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# Two-radius Support Theorems for Spherical Radon Transforms on Manifolds

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ABSTRACT. Support theorems and injectivity are proven for the Radon transform for spheres of two fixed radii with arbitrary nowhere zero real-analytic weights in real-analytic Riemannian manifolds. The ratio of the radii of the spheres is assumed to be irrational and the radii are assumed to be sufficiently small in relation to the injectivity radius of the manifold. If a function or distribution is zero on a small set and its spherical Radon transform is zero on an open connected set of centers $\mathcal{A}$ that is large enough, then the function is zero for all points on the spheres of these two radii with centers in $\mathcal{A}$. The proof of the theorem is based on a microlocal regularity theorem for the sphere transform on manifolds [**20**] and a theorem of Hörmander, Kawai and Kashiwara [**16, 21**], and it generalizes [**30, 31**].

## 1. Introduction

This article provides a generalization of the main result in [**30, 31**], in which support theorems and injectivity are proven for restricted sets of spheres with arbitrary real-analytic measures on $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{RP}^n$. In these theorems we assume that the function (or distribution) $f$ is zero in a certain small open set and we assume $f$ has zero integral over spheres of two fixed radii $\frac{a}{2}$ and $\frac{b}{2}$ where $\frac{a}{b}$ irrational. We also assume the centers of the sphere lie in an open connected set which is not too small. In this article, we will prove two radius support theorems and injectivity for the general sphere transform in real-analytic Riemannian manifolds.

A number of people have worked on this problem on $\mathbb{R}^n$ and on manifolds. Fritz John [**17**], J. Delsarte and J. L. Lions [**7**], L. Flatto [**8**], and J. D. Smith [**24**] proved important uniqueness theorems for integrals over spheres of one fixed radius and of two well-chosen radii. Zalcman summarized previous results and proved the classical two-sphere theorem for $\mathbb{R}^n$; the sphere transform is injective under restrictions on radius $r$ in $\mathbb{R}^n$: if the integral of $f$ over every sphere in $\mathbb{R}^n$ of radii $a/2$ and $b/2$ is zero and $\frac{a}{b}$ is not a quotient of zeros of $J_{(n-2)/2}(z)$, the Bessel function with order $\frac{n-2}{2}$, then $f$ is zero [**7, 27, 28**]. The classical two-sphere theorem has

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weight function $\mu = 1$ and has the exceptional set the set of all quotients $\frac{z}{w}$, where $z$ and $w$ are positive zeros of this Bessel function $J_{(n-2)/2}(z)$ for $z \in \mathbb{R}$. This has been generalized to rank-one symmetric spaces in [**5**]. Berenstein, Gay, and Yger have proven strong local injectivity [**3**] and an inversion formula [**4**] for the transform on spheres of two well-chosen radii in $\mathbb{R}^n$. Schneider has a lovely theorem showing that integrals over spheres of *one* well-chosen radius on $\mathbb{S}^n$ (or $\mathbb{RP}^n$) are enough to determine uniqueness [**22**] (see also related work [**11, 15**]). Support theorems for related transforms have been proven on symmetric spaces [**5, 23**] and manifolds [**12, 20**]. In all of these cases, the spaces have a rich group structure and the measures are the canonical group-invariant ones. The problem is more subtle for the case we consider, the sphere transform with nowhere zero real-analytic weights on real-analytic manifolds, and we need to assume the function $f$ is zero on a small starter set to conclude $f$ is zero elsewhere. However, our theorems are valid quite generally.

Important work has been done on related problems, the Pompeiu problem on disks, in which the transform integrates over disks in canonical measures (e.g., [**27, 29**]), and the Morera problem over spheres, in which one integrates using complex forms (e.g., [**2, 12**]).

The spherical transform is defined in (2.5) and (2.6). Theorems 2.1 and 2.2 are local support theorems. Our global uniqueness theorem is Theorem 2.3.

