

SUPPORT THEOREMS FOR RADON TRANSFORMS ON REAL ANALYTIC LINE COMPLEXES IN THREE-SPACE

JAN BOMAN AND ERIC TODD QUINTO

ABSTRACT. In this article we prove support theorems for Radon transforms with arbitrary non-zero real analytic measures on line complexes (three dimensional sets of lines) in \mathbb{R}^3 . Let f be a distribution of compact support on \mathbb{R}^3 . Assume Y is a real analytic *admissible* line complex and Y_o is an open connected subset of Y with one line in Y_o disjoint from $\text{supp } f$. Under weak geometric assumptions, if the Radon transform of f is zero for all lines in Y_o , then $\text{supp } f$ intersects no line in Y_o . These theorems are more general than previous results, even for the classical transform. We also prove a support theorem for the Radon transform on a non-admissible line complex. Our proofs use analytic microlocal analysis and information about the analytic wave front set of a distribution at the boundary of its support.

1. Introduction.

A generalized Radon transform on lines in space, R_μ , integrates functions on \mathbb{R}^3 over lines using a specified integration weight, μ , on each line. For compactly supported functions, injectivity for this transform with positive C^2 weights [Finch 1986] follows from local invertibility for Radon transforms on lines in a *plane* (e.g., [Markoe and Quinto]). Injectivity follows from the plane result because line integrals over all lines in space provide line integrals over lines in any given plane. Injectivity of this transform can be reduced to a plane problem because this transform is *overdetermined*: the set of lines in space has greater dimension than \mathbb{R}^3 . As the transform is overdetermined, it is intriguing to find three dimensional submanifolds, Y , of lines, so called *line complexes*, for which R_μ restricted to Y is invertible.

The so called *admissible* complexes are: the set of lines tangent to a surface in space, the set of lines intersecting a curve in space, and the set of lines with direction vectors lying on a given curve on the sphere [Gelfand *et al.* 1961, 1966].

1991 *Mathematics Subject Classification.* Primary: 44A12, Secondary: 58G15.

Key words and phrases. Radon transform, admissible line complex, analytic microlocal analysis.

The first author is grateful to the Office of the Provost of Tufts University and to the Royal Swedish Academy of Sciences for their support. He was also partially supported by Swedish Natural Sciences Research Council grant F-FU4047-110. The second author thanks the Swedish Natural Sciences Research Council for its support; he was also partially supported by NSF grants MCS 8701415 and MCS 8901203. Both authors are grateful to the referee for helpful comments and a reference

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - \TeX

All of these complexes have similar geometric properties as reflected in Gelfand's admissibility condition (3.1.3).

In this article we prove injectivity and, more importantly, support theorems for real analytic admissible line complexes of real lines in \mathbb{R}^3 with arbitrary real analytic weights that are never zero. These support theorems guarantee uniqueness for tomography with sources on a curve. Counterexamples are given that show the theorems do not hold under weaker hypotheses. Our proofs are in the spirit of [Boman, Boman and Quinto] and involve the theory of analytic Fourier integral operators, microlocal analysis, and a theorem [Hörmander] about the analytic wave front set of a distribution at points on the boundary of its support. Even for the classical transform and real analytic manifolds, our theorems are stronger than previous results. Our theorems can be generalized to many other Radon transforms (*e.g.*, [Quinto 1991]), and the cases considered here exhibit the important ideas and are of some practical interest.

In example 4.3 we prove injectivity on functions of compact support for a Radon transform on a non-admissible complex Y with arbitrary nowhere zero real analytic measures. We also prove Helgason's support theorem for this transform. Guillemin [1985] discusses microlocal advantages of admissibility, so it is not surprising that the microlocal analysis is more difficult for non-admissible complexes and that the theorem is weaker. This example (with canonical measures) was described (without proof) in the complex category by [Gelfand, Graev] (see also [Greenleaf Uhlmann, pp. 231-237] for related results for \mathbb{R}^3).

Gelfand *et al.* [*ibid.*] have proven injectivity for admissible complexes of *complex* lines in \mathbb{C}^3 with standard weights and where the curves and surfaces are algebraic. Guillemin and Sternberg [1977] first considered the Radon transform as a Fourier integral operator (see also [Quinto 1980]), and Guillemin [1985] investigated admissibility using microlocal analysis. Using the calculus of Fourier integral operators, Greenleaf and Uhlmann [1989] considered admissible complexes of real geodesics on symmetric spaces in the C^∞ category and proved that $R_\mu^* R_\mu$ is in a class of Fourier integral operators associated with intersecting Lagrangian manifolds. For the transform with μ a non-zero constant and for the complex of lines through a curve, [Solmon] proved a support theorem and [Finch 1985] developed an inversion method. See [Helgason] for a proof of injectivity for the Radon transform with constant weights on all geodesics on symmetric spaces.

Section two of this article provides the important terms and the main theorems. Proofs are done in §3, and the counterexamples are given in §4.

2. Definitions and Main Theorems.

For $\theta \in \mathbf{S}^2$, and $x \in \mathbb{R}^3$ let $\ell(\theta, x) = \{x + s\theta \mid s \in \mathbb{R}\}$ be the line containing x and parallel to θ , and let ds be arc length measure on the line. Let Y be an immersed three dimensional real analytic submanifold of the set of lines in three-space. Such Y will be called a *line complex*. Let $X \subset \mathbb{R}^3$ be open and let $\mu(x, \ell)$ be a C^∞ function on

$$Z = \{(x, \ell) \in X \times Y \mid x \in \ell\}. \quad (2.1)$$

The generalized Radon transform with weight μ , $R_\mu : C_c(\mathbb{R}^3) \rightarrow C(Y)$, is defined for $\ell = \ell(\theta, x) \in Y$ by

$$R_\mu f(\ell) = R_\mu f(\theta, x) := \int_{z \in \ell(\theta, x)} f(z) \mu(z, \ell(\theta, x)) ds .$$

If W is a surface in \mathbb{R}^3 and ℓ is a line tangent to W at point of tangency w_o , we define the directional curvature of W along ℓ at w_o to be the curvature at w_o of the curve of intersection of W with the plane through ℓ and parallel to the normal to W at w_o .

The classical admissible complexes will be labeled as follows:

- Type I:** Given a non-planar real analytic surface W in space, Y is the set of all lines, ℓ , tangent to W , such that W has non-zero directional curvature along ℓ at every point of tangency;
- Type II:** Given a real analytic curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ that is non-singular ($\gamma'(t) \neq 0 \forall t$), Y is the set of lines intersecting that curve non-tangentially;
- Type III:** Y is the set of lines parallel to directions on a given real analytic curve on the sphere, \mathbf{S}^2 .

