract. We prove a new support theorem for the spherical Radon transform on manifolds using microlocal analysis, and we discuss the classical version of this theorem which was proved by Sigurdur Helgason for the spherical transform in $\mathbb{R}^n$. We use these theorems and their proofs to find similarities and differences between the classical and microlocal worlds, and we provide exercises and open problems.

1. Introduction

This article and much of the author's work has have been motivated by the beautiful results of Sigurdur Helgason for Radon transforms. Some of the author's proofs use classical techniques (e.g., [**47**]), but many of his proofs involve microlocal analysis to prove support theorems that are generalizations of Prof. Helgason's theorems to different spaces. In this article, we will present Helgason's proof of one of his support theorems and then a proof of a generalization by the author using microlocal analysis. With this as perspective, we will compare and contrast classical and microlocal proofs.

On the classical side, Prof. Helgason has created many beautiful theorems about the spherical Radon transform and results that grew from them, such as his generalizations of Ásgeirsson's Mean value theorem, generalized Pizetti formulas, Huygens' principle, and his theorems about the geodesic transforms on projective spaces. But one of his most fundamental and beautiful results is his support theorem for the hyperplane transform in $\mathbb{R}^n$. As I read his articles, I am reminded what gems they

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are. Each is crystal clear with all details, and he writes valuable overview articles that make his research even more accessible. The results are fundamental and beautiful.

Personally, Prof. Helgason has been a mentor to me over the years. He was my first professor at MIT, and he taught me real analysis. He was so inspiring that we graduate students voted him math professor of the year in 1974. He has always been generous of his time and ideas, when he was my Ph.D. co-advisor, as I was getting started in the field, and as I prove new theorems now. He is very kind and thoughtful, and I am lucky to number him among my friends.

This article is based on a talk the author gave at a conference at the University of Iceland in August, 2007, honoring Sigurdur Helgason on the occasion of his eightieth birthday.

Microlocal analysis is, basically, a precise study of singularities of functions and how they are perturbed under operators called Fourier integral operators (that include partial differential operators). The interplay between microlocal analysis and Radon transforms goes in both directions; authors have used microlocal analysis to understand Radon transforms and vice versa. Guillemin was the first to exploit the interplay in both directions. In [**29**] he developed the theory of Radon transforms as Fourier integral operator in broad generality. Then, in [**30**] Guillemin and Schaeffer developed a way of defining Fourier integral operators using push-forwards and pull-backs as Guillemin had done for the Radon transform (see our proof sketch of Theorem 6.2). This theory of Fourier integral operators was expounded in [**31**].

Subsequently authors have used Radon transforms to characterize Fourier integral operators and wavefront sets, basic objects of microlocal analysis. Authors in the Japanese school of microlocal analysis including Kaneko [**38**] used Radon transforms to understand wavefront sets of hyperfunctions. Takiguchi and Kaneko [**57**] proved a support theorem for the Radon transform of hyperfunctions. Candés and Demanet used curvelets, which are a sort of local Radon transform, to understand Fourier integral operators [**19**].

Many authors have used microlocal analysis to understand the pure mathematics of Radon transforms. Sample articles include Agranovsky (e.g., [**5**]), Boman [**14**], Greenleaf and Uhlmann [**27**], Ambartsoumian and Kuchment [**8**], Phong and Stein [**46**], Stefanov and Uhlmann [**55**] and many other authors. Authors including Candés [**18**], Finch [**23**] Katsevich [**39**] Nolan and Cheney [**43**], Quinto (e.g., [**50, 51, 52**]), have used microlocal analysis in tomography to understand singularity detection and to develop reconstruction methods.

We will start by describing Radon transforms and support theorems in general in §2. Then in §3, we state Helgason's classical support theorem, (Thm. 3.1) for the hyperplane Radon transform and give his reduction to a support theorem for the spherical transform, (Thm. 3.2). In §4 we will give a conjecture of his for the spherical transform and describe his answer. We will discuss the spherical transform on manifolds in §5 and the author's answer for two particular classes of spheres in Sections 5.1 and 5.2. Our proof is in §6, and it involves microlocal analysis. Having described classical and microlocal proofs, we will contrast them in §7 and give exercises and open problems in §8.

2. Support Theorems in General

We now give an overview of Radon transforms and support theorems. Let M be a Riemannian manifold and let Ξ be a collection of submanifolds of M all of which are diffeomorphic to each other. For example, M could be $\mathbb{R}^n$ and Ξ could be the set of hyperplanes in $\mathbb{R}^n$. Radon transforms are integral transforms integrating functions on M over the submanifolds in Ξ. For a function of compact support, $f \in C_c(M)$, and $\xi \in \Xi$, define **the Radon transform:**

$$Rf(\xi) = \int_{x\in\xi} f(x) dm_\xi \,.$$

In order for this to make sense, one needs to choose a set of manifolds, Ξ, and measures, dm_ξ on each manifold $\xi \in \Xi$ such that the manifolds and measures are regular enough so that the integral converges. Once this is true, then one can ask intriguing questions such as whether the transform is injective, what the range is, and whether support conditions hold. Prof. Helgason (e.g., [**32**]) has answered these questions for many different Radon transforms.

A support theorem specifies information about the support of f from information about the support of Rf. Typically it has the following form.

METATHEOREM 2.1. *Let $\mathcal{A}$ be a subset of Ξ and let $f \in C(M)$. Specify some hypotheses on f, Ξ, and $\mathcal{A}$. If $Rf(\xi) = 0$ for $\xi \in \mathcal{A}$, then* $\operatorname{supp} f$ *is disjoint from* $\bigcup_{\xi\in\mathcal{A}} \xi$.