## 2. Definitions and Main Theorems

Let $M$ be a real-analytic Riemannian manifold of dimension $n$ and let $d(x, y)$ be the geodesic distance on $M$. Let $y \in M$, and let $r > 0$. Define $S(y, r)$ to be the geodesic sphere of radius $r$ and centered at $y$ and define $B(y, r)$ to be the open geodesic disk of radius $r$:

$$S(y,r) = \{x \in M \,|\, d(x,y) = r\}, \quad B(y,r) = \{x \in M \,|\, d(x,y) < r\}. \tag{2.1}$$

For $\mathcal{A} \subset M$, define

$$B(\mathcal{A}, r) = \bigcup\nolimits_{y \in \mathcal{A}} B(y, r) \tag{2.2}$$

Recall that the *injectivity radius of $M$ at $x_0 \in M$* is the supremum, $I_{x_0} \in (0, \infty]$, of the set of $r > 0$ such that the exponential map $\exp_{x_0} : T_{x_0}(M) \to M$ is a surjective diffeomorphism from $B(0, r) \subset T_{x_0}(M)$ to $B(x_0, r) \subset M$. If $r < I_{x_0}$, then $S(x_0, r)$ is diffeomorphic to $\mathbb{S}^{n-1}$ and $S(x_0, r)$ is the boundary of $B(x_0, r)$. When $r < I_{x_0}$, we can define the *antipodal point to $x \in S(x_0, r)$ on $S(x_0, r)$* to be the unique point (besides $x$) on the geodesic containing $x$ and $x_0$ in $B(x_0, I_{x_0})$ that lies on the sphere $S(x_0, r)$. If one pulls the picture back to $T_{x_0}M$ using the exponential map, then $S(x_0, r)$ corresponds to a Euclidean sphere centered at the origin, the geodesic corresponds to a segment through the origin and $X = \exp_{x_0}^{-1} x$, and $x_a$ corresponds to the antipodal point to $X$ on the Euclidean sphere, $-X$ (e.g., [**18**], Chapter 1, Theorem 6.2 and [**26**] Chapter 3, Theorem 8.7).

We define the injectivity radius of $M$ to be

$$I_M = \inf\{I_{x_0} \,|\, x_0 \in M\}, \text{ and we assume } I_M > 0\,. \tag{2.3}$$

We now define the spherical Radon transform on spheres of radius $r > 0$. In order to ensure that all the spheres are diffeomorphic to Euclidean spheres, we will assume

$$I_M > r\,. \tag{2.4}$$

Under this assumption, the set

$$Z_r = \{(x,y) \in M \times M \,\big|\, d(x,y) = r\} \tag{2.5}$$

is an embedded real-analytic submanifold of $M \times M$.

Let $\mu$ be a nowhere zero real-analytic function on $Z_r$. For $f \in C(M)$, define the sphere transform on geodesic spheres in $M$ of radius $r$ to be

$$P_{\mu,r}f(y) = \int_{x\in S(y,r)} f(x)\mu(x,y)dx \tag{2.6}$$

when $y \in M$. Here $dx$ is the measure on $S(y,r)$ induced from the Riemannian measure on $M$. The assumption (2.4) ensures that $P^*_{\mu,r}$ is intrinsically defined as a dual operator, and it is a spherical transform with weight $\mu^*(x,y) = \mu(y,x)$. The operator $P_{\mu,r}$ can be defined locally under weaker assumptions. Let $y \in M$ and assume $r < R_0 < I_y$. Assume $r < I_z$ for all $z \in B(y,R_0)$. Then, $P_{\mu,r}$ and $P^*_{\mu,r}$ are well defined as spherical Radon transforms that are Fourier integral operators (see Lemma 3.2). In this case, the weight $\mu$ can be defined on an open subset of $Z_r$.

Our first two theorems are local support theorems.

THEOREM 2.1. *Let $M$ be a real-analytic manifold. Let $0 < a < b$ and assume $\frac{a}{b}$ is irrational. Let $y \in M$ and let $R_0 > 0$ and $\varepsilon > 0$ such that $R_0 + \varepsilon < I_y$. Assume $a + b < 2R_0$, and assume for all $z \in B(y, R_0 + \varepsilon)$, $2I_z > b$. Let $P_{\mu_a,a/2}$ be a spherical Radon transform on spheres of radius $a/2$ and nowhere zero real-analytic weight $\mu_a$, and let $P_{\mu_b,b/2}$ be a spherical Radon transform on spheres of radius $b/2$ and nowhere zero real-analytic weight $\mu_b$. Assume $f \in \mathcal{D}'(M)$ satisfies $P_{\mu_a,a/2}f = 0$ for all spheres of radius $a/2$ contained in $B(y,R_0+\varepsilon)$ and $P_{\mu_b,b/2}f = 0$ for all spheres of radius $b/2$ contained in $B(y,R_0+\varepsilon)$. Furthermore, assume $f$ is zero on a neighborhood of some sphere $S(y,r)$ for some $r \in [0,R_0]$. Then, $f = 0$ on $B(y,R_0+\varepsilon)$.*