The geometric conditions given for type I and II complexes ensure that the sets Y are *immersed* submanifolds of the set of all lines in \mathbb{R}^3 . Our theorems are easier to state for subsets $Y_o \subset Y$, which are *imbedded* in the set of all lines. For subsets of type I complexes, this implies that each line in Y_o is tangent to W in exactly one point. For subsets of type II complexes, this implies that the curve is simple and that no line in Y_o intersects the curve at more than one point. As our theorems are local, they can be applied locally to Y_o that are not imbedded. We shall not consider the complex of lines in a one parameter family of planes; this complex is similar to the complex of lines tangent to a surface.

The main theorems are now given.

Theorem 2.1. *Let W be a non-planar real analytic surface in \mathbb{R}^3 . Let Y_o be an open connected subset of the type I complex defined by W , and assume Y_o an imbedded submanifold of the set of all lines. In case there is a plane \mathcal{P} tangent to W at a non-discrete set of points, we assume no line in Y_o is contained in \mathcal{P} . Let X be an open set in \mathbb{R}^3 disjoint from W , and let $\mu(x, \ell)$ be a real analytic function on Z (see (2.1)) that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_{Y_o} = 0$ and if some line in Y_o is disjoint from $\text{supp } f$, then every line in Y_o is disjoint from $\text{supp } f$.*

If W is strictly positively curved, then the geometric assumptions in the theorem hold for all lines tangent to W and all functions f with support disjoint from W . Example 4.1 shows the necessity of the assumption that no line in Y_o (that intersects $\text{supp } f$) is contained in a plane that is tangent to W at a non-discrete set of points. This assumption about tangent planes to W is needed only for Proposition 3.2.3. Proposition 3.2.1 is a weaker support theorem that is valid without this assumption.

Theorem 2.2. *Let γ be a non-singular real analytic curve. Let Y_o be an open connected subset of the type II complex defined by γ , and assume Y_o is an imbedded*

submanifold of the set of all lines. Assume that, if γ is a plane curve, then no line in Y_o lies in a plane containing γ . Let X be an open set in \mathbb{R}^3 disjoint from γ , and let $\mu(x, \ell)$ be a real analytic function on Z (see (2.1)) that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_{Y_o} = 0$ and if some line in Y_o is disjoint from $\text{supp } f$, then every line in Y_o is disjoint from $\text{supp } f$.

Theorem 2.2 is true even if lines tangent to γ are included in Y_o . Example 4.2 shows that the conclusion of Theorem 2.2 is not true if γ is a plane curve and Y_o contains lines (that intersect $\text{supp } f$) in a plane that contains γ . A type II complex defined by the curve γ can be viewed as a limiting case as $\epsilon \rightarrow 0^+$ of type I complexes defined by surfaces of tubes of radius ϵ about γ .

In a very nice article, Solmon [1976] proves a result closely related to Theorem 2.2 that is valid for all continuous curves γ . He assumes that μ is constant; that Y_o is the set of all lines through γ with directions parallel a *fixed* open set, S , in \mathbf{S}^2 ; and that for every direction in S there is a line in Y_o in that direction that is disjoint from $\text{supp } f$. The conclusion is that every line in Y_o is disjoint from $\text{supp } f$. Theorem 2.2 is stronger than Solmon's theorem even for constant μ because the open set, S , of directions is not fixed but can vary with the point $x \in \gamma$. However, Theorem 2.2 does not apply when γ is a line and Solmon's theorem does. On the other hand, Proposition 3.3.1 for characteristic paths (3.3.1) with pivot points at infinity (which is true for all analytic curves) implies Solmon's theorem for all analytic curves, including lines.

Theorem 2.3. *Let $\theta : \mathbf{S}^1 \rightarrow \mathbf{S}^2$ be a closed simple non-singular real analytic curve of directions and let Y_o be an open connected subset of the type III complex of lines with directions on $\theta(\mathbf{S}^1)$. Assume $\theta(\mathbf{S}^1)$ is not a great circle of \mathbf{S}^2 . Let $\mu(x, \ell)$ be a real analytic function on Z (see (2.1)) that is never zero. Let $f \in \mathcal{E}'(\mathbb{R}^3)$. If $R_\mu f|_{Y_o} = 0$ and if some line in Y_o is disjoint from $\text{supp } f$, then every line in Y_o is disjoint from $\text{supp } f$.*

Theorem 2.3 follows from Theorem 2.2 together with the observation that a type III complex can be viewed as a type II complex with the curve at infinity. A similar construction to that in example 4.2 shows that the conclusion of Theorem 2.3 is not valid if $\theta(\mathbf{S}^1)$ is a great circle (and so Y is the set of lines in parallel slices).

In all of these theorems, μ needs only to be defined for x in some open set containing $\text{supp } f$.

The proofs use the calculus of the Fourier integral operator R_μ and Theorem 8.5.6 [Hörmander 1983] (quoted in §3.1) on analytic singularities. Let $\ell_1 \in Y_o$ be parallel to $\theta_1 \in \mathbf{S}^2$. The first key idea in the proof involves the microlocal analysis of R_μ : data $R_\mu f(\ell)$ for lines $\ell \in Y_o$ near ℓ_1 detect microlocal analytic singularities (wave front) of f at points on ℓ_1 in *almost* all cotangent directions perpendicular to θ_1 (Proposition 3.1.1, [Greenleaf Uhlmann] in the C^∞ category). Therefore, if $R_\mu f(\ell) = 0$ for lines in Y_o near ℓ_1 then f must be analytic at all points $x \in \ell_1 \cap X$ in *almost all* directions perpendicular to θ_1 . (This is a slight abuse of notation: precisely, f is analytic at x in direction ξ if $(x, \xi) \notin WF_A(f)$, the analytic wave front set of f [Hörmander 1983].) The second key idea involves a theorem (given

in §3.1) about analytic wave front sets; under certain assumptions, this theorem implies vanishing of a function at y if the function is analytic in a specified direction at y . The proofs of our support theorems proceed as follows. Assume $\text{supp } f$ does not intersect the line $\ell_o \in Y_o$ and let $\ell_1 \in Y_o$. Using the assumptions that Y_o is open and connected we construct a sequence of *characteristic* paths of lines in Y_o from ℓ_o to ℓ_1 (for type I complexes: Propositions 3.2.1 & 3.2.3). The first key idea, about R_μ , implies f is analytic in most directions at all points on all lines in the path because $R_\mu f$ is zero near these lines. The second key step in the proof is to use Theorem 8.5.6 [Hörmander 1983] to show that f must be zero near those lines. This proves the theorem. The proofs are more subtle than those in [Boman and Quinto] for the Radon transform on hyperplanes in \mathbb{R}^n since, in the case of lines, some important wave front directions of f are not detected by the data, $R_\mu f$. The heart of the article is §§3.2-3 in which these more subtle arguments are given.