That is, if f integrates to zero over all submanifolds in $\mathcal{A}$ and there are appropriate hypotheses on $\mathcal{A}$, Ξ, and f, then f itself must be zero on all the points on all $\xi \in \mathcal{A}$. Typically, $\mathcal{A}$ is open and connected. If M is not compact, then typically f is required to satisfy some growth condition at infinity (e.g., Theorem 3.1 given below) or be supported away from one of the manifolds in $\mathcal{A}$ (e.g., Theorem 5.2).

Support theorems are important in theory and for applications. Prof. Helgason used one of his support theorems [**32**, Theorem 2.1] to prove a range theorem for the hyperplane transform on $C_c^\infty(\mathbb{R}^n)$ [**32**, Corollary 4.3]. He used a support theorem for the horocycle transform on symmetric space [**33**, Lemma 8.1] to prove another beautiful result, the solvability of invariant differential operators on symmetric spaces [**33**, Theorem 8.2]. Then, in 1987, he used a support theorem for the hyperbolic space [**35**, Lemma 2.3] to prove a converse to Huygens principle [**35**, Proposition 5.2]. This is just a sample of his work in this area.

Support theorems provide a guide for inversion methods for limited data problems in tomography. Many problems in tomography are modelled by Radon transforms, the most common of which is the Radon line transform. Allan Cormack, one of the fathers of tomography, got me interested in the *exterior problem*: can one reconstruct a function in the plane using only integrals over lines outside a convex set? Allan explained this is a problem in X-ray tomography in which one wants to image the area surrounding the beating heart; in the 1980s, CT scanners took data slowly enough that CT data through the heart would be blurry because of the movement of the heart. This would add artifacts to the reconstructions using standard methods since they need data through the heart even to reconstruct the density outside the heart. Helgason's support Theorem 3.1 [**32**] shows that the exterior problem has a solution for rapidly decreasing functions. Although it does not provide an inversion method, the support theorem does show inversion is possible.

3. Radon Transforms on $\mathbb{R}^n$

First, we introduce notation for the spherical Radon transform. The sphere and disk centered at $y \in \mathbb{R}^n$ and of radius $r > 0$ are respectively

$$\begin{aligned} S(y,r) &:= \{x \in \mathbb{R}^n \,\big|\, d(y,x) = r\} \\ D(y,r) &:= \{x \in \mathbb{R}^n \,\big|\, d(y,x) \le r\} \end{aligned} \tag{3.1}$$

where d is the standard Euclidean distance. In this case, the set of manifolds of integration,

$$\Xi_{\mathbb{R}^n} = \mathbb{R}^n \times (0,\infty)\,,$$

represents the set of all spheres in $\mathbb{R}^n$. We define the *classical spherical mean transform* to be

$$SMf(y,r) := \int_{x\in S(y,r)} f(x)dm_S \tag{3.2}$$

where dm_S is the area measure normalized so that $SM1 = 1$. This transform is trivially invertible if one is given integrals over all spheres since for continuous functions, $f(y) = \lim_{r\to 0^+} SMf(y,r)$. The problem becomes more difficult when one restricts the set of spheres, and we will consider two such cases, restriction to a proper open subset, $\mathcal{A}$, of the set of all spheres $\Xi_{\mathbb{R}^n}$ (§5.1), and spheres of one fixed radius (§5.2).

We will also consider spherical transforms with arbitrary measures. If $m(x,y,r)$ is a smooth function, then we define the *generalized spherical transform* to be

$$SM_m f(y,r) := \int_{x\in S(y,r)} f(x)m(x,y,r)dm_S\,. \tag{3.3}$$

In this case, the measure of integration is $m(x,y,r)dm_S$. To prevent trivialities, we assume m is nowhere zero and smooth.

Now, we will consider one of the most important Radon transforms, the classical hyperplane Radon transform. The set of submanifolds, Ξ_H, will be the set of $n-1$-dimensional hyperplanes in $\mathbb{R}^n$. For $f \in C_c(\mathbb{R}^n)$ and $\xi \in \Xi_H$ the *classical Radon hyperplane transform* is defined to be

$$Rf(\xi) = \int_{x\in\xi} f(x)dm_H$$

where dm_H is the measure on ξ induced from Lebesgue measure on $\mathbb{R}^n$. Here is Helgason's famous theorem.

THEOREM 3.1 (Helgason [**32**, Theorem 2.1]). *Let $f \in C(\mathbb{R}^n)$ satisfy the following conditions:*

i. For each integer $k > 0$, $|x|^k f(x)$ is bounded
ii. There is a constant $A > 0$ such that $Rf(\xi) = 0$ for all hyperplanes ξ outside the disk $D(0,A)$.

Then $f(x) = 0$ outside of the disk $D(0,A)$.

This is an example of Metatheorem 2.1 where $\mathcal{A}$ is the set of hyperplanes not intersecting $D(0,A)$ and the growth condition on f is rapid decrease at infinity. The theorem also holds for hyperplanes outside of a compact convex set because such sets are the intersection of the disks that contain them. The proof is beautiful and it involves a support theorem for the spherical transform, so we will sketch it and focus on the theorem about the spherical transform. See [**25, 56, 15**] for other proofs.

Proof. First Prof. Helgason reduces from $f \in C(\mathbb{R}^n)$ to $f \in C^\infty(\mathbb{R}^n)$ by convolving with $\phi \in C_c^\infty(\mathbb{R}^n)$ supported in $D(0,\epsilon)$: $R(f*\phi) = (Rf) *_p (R\phi)$.