An example in [**30**] shows the necessity of the requirement $a/b$ is irrational. The condition $a + b < 2R_0$ is important. If $b > a > R_0$ and $\varepsilon$ is chosen so small that $a > R_0 + \varepsilon$, then any function supported in $B(y, a - R_0 - \varepsilon)$ is a null function.

This theorem has the same flavor as the local Pompeiu theorems of Berenstein and Gay [**3**] in $\mathbb{R}^n$. They assume integrals of $f$ over disks of diameter $a$ and $b$ are zero (with measure $\mu = 1$) for all disks contained in $B(y,R_0)$. They assume $a/b$ is not the ratio of zeros of the Bessel function $J_{(n-2)/2}$ and $a + b < 2R_0$. Then, they conclude $f = 0$ on $B(y,R_0)$. Their theorem does not require $f$ to be zero on a small starter set, as does our theorem, but it has been proven only for canonical measures on $\mathbb{R}^n$ [**3**]. Their theorem has a different set of excluded radii than ours, but their assumptions are different from ours, and their proofs use specific harmonic analysis of $\mathbb{R}^n$. Global injectivity has been proven for the canonical transform on on rank-one symmetric spaces [**5**].

Theorem 2.1 can be generalized from one center, $y$, and one set $B(y,R_0)$ to open connected sets of centers $\mathcal{A}$.

THEOREM 2.2. *Let $M$ be a real-analytic manifold. Let $\mathcal{A}$ be a connected, open subset of $M$ and assume $R_0 > 0$ is chosen so that for each $y \in \mathcal{A}$, $R_0 \in (0, I_y)$. Let $0 < a < b$ and assume $\frac{a}{b}$ is irrational. Assume for all $z \in B(\mathcal{A}, R_0)$, $2I_z > b$,* *and assume $a+b<2R_0$. Let $P_{\mu_a,a/2}$ be a spherical Radon transform on spheres of radius $a/2$ and nowhere zero real-analytic weight $\mu_a$, and let $P_{\mu_b,b/2}$ be a spherical Radon transform on spheres of radius $b/2$ and nowhere zero real-analytic weight $\mu_b$. Assume $f\in\mathcal{D}'(M)$ satisfies $P_{\mu_a,a/2}f=0$ for all spheres of radius $a/2$ contained in $B(\mathcal{A},R_0)$ and $P_{\mu_b,b/2}f=0$ for all spheres of radius $b/2$ contained in $B(\mathcal{A},R_0)$. Furthermore, assume $f$ is zero on a neighborhood of some sphere $S(y,r)$ for some $r\in[0,R_0]$ and some $y\in\mathcal{A}$. Then, $f=0$ on $B(\mathcal{A},R_0)$.*

Theorem 2.1 and Theorem 2.2 are valid even if $I_M\not> b$, and $\mu_a$ and $\mu_b$ do not need to be defined globally. For Theorem 2.2, one must assume $\forall y\in B(\mathcal{A},R_0)$, $I_y>b$ and one can define $\mu_b$ on $\{(x,y)\in M^2\,|\,d(x,y)=b/2,\ x\in B(\mathcal{A},R_0)\}$ and similarly for $\mu_a$. This is true because the basic microlocal regularity theorem, Lemma 3.2, is valid under these assumptions.

Our global uniqueness theorem follows from Theorem 2.2.