Using ideas related to those in [Boman and Quinto], Globevnik [1990] has proved a clever local support theorem for the X-ray transform with $\mu = 1$ and data on all lines; similar theorems are true for real analytic transforms on admissible line complexes, and the proofs are local versions of the proofs below.

3. Proofs.

Throughout this section we assume that μ is real analytic and never zero and that all manifolds are real analytic. One key to the proofs of all our support theorems is the following important theorem of Hörmander, (generalized by Kashiwara and Kawai [Kaneko] for hyperfunctions [Sato, Kawai, and Kashiwara]).

Theorem 8.5.6 [Hörmander 1983]. *Let X be an open subset of \mathbb{R}^n and $f \in \mathcal{D}'(X)$, let x_o be a boundary point of the support of f , and assume that there is a C^2 function F such that $F(x_o) = 0$, $dF(x_o) \neq 0$, and $F(x) \leq 0$ on $\text{supp } f$. Then $(x_o, \pm dF(x_o)) \in WF_A(f)$ where $WF_A(f)$ is the analytic wave front set of f .*

Let x_o be a point on the boundary of the surface defined by the function F in Theorem 8.5.6 above, and let ξ_o be conormal to that boundary at x_o . The theorem implies that if $(x_o, \xi_o) \notin WF_A(f)$, then $x_o \notin \text{supp } f$.

3.1 The microlocal analysis of R_μ for admissible complexes.

The first key idea of the proofs requires the microlocal analysis of R_μ and it reflects the point of view of [Guillemin] and [Guillemin and Sternberg] that an understanding of the geometric properties of the double fibration (below left) and the corresponding map on the cotangent spaces (below right) is the appropriate way to discover the microlocal properties of R_μ as a Fourier integral operator:

$$\begin{array}{ccc}
 Z & \xrightarrow{\pi_Y} & Y & & \Lambda = N^*(Z) \setminus 0 & \xrightarrow{\pi_Y} & T^*(Y) \\
 \downarrow \pi_X & & & & \downarrow \pi_X & & \\
 X & & & & T^*(X) & &
 \end{array} \tag{3.1.1}$$

In (3.1.1), $N^*(Z)$ is the conormal bundle of Z in $T^*(X \times Y)$. The spaces (manifolds

or cotangent spaces) to which the projections π_X and π_Y apply will be clear from context. In order that (3.1.1) defines a double fibration, we require the *surjectivity condition*:

$$\pi_X : Z \rightarrow X \text{ has surjective differential} \quad (3.1.2)$$

If Y is a complex of type I or II then a calculation in local coordinates shows that (3.1.2) implies that X is disjoint from the surface or curve defining Y . For $x \in X$ define $\Sigma_x = \bigcup \pi_Y(\pi_X^{-1}(x))$, the set of all points on the lines in the complex Y that pass through x . Admissibility is defined by Gelfand [*et al.* 1961, p. 810] in terms of the Σ_x ; admissible complexes of type I, II, and III all satisfy his *admissibility condition*:

$$\text{if } x_1 \in \ell \in Y \text{ and } x_2 \in \ell, \text{ then } \Sigma_{x_1} \text{ and } \Sigma_{x_2} \text{ have the same tangent plane along } \ell. \quad (3.1.3)$$

Proposition 3.1.1. *Let Y be a real analytic line complex such that the surjectivity (3.1.2) and admissibility (3.1.3) conditions hold for all $x \in X$ and all $\ell \in Y$. Let $\ell_o \in Y$ and assume $f \in \mathcal{E}'(X)$ and $R_\mu f(\ell) = 0$ for all $\ell \in Y$ in a neighborhood of ℓ_o . Let $x \in \ell_o \cap X$ and let $\xi \in T_x^*(X)$ be conormal to ℓ_o but not conormal to the tangent plane to Σ_x along ℓ_o . Then $(x, \xi) \notin WF_A(f)$.*

Let $\ell_o \in Y_o$. For $x \in \ell_o$, let $N_x^*(\ell_o) \subset T_x^*(X)$ be the conormal space of ℓ_o at x . The conclusion of Proposition 3.1.1 is that the cotangent directions in $N_x^*(\ell_o)$ that are not detected by R_μ are exactly those conormal at x to the tangent plane to Σ_x along ℓ_o . By Gelfand's admissibility condition, the undetected cotangent directions are the same for every point on ℓ_o . For type I complexes, these undetected directions are exactly those ξ that are conormal to W at the point of tangency of ℓ_o with W ; for type II complexes, they are the ξ that are conormal to the curve γ at the point of intersection of ℓ_o with γ .

Proposition 3.1.1 is proven for the C^∞ category in [Greenleaf Uhlmann], and the arguments are essentially the same as in the analytic category so they will only be sketched here. Let Λ_o be the open subset of Λ consisting of all $(x, \xi, \ell, \eta) \in \Lambda$ for which ξ is not conormal to Σ_x along ℓ_o . Using essentially only the surjectivity and admissibility conditions, one proves: R_μ is a Fourier integral operator with Lagrangian manifold Λ ; Λ_o is a local canonical graph; and R_μ is analytic elliptic when microlocally restricted to Λ_o . One finishes the proof of the proposition by using the calculus of Fourier integral operators to show that $R_\mu f = 0$ near ℓ implies that the specified directions are not in $WF_A f$. Finally, one needs to check using local coordinates that complexes satisfying the hypotheses of Theorems 2.1 and 2.2 satisfy the admissibility and surjectivity conditions.

3.2. Characteristic paths for type I complexes and the proof of Theorem 2.1.

Let $\ell_o \in Y_o$ be a line disjoint from $\text{supp } f$, and let $\ell_1 \in Y_o$. We will construct a path in Y_o from ℓ_o to ℓ_1 and show f is zero near all lines in the path. Since the analytic wave front set of f in the exceptional directions, ξ conormal to Σ_x , is not detected by data $R_\mu f$, "characteristic" paths for R_μ must be constructed for which these directions are not problematic.

Let $x_o \in \mathbb{R}^3 \setminus W$. A *characteristic path with pivot point x_o* is a smooth path of lines in the set $\pi_Y(\pi_X^{-1}(x_o))$ where these maps are defined by the double fibration on the left of (3.1.1). For type I complexes, characteristic paths consist of lines that intersect x_o and are tangent to W . A characteristic path with pivot point at infinity is a smooth path of parallel lines tangent to W .