Next, he proves the support theorem for smooth radial functions that decrease rapidly at infinity. This is proven by showing the Radon transform of a radial function is an generalized Abel or Volterra type integral operator on that function. The integral operator is easily inverted, and the inversion formula involves a similar integral equation. The inversion formula implies that if $Rf(\xi) = 0$ for hyperplanes of distance greater than A from the origin, then the radial function is zero for $r = |x| > A$.

This allows one to reduce to a support theorem for the spherical mean transform. Let $y \in \mathbb{R}^n$ and $r > 0$. Now let $O(n)$ be the orthogonal group in $\mathbb{R}^n$ and let dk be the normalized Haar measure on $O(n)$. Define

$$g_y(z) = \int_{k\in O(n)} f(y+kz)dk = SMf(y,|z|)\,.$$

Then g is radial in z and $R(g_y)(\xi) = \int_{k\in O(n)} Rf(y+k\xi)dk$ since the integrations over the hyperplane and over $O(n)$ can be commuted. Since $Rf(y+k\xi) = 0$, if $d(0,\xi) > |y| + A$, the radial function g_y satisfies $R(g_y)(\xi) = 0$ for $d(0,\xi) > |y| + A$.

By his support theorem for the hyperplane transform and radial functions, $g_y(z) = 0$ for $|z| > |y| + A$. That is, $SMf(y,r) = 0$ if $S(y,r)$ encloses $D(0,A)$. The final step is the support theorem for the spherical transform that we will study thoroughly.

Theorem 3.2 (Helgason [**32**, Lemma 2.2]). *Assume $f \in C^\infty(\mathbb{R}^n)$ and $|x|^k f(x)$ is bounded for all $k \in \mathbb{N}$. Assume $SMf(y,r) = 0$ for all spheres $S(y,r)$ enclosing $D(0,A)$. Then, $f = 0$ outside of $D(0,A)$.*

We will prove a more local theorem that follows from Prof. Helgason's original proof. In [**32**, p. 157], he notes that his proof was inspired by a related proof of Fritz John [**37**] for spheres of one fixed radius. John's proof also inspired the author as he developed his results in [**49**] and in particular counterexample 3.2 in that article.

Theorem 3.3. *Assume $f \in C^\infty(\mathbb{R}^n)$ and $|x|^k f(x)$ is bounded for all $k \in \mathbb{N}$. Let $y_0 \in \mathbb{R}^n$ and U be an open neighborhood of y_0. Let $r_0 > 0$ and assume that $SMf(y,r) = 0$ for all $y \in U$ and all $r > r_0$. Then, $f = 0$ outside of the disk $D(y_0, r_0)$.*

Note that Theorem 3.3 implies Helgason's original statement, Theorem 3.2. This is another example of Metatheorem 2.1 with $\mathcal{A} = U \times (r_0,\infty)$ and f decreasing rapidly at infinity.

One might wonder why the rapid decrease condition is needed for Theorem 3.3, since the spheres being integrated over all have compact support. A simple example in the plane illustrates why. Let $k \in \{3,4,\dots\}$ and let $f(x_1,x_2) = 1/(x_1+ix_2)^k$ for $|(x_1,x_2)| > 1$ and altered in the unit disk to be smooth. Then one can show that $SMf = 0$ for all circles enclosing the unit disk [**26**]. In fact this same example shows that the rapid decrease condition is necessary for Theorem 3.1.

Proof. Prof. Helgason's idea is to perturb the center of the sphere and show that integrals of f times polynomials over the sphere are zero. We will give his proof but with the trivial additions that allow one to prove the more local version, Theorem 3.3. Let $y_0 \in \mathbb{R}^n$ and let U be an open neighborhood of y_0. Let $y \in U$, $r > r_0$ then we know

$$\text{constant} = \int_{\mathbb{R}^n} f = \int_{D(y,r)} f = \int_{z \in D(0,r)} f(y+z) \tag{3.4}$$

since integrals of f on spheres $S(y,s)$ are zero for $s > r$. If we take derivative with respect to the i^{th} coordinate we get

$$0 = \int_{D(0,r)} \frac{\partial}{\partial y_i} f(y+z)dz = \int_{D(0,r)} \frac{\partial}{\partial z_i} f(y+z)dz\,. \tag{3.5}$$

Defining the vector field $F(z) = f(y+z)\frac{\partial}{\partial z_i}$ and using the Divergence Theorem, we see

$$\begin{aligned} 0 = \int_{D(0,r)} \operatorname{Div} F dz &= \operatorname{Vol}(S(0,r)) \int_{S(0,r)} \langle F, \vec{n} \rangle dm_S \\ &= \operatorname{Vol}(S(0,r)) \int_{S(0,r)} f(y+z) \frac{z_i}{r} dm_S\,. \end{aligned} \tag{3.6}$$

Since also $SMf(y,r) = 0$, we can now assert that the integral of f times any first-order polynomial over $S(y,r)$ is zero. Note that, in order to prove this, we need f to be integrable on $\mathbb{R}^n$ and $SMf(z,s)$ to be zero for $z \in U$ and $r > r_0$.