THEOREM 2.3. *Let $M$ be a connected real-analytic manifold with injectivity radius $I_M>0$. Let $0<a<b<a+b<2I_M$ and assume $\frac{a}{b}$ is irrational. Let $P_{\mu_a,a/2}$ be a spherical Radon transform on spheres of radius $a/2$ and nowhere zero real-analytic weight $\mu_a$ on $Z_{a/2}$, and let $P_{\mu_b,b/2}$ be a spherical Radon transform on spheres of radius $b/2$ and nowhere zero real-analytic weight $\mu_b$ on $Z_{b/2}$ (see (2.6) and (2.5)). Assume $f\in\mathcal{D}'(M)$ satisfies $P_{\mu_a,a/2}f=0$ for all spheres of radius $a/2$ in $M$ and $P_{\mu_b,b/2}f=0$ for all spheres of radius $b/2$ in $M$. Furthermore, assume $f$ is zero on a neighborhood of some sphere $S_0$ of radius $r\in[0,I_M)$. Then, $f=0$ on $M$.*

These theorems can be applied to spheres of any two radii (with irrational ratio) on noncompact symmetric spaces since these spaces have infinite injectivity radius.

## 3. Lemmas and Proofs

In order to prove the main theorems, we need some preliminary results, the first of which is a theorem of Hörmander, Kawai and Kashiwara. For $f\in\mathcal{D}'(M)$ we let $\mathrm{WF}_A(f)$ be the analytic wavefront set of $f$ [**16, 25**]. If $S$ is a submanifold of $M$, then $N^*(S)$ denotes the conormal bundle of $S$ in $T^*(M)$, and for $x_0\in S$, $N^*_{x_0}(S)$ is the fiber above $x_0$ (that is, all covectors conormal to $T_{x_0}(S)$).

LEMMA 3.1 (Microlocal Analytic Continuation Theorem [**16, 21**]). *Let $f\in C(M)$ (or $f\in\mathcal{D}'(M)$). Let $x_0\in M$ and let $U$ be a neighborhood of $x_0$. Assume $S$ is a $C^2$ submanifold of $U$ and $x_0\in S$. Assume $S$ divides $U$ into two nonempty open connected sets, and assume $f$ equals zero on one of these open sets. Let $\eta\in N^*_{x_0}S\setminus 0$, then $(x_0,\eta)\in\mathrm{WF}_A(f)$.*

This statement is a strengthening of the fact that if $f$ is zero on one of the connected parts of $U$ and supp $f$ meets $S$ at the point $x_0$, then $f$ cannot be real-analytic near $x_0$. Under these assumptions, not only is $f$ not real-analytic, but the conormal directions to $S$ at $x_0$ are in $\mathrm{WF}_A(f)$.

LEMMA 3.2 (Microlocal Regularity of $P_{\mu,r}$ [**20**]). *let $M$ be a real-analytic Riemannian manifold and let $\mathcal{A}\subset M$ be open. Let $r>0$ and assume for each $y\in B(\mathcal{A},r)$, $r<I_y$. Let $f\in C(M)$ or $f\in\mathcal{D}'(M)$. Let $y\in\mathcal{A}$, assume $P_{\mu,r}f(y)=0$ in a neighborhood of $y$ where $\mu_r$ is a nowhere zero real-analytic function on $\{(x,y)\in M^2\,|\,d(x,y)=r,\ x\in B(\mathcal{A},r)\}$. Let $x_0\in S(y,r)$ and let $x_a$ be the*

*antipodal point to* $x_0$ *on* $S(y,r)$*. Then,*

$$\text{(3.1)}\qquad \mathrm{WF}_A(f)\cap N^*_{x_a}S(y,r)=\emptyset \text{ if and only if } \mathrm{WF}_A(f)\cap N^*_{x_0}S(y,r)=\emptyset\,.$$

Note that Lemma 3.2 implies that if $f$ is zero in a neighborhood of $x_a$, then $\mathrm{WF}_A(f)\cap N^*_{x_0}(S(y,r))=\emptyset$. This theorem is a special case of a broad range of microlocal theorems about Radon transforms (e.g., [**1**]) that are based on a fundamental result of Guillemin and Sternberg [**13**]. They proved that many Radon transforms defined by double fibrations [**14, 9**] are Fourier integral operators. For this reason, the behavior of wavefront sets of distributions under such operators is well understood, and this has been used to prove support theorems like the ones in this article (e.g., [**6, 12, 10**]).

Now we are going to prove a density theorem in $\mathbb{R}$. We will use this theorem to show how to cover geodesic intervals in $M$ by spheres.