Proposition 3.2.1. *Assume that the hypotheses of Theorem 2.1 hold for X , W , and Y_o . Let $f \in \mathcal{E}'(X)$ and assume $R_\mu f = 0$ on Y_o . Let $l : [a, b] \rightarrow Y_o$ be a characteristic path and assume $l(a)$ does not meet $\text{supp } f$ and the pivot point of the path $l(s)$ is disjoint from $\text{supp } f$. Then*

$$l(s) \cap \text{supp } f = \emptyset \text{ for } a \leq s \leq b. \quad (3.2.1)$$

Proof. We can assume the characteristic path has pivot point at infinity. This reduction uses an affine transformation of $RP^3 \supset \mathbb{R}^3$.

The reduction is as follows. Consider the projective space RP^3 to be the union of \mathbb{R}^3 plus a plane, H_∞ , of points at infinity. Choose a projective transformation $\Psi : RP^3 \rightarrow RP^3$ such that $\Psi(x_o) \in H_\infty$. Let $H = \Psi^{-1}(H_\infty)$. By breaking up the characteristic path in (3.2.1) into a finite number of paths, if necessary, we can assume that H contains none of the lines $l(s)$ in the path. Those parts of W and $\text{supp } f$ that are far from the union of all $l(s)$ are clearly irrelevant to this argument, so we can assume $\text{supp } f$ and W do not meet H ; thus we can assume Ψ maps W into an analytic surface $\Psi(W)$ in \mathbb{R}^3 and that Ψ maps f into a distribution $\tilde{f} \in \mathcal{E}'(\mathbb{R}^3)$. Since Ψ is real analytic and preserves order of contact, the hypotheses of Proposition 3.2.1 hold on \tilde{f} , $\Psi(W)$, and the Radon transform induced by Ψ on R_μ and the path of lines $\Psi(l(s))$ has pivot point at infinity.

We now define “cones” of lines in Y_o near $l(s)$ that will be used to eat away at the support of f and thus prove (3.2.1). We assume W is tangent to the $x_1 - x_3$ plane at $w_o = (0, 0, -d)$ and use the equation $y_2 = h(y_1, y_3)$ to define W near w_o , where $h(0, -d) = 0 = [\partial h / \partial y_1](0, -d) = [\partial h / \partial y_3](0, -d)$. In this proof, points of tangency to W will be written $(y_1, h(y_1, y_3), y_3)$ and points on lines will be written (x_1, x_2, x_3) . We can assume that $a = 0$ in (3.2.1), that $l(0) = \ell_o$ is the x_3 axis, and that $l(s)$ lies in the plane $x_1 = s$. The curvature assumption given in the definition of type I complex implies

$$\frac{\partial^2 h}{\partial y_3^2}(0, -d) \neq 0. \quad (3.2.2)$$

The set of lines tangent to W and parallel to the $x_2 - x_3$ plane can be locally parameterized as follows. Let $l(y_1, \alpha)$ be the line in Y that lies in the plane $x_1 = y_1$ and is parallel to $(0, \alpha, 1)$. The y_3 -coordinate of the point of tangency to W of this line satisfies

$$\frac{\partial h}{\partial y_3}(y_1, y_3) = \alpha. \quad (3.2.3)$$

By (3.2.2) and the Implicit Function Theorem, there is a constant $C > 0$ such that for any $|y_1| \leq 2C$, $|\alpha| \leq C$, (3.2.3) has an analytic solution $y_3 = y_3(y_1, \alpha)$.

Let $\epsilon(s)$ be a smooth positive function with negative derivative for all s . For s near zero define $D(s)$ to be the ‘‘cone’’ of all lines $l(y_1, \alpha)$ satisfying

$$(y_1 - s)^2 + \alpha^2 \leq \epsilon^2(s). \quad (3.2.4)$$

Assume that $\epsilon(s)$ is so small that for each $s \in [0, C]$, $\epsilon(s) \leq C$. This will ensure that the parameterization of lines $l(y_1, \alpha)$ will be valid for all (y_1, α) satisfying (3.2.4). By making $\epsilon(s)$ sufficiently small we can assume that the non-smooth part of $\partial D(s)$ (near W) does not meet $\text{supp } f$ and that for all points (x_1, x_2, x_3) in $(\text{supp } f) \cap D(s)$,

$$x_3 \neq y_3(x_1, \alpha) \text{ if } (x_1 - s)^2 + \alpha^2 \leq \epsilon^2(s). \quad (3.2.5)$$

Finally, we can assume that $D(0)$ does not intersect $\text{supp } f$. Let

$$\bar{s} = \sup\{s_1 \in [0, C] \mid D(s) \cap \text{supp } f = \emptyset \text{ for } 0 \leq s < s_1\}. \quad (3.2.6)$$

The set in (3.2.6) is non-empty as $D(0)$ does not meet $\text{supp } f$. Assume $\bar{s} < C$. Then $D(\bar{s})$ meets $\text{supp } f$ at some point \bar{x} . By the definition of \bar{s} , \bar{x} must be a boundary point of $D(\bar{s})$. Let \bar{w} be the point of intersection of W and the line, $\bar{\ell}$, in $D(\bar{s})$ containing \bar{x} .

Lemma 3.2.2. *No conormal, $\bar{\xi}$, to $\partial D(\bar{s})$ at \bar{x} is conormal to $\Sigma_{\bar{x}}$ along $\bar{\ell}$.*

Assuming Lemma 3.2.2 for the moment we complete the proof of Proposition 3.2.1. As $R_\mu f = 0$ for all lines in a neighborhood of $\bar{\ell}$ and $\bar{\xi}$ is not conormal to $\Sigma_{\bar{x}}$ along $\bar{\ell}$, Proposition 3.1.1 shows that $(\bar{x}, \bar{\xi}) \notin WF_A(f)$. As $\bar{\xi}$ is conormal to $\partial D(\bar{s})$ at \bar{x} and $f = 0$ to one side of this boundary, Theorem 8.5.6 [Hörmander 1983] implies that $f = 0$ in a neighborhood of \bar{x} . This proves $\bar{s} = C$. This argument can clearly be repeated so that after a finite number of steps one concludes $\bar{s} = b$.

Proof of Lemma 3.2.2. The basic idea is that $\mathcal{W} = \cup_{0 \leq s \leq \bar{s}} D(s)$ is a wedge with decreasing width because $\epsilon(s)$ strictly decreases. The conormal $\bar{\xi}$ to $D(\bar{s})$ at \bar{x} is a conormal to this wedge. If $\bar{\xi}$ were also conormal to W at \bar{w} , then the width of the wedge, \mathcal{W} , would not be decreasing at \bar{s} . The rigorous proof is as follows.

