SUPPORT THEOREMS FOR REAL ANALYTIC RADON TRANSFORMS

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1. Introduction.

A generalized Radon transform, R , integrates functions on a manifold X over each member of a class of submanifolds, Y, using a specified measure of integration for each submanifold in Y. For both practical [Tretiak] and theoretical reasons [Helgason 1973], fairly arbitrary classes of submanifolds, Y, and fairly arbitrary measures are considered. Gelfand [1966], Helgason [1970], Grinberg [1983], Quinto [1983], and many others have investigated these transforms using techniques from group representations to integral equations. Our work is based on the seminal work of Guillemin. Guillemin and Sternberg [1977] have proven that many generalized Radon transforms are elliptic Fourier integral operators and that composition with their adjoints, R^*R , are elliptic pseudodifferential operators in the C^∞ category. The role the measures play in this pseudodifferential operator has been investigated [Quinto 1980]. Guillemin and Sternberg [1979] have proven range theorems for these transforms, and Guillemin [1985] has proven the characterization of admissable line complexes in C^3 using microlocal analysis. If the submanifolds and the measures are real analytic, R^*R has been proven to be an analytic pseudodifferential operator in certain cases, which implies invertibility [Boman 1984,1986]. In contrast, Boman [1985] discovered counterexamples to invertibility for positive C^∞ measures.

Many Radon transforms satisfy support theorems. Given appropriate functions f and appropriate subsets A of Y, if $Rf(y) = 0$ for each submanifold y in A , then f is zero on the union of the submanifolds in A. Helgason [1965,1973] proved this for many group invariant Radon transforms including the classical transform integrating over hyperplanes in $\mathbf{R}^n$ in Lebesgue measure. Cormack [1981, 1982], Solmon [1976], and others have proven support theorems for transforms integrating over various curves and surfaces and with non-standard measures (e.g., [Quinto 1983], [Hertle 1984], [Finch 1985]). Support theorems are useful in partial differential equations [Helgason 1973] and can have implications in tomography [Shepp and Kruskal]. However there are examples depending on the function class (e.g., [Shepp and Kruskal]) or measure [Boman 1985] for which support theorems do not hold.

Our goal is to prove support theorems for Radon transforms with positive real analytic measures on hyperplanes in $\mathbf{R}^n$. We will use the theory of analytic pseudodifferential operators and a lovely theorem of Kawai-Kashiwara-Hörmander about analytic wave front sets. Even for the classical transform, our theorems are as strong as presently known. Our theorems can be generalized to many other Radon transforms [Boman and Quinto, to appear]. The case considered here exhibits the important ideas, and the arguments in general, involving Fourier integral operators, are more esoteric.

Section Two defines the important terms and gives the theorems. Proofs are given in Section Three.

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2. Definitions and Main Theorems.

Let $\cdot$ be the inner product and let $|\ |$ be the norm on $\mathbf{R}^n$. For $\omega \in S^{n-1}$, and $p \in \mathbf{R}$ let $H(\omega, p) = \{x \in \mathbf{R}^n | x \cdot \omega = p\}$ be the hyperplane containing $p\omega$ and perpendicular to ω. Let dx_H be Lebesgue measure on the hyperplane. Let $\mu(x, \omega)$ be a C^∞ function on $\mathbf{R}^n \times S^{n-1}$. The generalized Radon transform $R_\mu : C_c^\infty(\mathbf{R}^n) \to C_c^\infty(S^{n-1} \times \mathbf{R})$ is defined by

$$R_\mu f(\omega, p) = \int_{H(\omega,p)} f(x)\mu(x, \omega) dx_H \quad .$$

R_μ is extended continuously to domain $\mathcal{E}'(\mathbf{R}^n)$ by continuity of its adjoint on C^∞.

If $\mu(x,\omega)$ is even in ω, then $R_\mu f(\omega,p)$ is even in (ω,p) and therefore can be considered to be a function on the set of hyperplanes.

Theorem 2.1. *Assume $(\omega_o,p_o) \in S^{n-1}\times \mathbf{R}$ and $f \in \mathcal{E}'(\mathbf{R}^n)$. Let $\mu(x,\omega)$ be a strictly positive real analytic function on $\mathbf{R}^n \times S^{n-1}$ that is even in ω. Let V be an open neighborhood of ω_o. Finally assume $R_\mu f(\omega,p)=0$ for $p>p_o$ and $\omega \in V$. Then $f=0$ on the half space $x\cdot\omega_o > p_o$.*

The proof of theorem 2.1 is the heart of this article. It involves analytic pseudo-differential operators and analytic microlocal analysis. The first key idea to the proof is that $R_\mu f(\omega,p)$ for ω near ω_o picks up all analytic singularities of f in direction ω_o . If $R_\mu f(\omega,p)=0$ for ω near ω_o and $p>p_o$, then f must be analytic in direction ω_o at all points x in the half space $x\cdot\omega_o>p_o$. (This is a slight abuse of notation; precisely, f is analytic at x in direction ω_o if (x,ω_o) is not in the analytic wave front set [Hörmander] of f.) The second key idea is a theorem of Kawai, Kashiwara and Hörmander [Theorem 8.5.6] that implies the result: if x_o is a boundary point of the support of a function, h, and h is zero on one side of the hyperplane through x_o perpendicular to ω_o, then h is *not* analytic at x_o in direction ω_o. Since the f above has compact support and is analytic in direction ω_o at all points in the half space, Hörmander's theorem implies f must be zero on this half space.