The proof then proceeds by induction. Since f is rapidly decreasing at infinity, we know that fP is integrable for each first order polynomial P. We can repeat this argument to show that $SM\big(fy_iP\big)(z,s) = 0$ for $(z,s) \in U \times (r_0, \infty)$, $i = 1, \dots, n$, and this implies $SM(fP)(z,s) = 0$ for all *second* order polynomials, P. Since f is rapidly decreasing at infinity, we can continue this process and prove

$$SM\big(fP\big)(z,s) = 0 \quad \forall (z,s) \in U \times (r_0, \infty) \tag{3.7}$$

for *all* polynomials P. By the Stone-Weierstrass Theorem,

$$f = 0 \ \text{ on } \bigcup_{(z,s) \in U \times (r_0,\infty)} S(z,s)\,. \tag{3.8}$$

Since $z \in U$ was arbitrary and $r > r_0$, this shows f is zero outside of $D(y_0, r_0)$. □

Remark 3.4. Prof. Helgason's proof of his support theorem, Thm. 3.2, demonstrates how important the symmetry of the situation is. First, by symmetry, he can reduce to assuming f is smooth. Then, because of the symmetry, the integral equation for radial functions is easy to solve. The radial function f does not need to be rapidly decreasing at infinity only decreasing fast enough so that one can invert the integral equation to prove the support theorem for radial functions. As noted, this allows him to reduce to proving a support theorem for the spherical transform. Prof. Helgason uses the rapid decrease assumption in the proof of this second theorem.

The measure on the sphere is closely related to the measure on the whole space, so it is easy to reduce from an integral over $\mathbb{R}^n$ to an integral over disks with center near a specific point y and radius r in (3.4). Finally, he perturbs the center so that he can use Stokes' Theorem and get an integral over the sphere $S(y,r)$ in (3.6). To do this one needs the measure on the sphere to be sufficiently simple so that derivatives of the measure do not affect the calculation. If one considered the generalized transform (3.3) in (3.5), one would have to include the weight $m(x, y, r)$ in the calculation:

$$\frac{\partial f}{\partial y_i}(y + r\omega)m(y + r\omega, y, r) + f(y + r\omega)\frac{\partial m}{\partial y_i}(y + r\omega, y, s)$$

for $\omega \in S^{n-1}$, and one could not use a straightforward Stokes' Theorem argument to conclude an integral of $fy_i m$ over $S(y, r)$ is zero.

One can use classical proofs for some more general measures if the weight decomposes

$$m(x, \xi) = m_1(x)m_2(\xi)\,. \tag{3.9}$$

In this case, one can apply a classical theorem to $(m_1 f)$. Prof. Helgason [**34**, Lemma 2.3] (discussed in Section 4), Ehrenpreis [**20**], Palamodov [**44, 45**] and others (e.g., [**11**]), including the author, have considered such weights. The author has proven support theorems for rotation invariant transforms on hyperplanes using integral equations techniques [**47, 48**], and these are one step more general than measures that decompose.

Another point is that classical proofs also can be used to conclude more local statements than what might be given, as in his original Theorem 3.2 and the more local Theorem 3.3. However, often these stronger statements aren't needed for the goal at hand (e.g., to prove the support theorem for the hyperplane transform).

4. Helgason's Problem and Generalizations

Soon after Prof. Helgason proved Theorem 3.2 he posed an intriguing conjecture. He wondered if his Theorem 3.2 was true in a more general setting. The definitions all make sense on a Riemannian manifold where, in this setting, d is the Riemannian distance and spheres are geodesic spheres.

We now introduce the notation needed to describe the problem on manifolds. Let M be a Riemannian manifold and let $d(\cdot, \cdot)$ be the geodesic distance on M. For $y \in M$, $r > 0$

$$S(y, r) = \{z \in M \,\big|\, d(z, y) = r\} \quad D(y, r) = \{z \in M \,\big|\, d(z, y) \le r\}$$

If $x \in M$, then we let I_x be the *injectivity radius of* M *at* x, then the exponential map at y is a Riemannian diffeomorphism from the open, origin centered disk of radius I_y in $T_y(M)$ to the interior of $D(y, I_y)$. We define $I_M = \inf\{I_x \,\big|\, x \in M\}$. It could happen that $I_M = 0$. Note that if $r < I_y$ then $S(y, r)$ is diffeomorphic to a Euclidean sphere. The set of spheres over which we integrate is

$$\Xi_M := \{(y, r) \in M \times (0, \infty) \,\big|\, \forall z \in D(y, r), r < I_z)\}\,. \tag{4.1}$$

We include the injectivity radius assumption so the spheres are all diffeomorphic to Euclidean spheres and that the Radon transform and its dual are well defined as integral operators.

Then, the *spherical Radon transform* for $f \in C(M)$ is

$$SMf(y, r) = \int_{x \in S(y,r)} f(x)\, dm_S(x).$$

This transform is in some sense "classical" since the measure $dm_S(x)$ is the measure on $S(y, x)$ induced from the Riemannian structure on M and the definition is formally the same as for $\mathbb{R}^n$ (3.2).

Problem 4.1 (Helgason [**40**, §6, Problem 1]). *Let M be a complete simply connected Riemannian manifold of negative curvature and D a closed ball in M. Let $f \in C_c^\infty(M)$. Assume f has surface integral 0 over every sphere enclosing D ($SMf = 0$ over such spheres). Is $f = 0$ on $M \setminus D$?*

Prof. Helgason answered this problem affirmatively for M of constant negative curvature, that is in hyperbolic space, in [**34**, Lemma 2.3]. He assumes the function decreases exponentially at infinity. The proof is intriguing because he essentially reduces the result to his Theorem 3.2 for the spherical transform on Euclidean space. This is possible since, using the disk model of hyperbolic space, hyperbolic spheres are Euclidean spheres with different centers and radii. The important issue is to check that the canonical measure on spheres in the hyperbolic geometry decomposes as in (3.9). Namely, the hyperbolic measure on the sphere dm_S at $x \in S(y,r)$ can be written $m_1(x)dm_S^E$ for a nonzero real-analytic function m_1 where dm_S^E is the Euclidean measure on the sphere. This allows one to write $SMf(y,r) = \int_{S(y,r)}[f(x)m_1(x)]dm_S^E$, and it reduces the problem to one on $\mathbb{R}^n$. Then, he uses a clever perturbation argument for the Euclidean sphere similar to the one in the proof of Theorem 3.3. He uses this theorem to prove a support theorem the geodesic hyperplane (horosphere, or $n-1$-dimensional totally-geodesic) transform for exponentially decreasing functions [**34**, Theorem 2.1].