The proof is done in two steps. First we find conditions on $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ for conormals to $\partial D(s)$ at \bar{x} to be conormal to $\Sigma_{\bar{x}}$ along $\bar{\ell}$ (equivalently: to be conormal to W at \bar{w}). The line $l(y_1, \alpha)$ has equation

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= H(\alpha, y_1, x_3, y_3(y_1, \alpha)) \quad \text{where} \\ H(\alpha, y_1, x_3, y_3) &:= \alpha(x_3 - y_3) + h(y_1, y_3) \end{aligned} \quad (3.2.7)$$

Near \bar{x} , the surface $\partial D(\bar{s})$ is generated by the one-parameter family of lines $l(y_1(t), \alpha(t))$ where

$$\begin{aligned} y_1(t) &:= \bar{s} + \epsilon(\bar{s}) \cos t \\ \alpha(t) &:= \epsilon(\bar{s}) \sin t \end{aligned} \quad (3.2.8)$$

for $t \in [0, 2\pi]$.¹ Hence the intersection of $\partial D(\bar{s})$ with the plane $x_3 = \bar{x}_3$ can be parameterized by $x_1 = y_1(t)$; $x_2 = x_2(t) := H(\alpha(t), y_1(t), \bar{x}_3, y_3(y_1(t), \alpha(t)))$; $x_3 = \bar{x}_3$. Since $\partial H/\partial y_3 = 0$ for $y_3 = y_3(y_1, \alpha)$ by (3.2.3), we see

$$x'_2(t) = \alpha'(t)(\bar{x}_3 - y_3) + \partial h/\partial y_1 y'_1(t). \quad (3.2.9)$$

Let \bar{x} be on the line parameterized by \bar{t} in (3.2.8). If the normal to $\partial D(\bar{s})$ at \bar{x} were normal to W at \bar{w} , then $(y'_1(\bar{t}), x'_2(\bar{t}), 0) \cdot (-\partial h/\partial y_1, 1, -\partial h/\partial y_3) = 0$. This would imply that $x'_2(\bar{t}) - y'_1(\bar{t})\partial h/\partial y_1 = 0$. But because of (3.2.9), $0 = \alpha'(\bar{t})[\bar{x}_3 - y_3(y_1(\bar{t}), \alpha(\bar{t}))]$. Because of (3.2.5), $\alpha'(\bar{t}) = 0$ so $\bar{t} = \pi/2, 3\pi/2$. This corresponds to lines $l(\bar{s}, \pm\epsilon(\bar{s}))$ and points

$$\bar{x} \text{ with } \bar{x}_1 = \bar{s} \quad (3.2.10)$$

on the “left” and “right” ends of $\partial D(\bar{s})$.

Second we show points (3.2.10) are not in the boundary of the wedge \mathcal{W} . Let $g(x_1, \alpha, s) = (x_1 - s)^2 + \alpha^2 - \epsilon^2(s)$. Then $\frac{\partial g}{\partial s}(\bar{s}, \pm\epsilon(\bar{s}), \bar{s}) = -2\epsilon(\bar{s})\epsilon'(\bar{s}) > 0$. As $g(\bar{s}, \pm\epsilon(\bar{s}), \bar{s}) = 0$, for some $s_o < \bar{s}$, $g(\bar{s}, \pm\epsilon(\bar{s}), s_o) < 0$. This shows that if \bar{x} satisfies (3.2.10), then it is inside $D(s_o)$. Since $s_o < \bar{s}$, this contradicts the definition of \bar{s} . ■

Proposition 3.2.3. *Assume W , Y_o , and f satisfy the assumptions of Theorem 2.1. Then any two lines in Y_o can be joined in Y_o by a finite number of characteristic paths with pivot points away from $\text{supp } f$.*

Proof. For $\ell \in Y_o$ define $E(\ell)$ to be the set of all lines in Y_o that can be joined with ℓ by a finite sequence of characteristic paths with pivot points away from $\text{supp } f$ and W . We will show $E(\ell)$ contains a neighborhood of ℓ . This implies the lemma; for if Γ is a path in Y_o joining two arbitrary lines ℓ_o and ℓ_1 in Y_o , the collection $\{E(\ell) | \ell \in \Gamma\}$ forms an open cover of Γ , so an application of the Heine-Borel Theorem gives a finite subcovering completing the proof.

Let \mathcal{P} be a plane perpendicular to ℓ_o and away from $\text{supp } f$, and let p_o be the point of intersection of ℓ_o and \mathcal{P} . Let σ be the map that to any line ℓ in Y near ℓ_o associates its point of intersection with \mathcal{P} . Then σ is a smooth fibering of a neighborhood of ℓ_o and the fiber above $x \in \mathcal{P}$ is the characteristic path with pivot point x . We will prove:

$$\begin{aligned} &\text{projections under } \sigma \text{ of characteristic paths connect } p_o \text{ with} \\ &\text{every point in some neighborhood, } N, \text{ of } p_o \text{ in } \mathcal{P}. \end{aligned} \quad (3.2.11)$$

This statement will finish the proof of the proposition by allowing one to connect ℓ_o to any nearby line by at most two characteristic paths.

To prove (3.2.11), we first construct a smooth foliation of \mathcal{P} near p_o by projections of characteristic paths under σ . Let \mathcal{Q} be another plane perpendicular to ℓ_o and

¹By (3.2.3) and (3.2.5) $\frac{\partial x_2}{\partial \alpha}$ given from (3.2.7) has the proper sign so that points near \bar{x} on $l(y_1(t), \alpha(t))$ will be in $\partial D(\bar{s})$.

not intersecting $\text{supp } f$. Let q_o be the point of intersection of ℓ_o with \mathcal{Q} . Choose a small segment S through q_o on \mathcal{Q} that is perpendicular to the tangent plane of W at w_o , the point of tangency of ℓ_o with W . Each $q \in S$ is the pivot point of a characteristic path. If λ_q is the projection under σ of the characteristic path with pivot point q , then the λ_q for $q \in S$, give a one-parameter family of smooth curves in \mathcal{P} that is a foliation of a neighborhood of p_o . (This can be shown by proving that $(q, t) \rightarrow \lambda_q(t)$ for $q \in S$ and t a real parameter define local coordinates on \mathcal{P} .) Let v_o be a tangent vector at p_o to the curve λ_{q_o} . The vector $v_o \neq 0$ as $q_o \neq p_o$.

We now construct another characteristic path with projection that is transversal to the family λ_q at p_o . Let α be the curve traced out on \mathcal{Q} by the characteristic path of lines with pivot point p_o . The tangent line at p_o to the curve in \mathcal{P} traced out by pivoting at $q \in \alpha$ is the intersection of \mathcal{P} and the tangent plane to W through the line $\overline{p_o q}$ connecting p_o and q . The only way that all such tangent lines at p_o , for different $q \in \alpha$ can be parallel to v_o is if all the corresponding tangent planes to W of lines $\overline{p_o q}$ are parallel and hence the same plane. This would contradict the assumption that every line in Y_o is tangent to W along a tangent plane that is tangent to W in only a discrete set of points. Therefore, for some $q \in \alpha$, the curve on \mathcal{P} traced out by pivoting at q is transversal to v_o . This transversal curve allows one to connect p_o with any curve in the foliation that is sufficiently near p_o . The curve itself is the projection of a characteristic path (the one pivoting at q), so as the foliation fills out a neighborhood of p_o , (3.2.11) is proved. ■

Proposition 3.2.1 implies that if f is zero near a line $\ell_o \in Y_o$ then f is zero near points on lines in every characteristic path starting at ℓ_o . Proposition 3.2.3 implies that ℓ_o can be connected to any line in Y_o by a sequence of characteristic paths. This finishes the proof of Theorem 2.1.