LEMMA 3.3 (Density Theorem 1). *Let* $D_0>0$*. Let* $a$ *and* $b$ *be positive real numbers such that* $a<b<a+b<D_0$*, and* $\frac{a}{b}$ *is irrational. Let*

$$\text{(3.2)}\qquad S=\{na-mb\,\big|\,n\geq 0, n,m\in\mathbb{Z} \text{ and } na-mb\in[0,D_0]\}.$$

(*i*) *The set* $S$ *is dense in* $[0,D_0]$*.*

(*ii*) *For any two points in* $S$*, a finite sequence of segments* $\mathcal{J}_1,\dots,\mathcal{J}_k$*, each of length* $a$ *or* $b$*, can be chosen to join these points and such that each* $\mathcal{J}_i$ $(i=1,\dots,k)$ *lies in* $[0,D_0]$ *and one endpoint of each* $\mathcal{J}_i$ $(i=1,\dots,k-1)$ *is also an endpoint of next* $\mathcal{J}_{i+1}$*, and both endpoints of* $\mathcal{J}_i$ $(i=1,\dots,k)$ *lie in* $S$*.*

PROOF. Let $a'\in(0,1)$ be irrational, and let

$$\text{(3.3)}\qquad A=\{na'-[na']\,\big|\,n\in\mathbb{N}\}\,.$$

G. Polya and G. Szegö prove that $A$ is equidistributed in $[0,1]$ ([**19**], pp. 88, No. 166), and this implies $A$ is dense in $[0,1]$ by the definition of the equidistribution ([**19**], pp. 88).

In general, let $a'=\frac{a}{b}$, then $a'\in(0,1)$ and $a'$ is irrational. Applying Polya and Szegö's result to $a'$, and dilating $[0,1]$, we have $\{na'-m\,\big|\,n\geq 0,\ m\in\mathbb{Z},\ na'-m\geq 0\}$ is dense in $[0,\infty)$. This shows, for $t=\frac{D_0}{b}$ that $\{na'-m\,\big|\,n\geq 0,\ m\in\mathbb{Z},\ na'-m\in[0,t]\}$ is dense in $[0,t]$. Therefore, $S$ is dense in $[0,D_0]$ and (*i*) is proven.

Now, we show we can connect 0 to any other point in $S$ using segments as described in (*ii*). Let $s=na-mb\in S$. If either $n=0$ or $m\leq 0$, then one can join 0 and $s$ by an increasing sequence of intervals, $n$ of which are of length $a$ and $-m$ of which are length $b$. All of these segments lie in $[0,D_0]$ by construction. So, we assume $n>0$ and $m>0$. Since $a+b<D_0$, at least one of $(n-1)a-mb$ and $na-(m-1)b$ is in the interval $[0,D_0]$. This shows we can join $s$ to a point with smaller $n$ or smaller $m$ by a segment in $[0,D_0]$ of length $a$ or $b$ and with endpoints in $S$. We finish the proof by induction. □

We use Lemma 3.3 and the exponential map to transfer the segment $[0,D_0]$ to a geodesic segment on the manifold $M$ and to transfer the dense set $S$ to a dense set on this geodesic segment. We will use this dense set and spheres associated to it to prove our main microlocal smoothness theorem on $M$. To do this, we need to introduce some notation.

DEFINITION 3.4. Let $y\in M$ and let $R_0\in(0,I_y)$. Let $x\in S(y,R_0)$, and let $\ell_{xy}$ be the unique geodesic in $B(y,I_y)$ through $x$ and $y$. For $s\in(R_0-I_y,R_0+I_y)$

define $g_{xy}(s)$ to be the unique point $|s|$ units from $x$ on $\ell_{xy}$, in the direction of $y$ if $s > 0$ and in the opposite direction if $s < 0$. Let $a < D_0$ and $b < D_0$. We define

$$L_{xy} = g_{xy}([0, D_0]), \quad L_a = g_{xy}([a/2, D_0 - a/2]), \quad L_b = g_{xy}([b/2, D_0 - b/2])\,. \tag{3.4}$$

If $z_1$ and $z_2$ are in $\ell_{xy}$ then we define $\overline{z_1 z_2}$ to be the geodesic segment between $z_1$ and $z_2$ in $\ell_{xy}$. Finally, if $p \in \ell_{xy}$, then we say $\eta_p \in T_p^* M$ is **cotangent to $\ell_{xy}$ at** $p$ if and only if $\eta_p$ is conormal to any sphere through $p$ in $B(y, I_y)$ that is centered at a point of $\ell_{xy}$.