5. The generalized Spherical Transform on Riemannian Manifolds.

Many years later, the author answered Problem 4.1 for the generalized spherical transform on real-analytic Riemannian manifolds with real-analytic weights. His original answer, [**28**, Theorem 3.2.5], follows from the main theorem in [**49**]. That theorem is for spheres of fixed radius (see Theorem 5.3 below). Now we will provide a different answer for spheres of varying radius. It is a generalization of Helgason's local support theorem, Theorem 3.3, to real-analytic Riemannian manifolds, M. Let Ξ_M be the set of spheres satisfying the condition of (4.1). The *incidence relation* of points and spheres in M is

$$Z = \{(x,y,r) \,\big|\, (y,r) \in \Xi_M,\ x \in S(y,r)\}\,. \tag{5.1}$$

We let $m : Z \to \mathbb{C}$ be a nowhere zero real-analytic weight. Let $f \in C(M)$. We define the *generalized spherical transform* of $f \in C(M)$ at $(y,r) \in \Xi_M$ to be

$$SM_m f(y,r) := \int_{x \in S(y,r)} f(x)m(x,y,r)dm_S\,. \tag{5.2}$$

We include the nowhere zero real-analytic weight $m(x,y,r)$ on Z since measures aren't canonical in general, and our proofs don't use any symmetry. This is the natural generalization of the transform (3.3) on $\mathbb{R}^n$.

If we consider the spherical transform on all spheres in M, then injectivity follows by continuity. We will assume that $m(x,y,r) \neq 0$. If for each fixed y

$$\lim_{\substack{r\to 0^+\\ (x,y,r)\in Z}} m(x,y,r) = g(y) \quad \text{for a nowhere-zero function } g \tag{5.3}$$

One just takes the limit as $r \to 0^+$ and sees for continuous f that $SM_m f(y,r) \to g(y)f(y)$. This trivial inversion formula justifies restricting the set of spheres. It is worth pointing out that if M has dimension n, then Ξ_M has dimension $n+1$, so in some geometric sense, finding f from $SM_m f$ is an overdetermined problem.

There are several ways to restrict the set of spheres. First, one can consider support theorems of the form of Metatheorem 2.1 for this set Ξ_M, which we do in §5.1. Second, one can consider a restricted set of spheres that has dimension n, as we do in §5.2.

5.1. Radon transforms over spheres of arbitrary radius and center. Since, under reasonable assumptions (5.3), the spherical transform is trivially injective, one should look for support theorems. A general support theorem was given in [**28**]. As noted in Remark 3.4, Prof. Helgason's proof of Theorem 3.2 really implies a more local theorem, and here is the exact generalization of that local result, Theorem 3.3.

THEOREM 5.1. *Let M be a real-analytic Riemannian manifold with infinite injectivity radius and let $y_0 \in M$ and let U be an open neighborhood of y_0. Let $r_0 > 0$ and let $\mathcal{A} = U \times (r_0, \infty)$. Let $f \in C_c(M)$ and let m be a nowhere zero real-analytic weight on $Z = M \times (0, \infty)$. Assume $SM_m f(y, r) = 0$ for all $(y, r) \in \mathcal{A}$. Then,* $\operatorname{supp} f \subset D(y_0, r_0)$.

The proof will be given in §6 and then it will be contrasted with Prof. Helgason's classical proof for $\mathbb{R}^n$ in §7. Here is a very general support theorem that follows from the same arguments.

THEOREM 5.2. *Let M be a real-analytic Riemannian manifold and let Ξ_M be the set of spheres in* (4.1). *Let $\mathcal{A} \subset \Xi_M$ be open and connected. Let f be a continuous function on M and let m be a nowhere zero real-analytic weight on Z* (5.1). *Assume $SM_m f(y, r) = 0$ for all $(y, r) \in \mathcal{A}$, and assume for some $(y_1, r_1) \in \Xi_M$, $S(y_1, r_1)$ is disjoint from* $\operatorname{supp} f$. *Then,* $\operatorname{supp} f \subset D(y_0, r_0)$.

This theorem is exactly the form of Metatheorem 2.1 where the condition on f is that f is zero near a sphere $S(y_1, r_1)$ and $\mathcal{A}$ is open and connected. The proof of Theorem 5.2 uses the same key ingredients as in the proof of Theorem 5.1 given in Section 6, and it is left to the reader. See [**28**].

5.2. Spherical transforms over spheres of one fixed radius. The author has proven support theorems for spherical Radon transforms in several settings in which the set of spheres, Ξ, has the same dimension as M, and we now discuss the set of spheres of one fixed radius. The problem is not overdetermined since this set of spheres has dimension the same as M. There are many classical results for the set of spheres with a finite number of radii on $\mathbb{R}^n$ [**60, 9, 10**] and related problems on hyperbolic spaces [**12**]. We consider on fixed radius, $r = r_0$, and we define

$$\Xi_{r_0} = \{y \in M \,\big|\, \forall x \in D(y, r_0), I_x > r_0\}$$
$$Z = \{(x, y) \,\big|\, y \in \Xi_{r_0}, x \in S(y, r_0)\}$$

and let m be a real-analytic nowhere zero function on Z. Because of the relation to the Pompeiu problem (e.g., [**60**]), we denote the spherical transform with fixed radius

$$P_m f(y) := SM_m f(y, r_0) = \int_{x \in S(y, r_0)} f(x) m(x, y) dm_S.$$

Here is a generalization of Prof. Helgason's Theorem 3.2 for the set of spheres of fixed radius. It follows from the proof of [**49**, Theorem 1.1].