3.3. Characteristic paths for type II complexes and the proof of Theorem 2.2.

The proof follows the general outline of the proof of Theorem 2.1. Let $\ell_o \in Y_o$ be a line disjoint from $\text{supp } f$ and let $\ell_1 \in Y_o$. We will construct a sequence of characteristic paths from ℓ_o to ℓ_1 in Y_o and show $\text{supp } f$ meets no line on the path.

A characteristic path with pivot point $x_o \in X$ is a smooth path of lines in $\pi_Y(\pi_X^{-1}(x_o))$ – a smooth path of lines through x_o and meeting γ . A characteristic path with pivot point at infinity is a path in Y

$$l(s) = \ell(\theta_o, \gamma(s)) \text{ for fixed } \theta_o, a \leq s \leq b. \quad (3.3.1)$$

Proposition 3.3.1. *Assume that the hypotheses of Theorem 2.2 hold for X , γ , and Y_o . Let $f \in \mathcal{E}'(X)$ and assume $R_\mu f = 0$ on Y_o . Let $l : [a, b] \rightarrow Y_o$ be a characteristic path and assume $l(a)$ does not meet $\text{supp } f$ and the pivot point of the path $l(s)$ is disjoint from $\text{supp } f$. Then*

$$l(s) \cap \text{supp } f = \emptyset \text{ for } a \leq s \leq b. \quad (3.3.2)$$

Proof. As in the proof of Proposition 3.2.1, we can assume the pivot point is at infinity. For such paths we use the family of circular cones

$$D(s) = \bigcup \{ \ell(\theta, \gamma(s)) \mid \theta \in \mathbf{S}^2, |\theta - \theta_o| \leq \epsilon(s) \} \text{ for } s \in [a, b]$$

where $\epsilon(s)$ is a positive function with negative derivative such that $D(a)$ is so small that it does not intersect $\text{supp } f$, and for all $s \in [a, b]$, $D(s)$ contains only lines in Y_o . Furthermore ϵ must decrease fast enough so that in each planar cross-section C of $\text{supp } f$ that is perpendicular to θ_o , $D(s) \cap C$ has decreasing radius as s increases. This can be done because $\text{supp } f$ is compact and lies away from $\gamma(\mathbb{R})$.

One way is as follows: For some $\rho > 0$ the tube about γ of radius ρ is disjoint from $\text{supp } f$. Let \mathcal{P}_o be the plane perpendicular to θ_o and through $\gamma(a)$. Assume that for $s > a$ and s sufficiently near a , $\gamma(s)$ and $\gamma'(s)$ lie “above” \mathcal{P}_o (that is $(\gamma(s) - \gamma(a)) \cdot \theta_o$ and $\gamma'(s) \cdot \theta_o$ both have the same sign or are 0). Let \mathcal{P}_1 be the plane $\rho/2$ units “below” \mathcal{P}_o . Let $g(s)$ be the projection of $\gamma(s)$ onto \mathcal{P}_1 . Let $e(s)$ be a positive decreasing function for $s \geq a$. Define $\epsilon(s)$ and $D(s)$ so that $D(s) \cap \mathcal{P}_1$ is the disc about $g(s)$ of radius $e(s)$ ($e(s)$ must be so small that the other requirements on $D(s)$ are satisfied, too). Because locally near $\gamma(a)$, $D(s) \cap \mathcal{P}_1$ does not touch $\text{supp } f$ and since γ and γ' lie above \mathcal{P}_o , ϵ decreases fast enough so that the radius of $D(s) \cap C$ decreases for any cross-section C of $\text{supp } f$ perpendicular to θ_o . This allows one to move $D(a)$ to $D(a_1)$ for some $a_1 > a$ and have the radius of $D(s) \cap C$ decrease as s increases to a_1 . This process can be continued to b because ρ is fixed, γ' is never parallel to θ_o , and $\gamma([a, b])$ is compact.

Because the radii of the cross-sections are decreasing, as in the proof of Proposition 3.2.1, no cotangent vector conormal to $\gamma'(s)$ is conormal to the boundary of $\cup_{a \leq s \leq \bar{s}} D(s)$. Now Proposition 3.1.1 and Theorem 8.5.6 [Hörmander 1983] can be used to finish the proof. ■

Proposition 3.3.2. *Assume that the hypotheses of Theorem 2.2 hold on γ and Y_o . Then any two lines in Y_o can be joined by a finite number of characteristic paths.*

Proof. We use the notation of Proposition 3.2.3. The set of curves γ_q is a foliation on \mathcal{P} near p_o because the curves γ_q are defined in terms of local coordinates on Y . The fact that Y_o contains no line in any plane containing γ (if γ is a plane curve) is used to construct a curve in the plane \mathcal{P} transversal to the foliation γ_q . The rest of the proof is essentially the same as the proof of Proposition 3.2.3. ■

The conclusion of Theorem 2.2 is true if lines tangent to γ are included in Y_o . Under the hypotheses of Theorem 2.2, γ is not a line; so, if ℓ is a line tangent to γ and $x \in \ell \setminus \gamma$, then x is also on lines in Y that are arbitrarily close to ℓ and that are not tangent to γ . Thus, Theorem 2.2 for non-tangent lines can be applied to eat away at $\text{supp } f \cap \ell$.

3.4. Proof of Theorem 2.3.

We prove this by reducing to Theorem 2.2 using an affine motion of RP^3 . The curve of directions on \mathbf{S}^2 , $\theta(t)$ for $t \in \mathbf{S}^1$ can be viewed as a curve in the plane at infinity in RP^3 . Each line with direction vector $\theta(t)$ intersects the same point on this curve at infinity. There is an affine motion, Ψ , of RP^3 that moves this curve at infinity (with at most a finite number of points removed) to an analytic plane curve, γ , in \mathbb{R}^3 . As $\theta(\mathbf{S}^1)$ is not a great circle of \mathbf{S}^2 , γ is not a line and so $\text{supp } (f \circ \Psi^{-1})$ does not meet any plane containing γ . As f has compact support

Ψ can be chosen so $\text{supp}(f \circ \Psi^{-1})$ stays in \mathbb{R}^3 . Now Theorem 2.2 applied to γ and $f \circ \Psi^{-1}$ can be used to prove Theorem 2.3.