Theorem 2.1 implies

Theorem 2.2. *Let W be an open, unbounded, connected subset of $S^{n-1}\times\mathbf{R}$. Let $\mu(x,\omega)$ be a strictly positive real analytic function that is even in ω. Let $f\in\mathcal{E}'(\mathbf{R}^n)$ be such that $R_\mu f(\omega,p)=0$ for $(\omega,p)\in W$. Then $f=0$ on $\bigcup\{H(\omega,p) | (\omega,p)\in W\}$.*

Theorem 2.2 implies that if $R_\mu f(\omega,p)=0$ for hyperplanes outside the convex, compact set K , then f is supported in K . This support theorem is well known for the classical Radon transform, (herein denoted by R) [Helgason 1965]. Theorem 2.2 is local in the sense that the behavior of f away from points in hyperplanes in W does not play a role in the result.

Our next theorem generalizes the limited angle uniqueness theorems for the classical Radon transform.

Theorem 2.3. *Assume $J \subset S^{n-1}$ is contained in no proper real analytic variety in S^{n-1}. Let $\mu(x,\omega)$ be strictly positive and real analytic. Let $f \in \mathcal{E}'(\mathbf{R}^n)$ satisfy $R_\mu f(\omega,p) = 0$ for $\omega \in J$ and all p. Then $f = 0$.*

If Ω is a convex set, these theorems hold for $f \in \mathcal{E}'(\Omega)$, if $\mu(x,\omega)$ is positive and real analytic for $x \in \Omega, \omega \in S^{n-1}$. In this case, too, (3.4) is an analytic elliptic psuedodifferential operator. However, the theorems do not generally hold for piecewise analytic measures [Markoe and Quinto, Example 2], even though injectivity does hold in many cases [Boman 1984, 1986].

3. Proofs.

Throughout this section we assume $\mu(x,\omega)$ is strictly positive and real analytic for $(x,\omega) \in \mathbf{R}^n \times S^{n-1}$. We assume $f \in \mathcal{E}'(\mathbf{R}^n)$. For Theorems 2.1 and 2.2, we assume μ is even in ω.

Proof of Theorem 2.1. We first construct an elliptic pseudodifferential operator, (3.3), related to R_μ. Let F_n be the Fourier transform on $\mathbf{R}^n$,

$$F_n f(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)dx \ .$$

For $g \in \mathcal{S}(S^{n-1} \times \mathbf{R})$ let $F_1 g(\omega,\tau)$ be the one dimensional Fourier transform of g in the second variable. Let R^* be the classical dual Radon transform

$$R^* g(x) = \int_{\omega \in S^{n-1}} g(\omega, x\cdot\omega)d\omega.$$

Let $\lambda : S^{n-1} \times \mathbf{R} \to \mathbf{R}^n$ be defined by $\lambda(\omega,p) = p\omega$. This induces a map $\Lambda : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(S^{n-1} \times \mathbf{R})$, $\Lambda(f) = f \circ \lambda$. The projection slice theorem, $F_1 \circ R = \Lambda \circ F_n$, shows that the diagram

$$\begin{array}{ccc} \mathcal{S}(\mathbf{R}^n) & \xrightarrow{F_n} & \mathcal{S}(\mathbf{R}^n) \\ \downarrow R & & \downarrow \Lambda \\ \mathcal{S}(S^{n-1}\times\mathbf{R}) & \xrightarrow{F_1} & \mathcal{S}(S^{n-1}\times\mathbf{R}) \end{array} \qquad (3.1)$$

commutes. The adjoint of Λ is easily seen to be

$$\Lambda^* g(x) = \frac{g(x/|x|,|x|) + g(-x/|x|,-|x|)}{|x|^{n-1}} \ , \tag{3.2}$$

where, in our applications, the evaluation of (3.2) is pointwise. The dual diagram to (3.1) implies $F_n^* \circ \Lambda^* = R^* \circ F_1^*$. Therefore

$$\begin{aligned} R^* R_\mu f(x) &= \frac{1}{2\pi} F_n^* \circ \Lambda^* \circ F_1 \circ R_\mu f(x) \\ &= \frac{1}{\pi} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} \ \frac{\mu(y,\xi/|\xi|)}{|\xi|^{n-1}} f(y) dy d\xi \ . \end{aligned} \tag{3.3}$$

The second equality in (3.3) holds by the evenness of μ in the second variable. This expression shows $R^* R_\mu$ is an elliptic analytic pseudodifferential operator [Boutet de Monvel and Krée, Treves].