Theorem 5.3 ([**28**, Theorem 3.2.5]). *Let M be a real-analytic Riemannian manifold with infinite injectivity radius and let $A > 0$ and $x_0 \in M$. Let $f \in C_c(M)$ and let $r_0 > A$ be chosen such that for some $y_0 \in M$,* $\operatorname{int}\big(D(y_0, r_0)\big)$ *contains* $(\operatorname{supp} f) \cup D(x_0, A)$*. Assume $P_m f(x) = 0$ for all $y \in M$ such that $D(y, r_0)$ contains $D(y_0, A)$. Then $f = 0$ outside of $D(y_0, A)$.*

The proof uses the same components as the proof of Theorem 5.1 and it is given in [**28**].

Zhou alone [**61, 62**] and with the author [**63**] proved two-radius theorems under weaker hypotheses. If one knows integrals of $SM_m f = 0$ for spheres of two radii the ratio of which is irrational, then the function needs to be zero only near one sphere to infer f is zero everywhere (rather than near one disk as is required for the theorems in [**49**]). This two-radius theorem is valid for $\mathbb{R}^n$ and for a real-analytic Riemannian manifold and nowhere zero real-analytic weight.

6. Proof of Theorem 5.1

Although this follows from [**28**, Theorem 3.2.4], the proof is easier in this case and it allows one to see the similarities and differences between classical and microlocal proofs. The elements of this proof are the ones in almost all of the author's proofs of support theorems in which microlocal ideas are used.

1. First, we use a good concept of singularity, the analytic wavefront set (Definition 6.1) which captures singularities at points and in (co)directions $x \in M$, $\eta \in T_x^*(M)$: $(x, \eta) \in \mathrm{WF}_A(f)$. Loosely, one takes a localized Fourier transform of f near the point and sees in which directions it does not decrease exponentially.
2. Next, we prove a microlocal regularity theorem, Theorem 6.2, that describes what SM_m does to analytic wavefront set. In particular, if $SM_m f$ is zero near a sphere $S(y, r)$ then f has no singularities at points on the sphere in conormal directions.
3. Then, we use a lovely theorem of Kawai, Kashiwara, and Hörmander, Thm. 6.4, that tells that one can extend a function that is zero locally on one side of a sphere to be zero locally on the other side when f has no singularities conormal to the sphere.
4. Once this is established, we let the spheres $S(y, r)$ for $(y, r) \in \mathcal{A}$ eat away at $\operatorname{supp} f$ using the microlocal regularity theorem (and the assumption $SM_m f = 0$ on $\mathcal{A}$) to show that f does not have wavefront at conormals to the sphere, and the Kawai, Kashiwara, and Hörmander theorem to prove that if it does not, it must be zero near the corresponding points.

We define analytic wavefront set as follows. This is related to the FBI transform as discussed in [**36**] and in particular, [**36**, Theorem 9.6.3].

Definition 6.1. Let f be a distribution of compact support, and let $x_1 \in \mathbb{R}^n$ and $\eta_1 \in T_{x_1}^*(\mathbb{R}^n) \setminus 0$. Then, $(x_1, \eta_1) \notin \mathrm{WF}_A(f)$ **iff** there are neighborhoods U of x_1 and V of η_1 and $\exists C > 0$, $\exists c > 0$ such that for all $x \in U$, $\eta \in V$: $\left| \int_{y \in \mathbb{R}^n} \big(f(y) e^{-\lambda |x-y|^2}\big) e^{-iy\cdot(\lambda\eta)} dy \right| \leq C e^{-c\lambda}$.

This definition extends to arbitrary distributions on $\mathbb{R}^n$ by localizing because the real-analytic wavefront set of f above a point x is the same as that of ϕf if ϕ is equal to one (or real-analytic) near x. Thus, one can extend the definition to Riemannian manifolds using local coordinates. To preserve invariance, one uses the natural identifications and the coordinates to define $\mathrm{WF}_A(f)$ on the cotangent bundle $T^*(M)\setminus 0$.

The Radon transform detects wavefront set in a precise way.

Theorem 6.2 (Microlocal Regularity Theorem [**28**, Thm. 3.2.4]). *Let M be a real-analytic Riemannian manifold $f \in \mathcal{D}'(M)$. Let m be a nowhere zero real-analytic weight on Z and let SM_m be the associated Radon transform on spheres in Ξ_M given by (4.1). Let $S_1 = S(y_1, r_1)$ for $(y_1, r_1) \in \Xi_M$. Let $x_1 \in S_1$ and let η_1 be conormal to S_1 at x_1. If $SM_m f$ is zero in a neighborhood of (y_1, r_1), then $(x_1, \eta_1) \notin \mathrm{WF}_A(f)$.*

This theorem is valid because SM_m is an elliptic Fourier integral operator associated to the Lagrangian manifold the conormal bundle of Z, $N^*(Z)\setminus 0$. This implies that SM_m detects only singularities conormal to spheres of integration. SM_m satisfies the microlocal Bolker assumption so $\big(SM_m{}^*\big)SM_m$ is an analytic elliptic pseudodifferential operator after one localizes f and Z so the composition is defined. In fact, Guillemin viewed Radon transforms as push-forward of pull-backs in [**29**] and then in [**31**]. Let

$$
\begin{array}{ccc}
Z & \xrightarrow{p_2} & \Xi_M \\
\downarrow{\scriptstyle p_1} & & \\
M & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
N^*(Z)\setminus 0 & \xrightarrow{\pi_2} & T^*(\Xi_M)\setminus 0 \\
\downarrow{\scriptstyle \pi_1} & & \\
T^*(M)\setminus 0 & &
\end{array}
\tag{6.1}
$$

where p_1 and p_2 (respectively π_1 and π_2) are the projections onto the respective factors. Given $f \in C(M)$ one can pull f back to Z using p_1 then multiply by a weight, m', related to $m(x, y, r)$ and then push forward by integrating over the fibers of π_2. Since these fibers are diffeomorphic to the manifolds ξ, that is equivalent to integrating with respect to dm_S on the fiber: $SM_m f = (p_2)_* m'(p_1)^*(f)$. In the real-analytic category, one can use this push-forward and pull back to show one can localize f near any point $x_1 \in S(y_1, r_1)$ [**53, 59, 36**].