4. Examples.

Counterexamples are given to Theorems 2.1, 2.2, and 2.3 if their geometric assumptions on the line complexes are not satisfied. An injective Radon transform on a non-admissible complex is given.

The Radon transform in examples 4.1-4.2 is the Radon transform with measure $\mu = 1$, $R = R_1$. The function used in the counterexamples will be

$$f(x_1, x_2, x_3) = \begin{cases} 1, & \text{if } (x_1, x_2, x_3) \in [-500, 500] \times [-1, 0] \times [2, 3] \\ -1, & \text{if } (x_1, x_2, x_3) \in [-500, 500] \times [-1, 0] \times [1, 2] \\ 0, & \text{otherwise.} \end{cases} \quad (4.1)$$

The function's crucial property is that $((x_1, 0, x_3), (0, 1, 0)) \in WF_A(f)$ for all $(x_1, x_3) \in [-500, 500] \times [1, 3]$. The counterexamples are constructed so that these singularities are not detected by data $R_\mu f|_{Y_o}$.

Example 4.1. *This example gives a type I complex Y defined by a surface that is tangent to a plane along a non-discrete set and for which the conclusion of Theorem 2.1 does not hold.*

Let W be a torus that is tangent to the $x_1 - x_3$ plane along a circle, lies in the bottom half space, $x_3 < 0$, and is symmetric about the $x_2 - x_3$ plane. Let Y be the type I complex defined by W , let $x^0 = (0, 0, x_3^0)$ be one of the two points where the x_3 -axis is tangent to W , and let $Y_o \subset Y$ be an open connected set of lines sufficiently close to the x_3 -axis and tangent to W near x^0 . The $x_1 - x_3$ plane is tangent to W in a non-discrete set of points (a circle), but all other hypotheses of Theorem 2.1 hold for Y_o . However, the conclusion of Theorem 2.1 does not hold because every line in Y_o intersects $[-500, 500] \times [-1, 0] \times [2, 3]$ in a segment of the same length as its intersection with $[-500, 500] \times [-1, 0] \times [1, 2]$

Example 4.2. *This example gives a type II complex Y defined by a curve (circle) that lies in a plane and for which the conclusion of Theorem 2.2 does not hold. A similar example contradicts the conclusion of Theorem 2.3 if the curve θ in S^2 is a great circle.*

Let γ be a circle in the bottom half of the $x_1 - x_3$ plane and symmetric about the x_3 -axis. Let Y be the type II complex defined by γ and let Y_o be an open connected subset of Y that is sufficiently close to the x_3 -axis. The support of the function f intersects the plane of γ but all other hypotheses of Theorem 2.2 hold. However, the conclusion of Theorem 2.2 does not hold for reasons similar to those given in example 4.1. Similarly, one can construct a counterexample to the conclusion of Theorem 2.3 using the function f above and the type III complex Y defined by the great circle of S^2 parallel to the $x_1 - x_3$ plane.

Example 4.3. *This example provides a Radon transform on a non-admissible line complex Y for which Helgason's support theorem holds for functions of compact support. This is a degree one complex as defined in §6 of [Greenleaf Uhlmann] but it is not admissible because Gelfand's admissibility condition, (3.1.3), fails.*

The complex, Y , we will consider will be parameterized by $y = (y_1, y_2, \theta_1)$; the line $l(y)$ will be defined to be the line intersecting $(y_1, y_2, 0)$ and parallel to $(\theta_1, \theta_2(y), 1)$ where

$$\theta_2(y) = y_1. \tag{4.2}$$

This example (with canonical measures) was described in the complex category by [Gelfand, Graev] (see also [Greenleaf Uhlmann pp. 231-237] for lines in \mathbb{R}^3).

Theorem 4.1. *Let Y be the line complex defined in (4.2), and let $\mu(x, \ell)$ be a real analytic function on Z (see (2.1)) that is never zero. Let $f \in \mathcal{E}'(\mathbb{R}^3)$. Assume $R_\mu f$ is zero for lines in Y that lie outside of the convex compact set K . Then f is supported in K .*

Proof of Theorem 4.1. We will only outline the microlocal analysis for this Radon transform as the calculations are similar to those in the admissible case. The set Z of (2.1) is defined by

$$x_1 = y_1 + x_3\theta_1 \quad \text{and} \quad x_2 = y_2 + x_3y_1, \tag{4.3}$$

and Λ is the set of all (x, ξ, y, η) such that

$$\begin{aligned} \xi_1 &= -a_1 & \eta_1 &= a_1 + x_3a_2 \\ \xi_2 &= -a_2 & \eta_2 &= a_2 \\ \xi_3 &= a_1\theta_1 + a_2y_1 & \eta_3 &= a_1x_3, \end{aligned} \tag{4.4}$$

where not both of a_1 and a_2 are zero. It follows that Λ is a subset of $T^*(X) \setminus 0 \times T^*(Y) \setminus 0$, and R_μ is an analytic Fourier integral operator associated with Λ . Let Λ_o be the set of all $(x, \xi, y, \eta) \in \Lambda$ satisfying

$$\xi_2 \neq 0 \quad \text{and} \quad \xi_1 - \xi_2x_3 \neq 0. \tag{4.5}$$

It follows that $\pi_X : \Lambda_o \rightarrow T^*X$ is an injective immersion, and all $(x, \xi) \in T^*(X) \setminus 0$ that satisfy (4.5) are in $\pi_X(\Lambda_o)$. This can be seen by solving for (y, η) in (4.3) and (4.4) using (4.5). To prove $\pi_Y : \Lambda_o \rightarrow T^*Y$ is two-to-one, one uses (4.4) to show x_3 satisfies

$$\eta_2x_3^2 - \eta_1x_3 + \eta_3 = 0. \tag{4.6}$$

Because of the first condition in (4.5), $\eta_2 \neq 0$ so (4.6) has two distinct solutions for x_3 if and only if $\eta_1^2 - 4\eta_2\eta_3 \neq 0$; by the relation between η and ξ given in (4.4) the latter expression is equal to $(\xi_1 - \xi_2x_3)^2$ on Λ . Now the second condition in (4.5) implies there are two distinct solutions to (4.6) on Λ_o and so π_Y is two-to-one on Λ_o . It is important to note that if $(y, \eta) \in \pi_Y(\Lambda_o)$ then the two points (x, ξ) and $(\hat{x}, \hat{\xi})$ associated to (y, η) in Λ_o satisfy $\hat{x}_3 = \xi_1/\xi_2$ and the dual relationship $x_3 = \hat{\xi}_1/\hat{\xi}_2$. These points $(\hat{x}, \hat{\xi})$ and (x, ξ) will be called *dual points*. The calculation showing π_Y is a local diffeomorphism on Λ_o is straight forward. These comments and the calculus of analytic Fourier integral operators prove the following lemma.