Now $R^* R_\mu$ is broken up into the sum of an operator, B, that is analytic regularizing in directions near ω_o, and an operator, C, that is zero for f as in the hypotheses of Theorem 2.1. Choose an even cut off function $\phi \in C^\infty(S^{n-1})$ with support in $V \cup -V$ and equal to one in a smaller neighborhood, V_1, of ω_o. Set $B = R^* R_{(1-\phi)\mu}$ and $C = R^* R_{\phi\mu}$. Then $B + C = R^* R_\mu$.

Lemma 3.4. *B is analytic regularizing in $\mathbf{R}^n \times V_1$. (For any $h \in \mathcal{E}'(\mathbf{R}^n)$ and any $(x,\omega) \in \mathbf{R}^n \times V_1$, Bh is analytic at x in direction ω.)*

Proof of Lemma 3.4. Let $h \in \mathcal{E}'(\mathbf{R}^n)$. As $R_{(1-\phi)\mu} h$ is a distribution of compact support, and $R^* : \mathcal{S}' \to \mathcal{S}'$, Bh is a tempered distribution. The Fourier transform of Bh can be calculated from (3.3) with μ replaced by $(1-\phi)\mu$. This Fourier transform is zero on the cone in $\mathbf{R}^n$ generated by V_1. By [Hörmander, Proposition 8.4.17], the analytic wave front set of Bh does not meet $\mathbf{R}^n \times V_1$. Thus B is analytic regularizing in $\mathbf{R}^n \times V_1$. ▌

Let f satisfy the hypotheses of Theorem 2.1. Let G be the half space $x \cdot \omega_o > p_o$. Assume supp$f$ has $H(\omega_o, p_1)$ as bounding hyperplane where $p_1 > p_o$. Let $x_1 \in H(\omega_o, p_1)$. By Lemma 3.4, Bf is analytic at x_1 in directions in V_1. By hypothesis, $Cf = 0$ in a neighborhood of x_1; hence Cf is analytic at x_1 in all directions. Therefore $R^* R_\mu f = Bf + Cf$ is real analytic at

x_1 in direction ω_o. As $R^* R_\mu$ is an elliptic analytic pseudodifferential operator, f is analytic at x_1 in direction ω_o. By [Hörmander, Theorem 8.5.1], f is zero in a neighborhood of x_1. As f has compact support, $p_1 \leq p_o$. ▌

Proof of Theorem 2.2. Assume W is open, unbounded, and connected in $S^{n-1} \times \mathbf{R}$. Assume $f \in \mathcal{E}'(\mathbf{R}^n)$ satisfies $R_\mu f(\omega, p) = 0$ for $(\omega, p) \in W$. Let K be the convex hull of the support of f, and let $C : [0,1] \to W$ be a continuous curve such that $H(C(1)) \cap K = \emptyset$. We will prove $f = 0$ on a neighborhood of $\bigcup\{H(C(t)) | t \in [0,1]\}$; this will prove the theorem.

Set $t_1 = \inf\{t | H(C(t_2)) \cap K = \emptyset \text{ for all } \; t_2 > t\}$. By the definition of t_1, the convex set K does not have points on both sides of $H(C(t_1))$. Since $R_\mu f$ is zero in a neighborhood of $C(t_1)$, Theorem 2.1 implies that f is zero in a neighborhood of $C(t_1)$. Thus $t_1 = 0$, and f is zero on the desired set. ▌

Theorem 2.3 is proven in a spirit similar to the classical proof. Essentially one shows that f is zero by showing its Fourier transform is analytic in too many directions for f to have compact support and be non-zero.

Proof of Theorem 2.3. For $\xi \in \mathbf{R}^n$ and $\theta \in S^{n-1}$, the function

$$G(\theta, \xi) = \int_{\mathbf{R}^n} f(x) \mu(x, \theta) e^{-ix \cdot \xi} dx$$

is analytic in both variables. So

$$G(\theta, \tau\theta) = F_1 R_\mu f(\theta, \tau)$$

is analytic in θ for every τ. By hypothesis, J is contained in no proper real analytic varety. As G is zero for all $\theta \in J$ and all τ, G is zero for all θ and τ; thus $R_\mu f \equiv 0$. The calculation giving (3.3) from (3.2) shows $R^* R_\mu$ is an elliptic analytic pseudodifferential operator whether μ is even in ω or not. Since $R^* R_\mu f \equiv 0$, so is f.▌

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