The right-hand diagram in (6.1) gives the microlocal properties of SM_m. In particular, π_2 is an injective immersion as shown in the proof of [**28**, Proposition 3.2.1] (in particular, equation (3.2.3) and the following text). This means that the localized $(SM_m{}^*)SM_m$ is a pseudodifferential operator. This was the fundamental idea that Guillemin developed in [**29, 31**]. One can also use either stationary phase [**54, 24**] or asymptotic arguments [Jan Boman, unpublished] using microlocally analytic elliptic operators [**13**] to show that the localized $(SM_m{}^*)SM_m$ is a microlocally analytic elliptic operator near (x_1, η_1) [**41, 42**]. Then, one can use this calculus of microlocally analytic elliptic pseudodifferential operators [**13, 58**] to finish the proof.

An important theorem of Kawai, Kashiwara and Hörmander gives precise information about analytic wavefront set at $\mathrm{bd}(\operatorname{supp} f)$. We need one definition, and then we state the theorem.

Definition 6.3. Let S_1 be a hypersurface in M. We say a function f is *zero on one side of S_1 near $x_1 \in S_1$* if there is a neighborhood W of x_1 such that $W \setminus S_1$ has two components and $W \cap \operatorname{supp} f$ is in one of those components union S_1. Analogously, *f is zero near x_1* if there is an open neighborhood W of x_1 such that f is zero on W.

If there is a neighborhood W of x_1 such that f is zero in all of W, then f is trivially zero on one side of S_1 near x_1. Our next theorem was proven by Kawai, Kashiwara, and Hörmander.

THEOREM 6.4 (Microlocal Extension Theorem [**36**]). *Let M be a real-analytic Riemannian manifold $f \in \mathcal{D}'(M)$. Let S_1 be a C^2 hypersurface and let $x_1 \in S_1$ and $\eta_1 \in N^*(S_1) \setminus 0$. Assume f is zero on one side of S_1 locally near x_1. If $(x_1, \eta_1) \notin \mathrm{WF}_A(f)$ then f is zero locally near x_1, that is $x_1 \notin \operatorname{supp} f$.*

This is a strengthening of the obvious fact that if f is real-analytic near x_1 and f is zero on one side of S_1 near x_1, then f must extend to be zero on the other side near x_1.

Now we have the tools to complete the proof. So we can discuss the proof in the next section, we will describe the proof in steps.

Step 1. Let $y_0 \in U$ and assume $\operatorname{supp} f \not\subset D(y_0, r_0)$. Let $r_1 > r_0$ be the largest radius such that $S_1 := S(y_0, r_1)$ meets $\operatorname{supp} f$ and let $x_1 \in S_1 \cap \operatorname{supp} f$. Here we use that, because f has compact support, we can find a sphere disjoint from $\operatorname{supp} f$ and then one that just touches $\operatorname{supp} f$. Because of the choice of r_1, $\operatorname{supp} f$ must be to one side of S_1 near x_1. Let η_1 be in the conormal bundle $N^*_{x_1}(S_1) \setminus 0$.

Step 2. Because $SM_m f(y, r) = 0$ for $(y, r) \in U \times (r_0, \infty)$, the Microlocal Regularity Theorem 6.2 implies that $(x_1, \eta_1) \notin \mathrm{WF}_A(f)$.

Step 3. Then by Theorem 6.4, x_1 cannot be a boundary point of $\operatorname{supp} f$.

Step 4. Step 3 provides a contradiction that finishes the proof.

7. Observations

7.1. General Observations. Microlocal proofs use local properties of the function and transform (namely how the transform changes wavefront set), and they are valid in a general setting: they apply to Radon transforms on real-analytic Riemannian manifolds and with nowhere zero real-analytic measures. The microlocal proofs require real-analyticity because the Kawai, Kashiwara, Hörmander theorem is not valid in the C^∞ category; there is only one real-analytic function of compact support (the zero function), but there are many such C^∞ functions. Also, injectivity is not true, in general, for Radon transforms with C^∞ measures as illustrated by a beautiful counterexample of Boman [**17**] for the line transform. Some classical proofs using integral equations techniques (e.g., [**47**]) and are valid for measures that are not even C^∞, but the measures do have symmetry (e.g., rotation invariance in [**47**]).

The cost of the generality is that the underlying theorems are technical and subtle. In order to apply in general, perhaps this has to be. In contrast, a classical proof like Prof. Helgason's is elegant and more elementary. It also generalizes nicely to cases such as hyperbolic space [**34**] and related but more general arguments prove support theorems on symmetric spaces [**33**]. But classical proofs cannot, in general, be used to prove support theorems for transforms with general measures on arbitrary spaces.