Lemma 4.2. *Let $(x, \xi) \in T^*(\mathbb{R}^3)$ satisfy (4.5). Let ℓ_o be the line in Y with $(x, \xi) \in N^*(\ell_o)$. Let $f \in \mathcal{E}'(\mathbb{R}^3)$ and assume $R_\mu f = 0$ for all lines in Y sufficiently near ℓ_o . Assume also that \hat{x} is not in $\text{supp } f$, where $(\hat{x}, \hat{\xi})$ is the dual point associated to (x, ξ) . Then $(x, \xi) \notin WF_A(f)$.*

Note that the assumption, $\hat{x} \notin \text{supp } f$, in Lemma 4.2 prevents singularities at \hat{x} and x from canceling when evaluating $R_\mu f$; precisely, in the proof we replace R_μ by a restricted operator, S_μ , for which the analogous projection to π_Y is injective and for which $S_\mu f = 0$.

To finish the proof of Theorem 4.1 we eat away at $\text{supp } f \setminus K$ by enclosing $\text{supp } f \cup K$ in slightly deformed balls. Let $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ be fixed and let $r > 0$; now define S_r to be the sphere centered at a of radius r . For small positive ϵ let C be the set of $x \in S_{r+\epsilon}$ such that for some covector $\xi \in N_x^*(S_{r+\epsilon})$, (x, ξ) does not satisfy the second condition in (4.5). Then C consists of a curve that “spirals” between the south and north poles of $S_{r+\epsilon}$. Let \mathcal{P}_t be the horizontal plane $x_3 = t$ and let D_r be the union of the convex hulls of $(C \cup S_r) \cap \mathcal{P}_t$ for $t \in [a_3 - r + \epsilon, a_3 + r - \epsilon]$. The set D_r consists of a deformed ball between flat top, T , and flat bottom, B . Finally, let L be the intersection of ∂D_r with the plane through a and parallel to the $x_1 - x_3$ plane; ϵ can be chosen so small that L is a longitudinal great circle of S_r flattened at the top and bottom. Therefore, covectors (x, ξ) conormal to ∂D_r along points $x \in L$ have $\xi_2 = 0$.

We claim that all points $(x, \xi) \in N^*(\partial D_r \setminus (C \cup T \cup B \cup L))$ satisfy (4.5). This is true because either such points are on S_r and away from “bad” points not satisfying (4.5); or they are near enough to C so that they lie on horizontal lines between S_r and C and have conormals ξ satisfying $\xi_1/\xi_2 < x_3$ or $\xi_1/\xi_2 > x_3$ (this second statement is true as C “juts” out from S_r along tangent lines to S_r).

Now assume the center, a , and radius r of D_r are chosen so that $\text{supp } f \cup K \subset D_r$ and so that $\text{supp } f \cup K$ is in one of the open half spaces determined by the plane $x_2 = a_2$. Therefore L does not meet $\text{supp } f$. The radius r can be chosen so large so that $\text{supp } f$ does not meet T or B as well. Therefore, if $\text{supp } f \setminus K$ meets ∂D_r then it must meet at a point $x_o \in C$ (if $x_o \notin C$, then Lemma 4.2 and Theorem 8.5.6 [Hörmander 1983] show $x_o \notin \text{supp } f$). As D_r is convex and is “pointed” at x_o , there is a plane through x_o tangent to D_r in conormal direction ξ_o where (x_o, ξ_o) does satisfy (4.5). Thus Hörmander’s theorem and Lemma 4.2 provide a contradiction. As such sets D_r can get arbitrarily close to K , $\text{supp } f \subset K$. ■

Bibliography.

- J. E. Björk 1979**, Rings of Differential Operators, North Holland, Amsterdam.
- J. Boman 1984**, Uniqueness theorems for generalized Radon transforms, in “Constructive theory of Functions ’84”, Sofia, pp. 173-176.
- J. Boman and E.T. Quinto 1987**, Support theorems for real analytic Radon transforms, *Duke Math. J.* **55**, 943-948.
- D.V. Finch 1985**, Cone beam reconstructions with sources on a curve, *SIAM J. Appl. Math.* **45**, 665-673.

- D.V. Finch 1986**, Uniqueness of the attenuated X-ray transform in the physical range, *Inverse Problems* **2**, 197-203.
- I.M. Gelfand and M. I. Graev 1961**, Integral transformations connected with straight line complexes in complex affine space, *Doklady* **2**, 809-812.
- I.M. Gelfand, M.I. Graev, and N. Vilenkin 1966**, Generalized functions, Vol. 5, Academic Press, New York.
- J. Globevnik 1990** A support theorem for the X-ray transform, *J. Math. Anal. Appl.* 165 (1992), 284-287.
- A. Greenleaf and G. Uhlmann 1989**, Non-local inversion formulas in integral geometry, *Duke J. Math.* **58**, 205-240.
- V. Guillemin 1985**, On some results of Gelfand in integral geometry, *Proc. Symposia in Pure Math.*, **43**, 149-155.
- V. Guillemin and S. Sternberg 1977**, Geometric Asymptotics, Amer. Math. Soc., Providence, RI.
- S. Helgason 1984**, Groups and Geometric Analysis, Academic Press, New York.
- L. Hörmander 1971**, Fourier integral operators I, *Acta Math.* **127**, 79-183.
- L. Hörmander 1983**, The Analysis of Linear Partial Differential Operators I, Springer, New York.
- A. Kaneko 1989**, Introduction to Hyperfunctions, Kluwer, New York, 1989.
- A. Markoe and E.T. Quinto 1985**, An elementary proof of local invertibility for generalized and attenuated Radon transforms, *SIAM J. Math. Anal.* **16**, 1114-1119.
- E.T. Quinto 1980**, The dependence of the generalized Radon transform on defining measures, *Trans. Amer. Math. Soc.* **257**, 331-346.
- E.T. Quinto 1991**, Real analytic Radon transforms on rank one symmetric spaces, *Proc. Amer. Math. Soc.*, to appear.
- M. Sato, T. Kawai, and M. Kashiwara 1973**, Hyperfunctions and pseudo-differential equations, Springer Lecture Notes in Math. **287**, pp. 265-529, New York.
- D.C. Solmon 1976**, The X-ray transform, *J. Math. Anal. Appl.* **56**, 61-83.
- F. Trèves 1980**, Introduction to Pseudodifferential and Fourier integral operators I, Plenum Press, New York.

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, S-11385 STOCKHOLM, SWEDEN

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155, USA

E-mail address: jabo@candida.matematik.su.se, equinto@math.tufts.edu