7.2. Comparing restrictions on f in classical and microlocal theorems. One other important distinction is the type of restriction on f that is needed for classical versus microlocal proofs. Only a decrease condition at infinity is needed for most classical proofs. For most microlocal proofs, one needs f to be zero near a $\xi_0 \in \Xi_M$. This is needed so that one can find the first ξ_1 that just touches $\operatorname{supp} f$ as in Step 1 of the proof of Theorem 5.1.

A generalization of Theorem 6.4 by Jan Boman [**16**] allows one to prove stronger theorems using microlocal techniques. He proved that if f is zero to infinite order on a submanifold S and no conormal to S is in $\mathrm{WF}_A(f)$ then f itself is zero near S. This was used in [**28**] to prove support theorems for the spherical transform assuming that f is zero to infinite order on one sphere ξ_0. Boman used his theorem in [**15**] in a microlocal analytic proof of Helgason's Support Theorem 3.1 for rapidly decreasing functions. He maps the hyperplane transform to the projective hyperplane transform on projective space. Then, the condition of rapid decrease at infinity becomes a condition of a being zero to infinite order on the projective hyperplane at infinity. It is not obvious that such a proof will work for the sphere transform because there is "no" sphere at infinity although there is a hyperplane at infinity.

7.3. Comparing classical and microlocal proofs. Now, we compare the methods in classical proof of Theorem 3.3 to those in proof of its generalization to Riemannian manifolds, Theorem 5.1.

The key to the classical proof is to perturb the sphere in order to show

$$\int_{S(y,r)} f(x)P(x)dm_S = 0 \tag{7.1}$$

for all polynomials. Then, one uses the Stone-Weierstrass theorem to assert f is zero on $S(y,r)$.

A loose equivalent for the perturbation argument is the combination of the Microlocal Regularity Theorem 6.2 in Step 2 with the Microlocal Extension Theorem 6.4 that is used in Step 3. In order to assert that f is real-analytic in all directions $(x_1,\eta_1) \in N^*\big(S(y_1,r_1)\big)$, one needs to be able to evaluate $SM_m f$ at nearby spheres. Then, one learns that for all $(x_1,\eta_1) \in N^*\big(S(y_1,r_1)\big)$, $(x_1,\eta_1) \notin \mathrm{WF}_A(f)$. This doesn't say f is zero, but at least that it is analytic in the directions that can be detected by integrals over $S(y_1,r_1)$. In Step 3 of the microlocal proof, one needs the Kawai Kashiwara Hörmander Theorem 6.4, in order to show f extends to be zero near x_1 since f is zero on one side of S_1 near x_1. so $x_1 \notin \operatorname{supp} f$.

Finally, we need to recall Step 1, namely that $x_1 \in \operatorname{supp} f$. This is why we structure the proof to get a contradiction in Step 4.

8. Exercises and Open Problems

Here are some exercises and open problems. The ones indicated by (*) are believed to be open problems. Have fun!

(1) Is the spherical transform injective if (5.3) does not hold. One could imagine a real-analytic measure that oscillates a lot. (Hint: one answer for complex valued weights uses complex analysis.)
(2) Prove an analogue of Theorem 5.1 if M has finite injectivity radius. WARNING: be careful to precisely state the support restriction on f.
(3) Prove Theorem 5.2.
(4) Find a counterexample to Theorem 5.3 when the disk $D(y_0,r_0)$ does not enclose $D(x_0,A) \cap \big(\operatorname{supp} f\big)$. Classical examples can be used [**60**] unless one assumes some sphere $S(y_0,r_0)$ is disjoint from $\operatorname{supp} f$. If one sphere

is disjoint from supp f, then one should study [**49**, Example 3.2] for a counterexample

(5) (*)Jan Boman [**16**] constructed a very clever example that shows the line transform is not injective on domain $C_c(\mathbb{R}^2)$ if the weight m is not real-analytic but only C^∞. The problem is to construct a counterexample to invertibility for the spherical transform with arbitrary centers and radius $r > 1$ on domain $C_c(\mathbb{R}^n)$. The author believes this could be difficult.

(6) (*)Is there a way to do microlocal proofs for the sphere transform with less restrictive conditions on f than that f is zero near one sphere in $\mathcal{A}$ (or zero to infinite order on one sphere)? For example, can one prove Theorem 5.1 with the condition that f is decreasing sufficiently quickly at infinity on M (which has infinite injectivity radius), or even for $M = \mathbb{R}^n$. Can one prove such a theorem without using microlocal analysis?

(7) (*) The author and Mark Agranovsky have studied the spherical mean transform with centers on a real-analytic hypersurface $C \subset M$ and arbitrary radii. The set of spheres we consider is $\Xi_C = \{(y, r) \,\big|\, y \in C,\ \forall x \in D(y, r),\ r < I_x\}$. Like the set of spheres of fixed radius, Ξ_C has dimension n, so injectivity is not overdetermined. The authors showed [**3**] that for f of compact support in $\mathbb{R}^2$ and the classical measure SM is injective for Ξ_C unless C is a finite set union a Coxeter system of lines through a point with equal angles. The authors have developed partial results in $\mathbb{R}^n$ [**4, 6**] as well as precise characterizations of the C for which the transform SM is invertible over Ξ_C for crystallographic domains. These results use microlocal analysis in fundamental ways. Other authors [**1, 2, 8, 7, 21, 22**] have used wave and Darboux equation techniques to understand this transform. However, as the author writes this, no one has a made characterization of the sets C in $\mathbb{R}^n$ for which SM restricted to centers on C is injective for domain $C_c(\mathbb{R}^n)$. The authors [**3**] conjectured that sets, C, on which SM is not injective are unions of zero sets of homogeneous harmonic polynomials and sets of algebraic codimension greater than one. This is a rephrasing of their result for $\mathbb{R}^2$.

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