So, $g_{xy} : (R_0 - I_y, R_0 + I_y) \to \ell_{xy}$ is an isometry. Note that $g_{xy}(0) = x$, $g_{xy}(R_0) = y$, and $g_{xy}(2R_0)$ is the antipodal point to $x$ on $S(y, R_0)$. The geodesic segments $L_a$ and $L_b$ correspond to the points on $L_{xy}$ at least $a/2$ units (respectively $b/2$ units) from the endpoints of $L_{xy}$, $x$ and its antipodal point. The concept that $\eta_p$ is cotangent to $\ell_{xy}$ is well defined independent of the specific sphere through $p$ centered at a point on $\ell_{xy}$ because all such spheres have the same tangent space at $p$ [**18**].

LEMMA 3.5 (Density Theorem 2). *Let $D_0 > 0$. Let $a$ and $b$ be positive real numbers such that $a < b < a + b < D_0$, and $\frac{a}{b}$ is irrational. Assume $M$ is a real-analytic manifold and $y \in M$ and $I_y > D_0/2 = R_0$. Let $x \in S(y, R_0)$ and let $g_{xy}$ and $\ell_{xy}$ be as in Definition 3.4. Let*

$$S_{xy} = \{g_{xy}(na - mb) \,\big|\, n \geq 0, n, m \in \mathbb{Z} \text{ and } na - mb \in [0, D_0]\}. \tag{3.5}$$

($i$) *The set $S_{xy}$ is dense in $L_{xy}$.*
($ii$) *For any two points in $S_{xy}$, a finite sequence of geodesic segments $J_1, ..., J_k$, each of length $a$ or $b$, can be chosen to join these points and such that each $J_i$ ($i = 1, ..., k$) lies in $L_{xy}$ and one endpoint of each $J_i$ ($i = 1, ..., k-1$) is also an endpoint of next $J_{i+1}$, and both endpoints of $J_i$ ($i = 1, ..., k$) lie in $S_{xy}$.*

PROOF. The proofs of ($i$) and ($ii$) follow from Lemma 3.3 and the fact $g_{xy}$ is an isometry from $[0, D_0]$ to $L_{xy}$. So, for example, $S$ is transferred to $S_{xy}$ and the segments $\mathcal{J}_i$ are transferred to the segments $J_i$. □

LEMMA 3.6 (Microlocal Smoothness Theorem). *Let $M$ be a real-analytic manifold and let $y \in M$. Assume $R_0 > 0$ and $\varepsilon > 0$ such that $R_0 + \varepsilon < I_y$. Let $D_0 = 2R_0$. Let $0 < a < b < a + b < D_0$ and assume $\frac{a}{b}$ is irrational. Assume $b < 2I_z$ for all $z \in B(y, R_0 + \varepsilon)$. Let $f \in \mathcal{D}'(M)$. We use the notation of Definition 3.4. Let $x \in S(y, R_0)$. Assume $f$ is zero in the epsilon neighborhood of $q_0$ for some $q_0 \in L_{xy}$. Let $\mu_a$ be a nowhere zero real-analytic function on $\{(x, y) \,\big|\, d(x, y) = a/2,\ x \in B(y, R_0 + \varepsilon)\}$ and let $\mu_b$ be a nowhere zero real-analytic function on $\{(x, y) \,\big|\, d(x, y) = b/2,\ x \in B(y, R_0 + \varepsilon)\}$. Assume $P_{\mu_a, a/2} f$ is zero in the epsilon neighborhood of $L_a$ and $P_{\mu_b, b/2} f$ is zero in the epsilon neighborhood of $L_b$. Then, for each $p \in L_{xy}$ and each $\eta_p$ cotangent to $\ell_{xy}$ at $p$, $(p, \eta_p) \notin \mathrm{WF}_A(f)$.*

PROOF. The proof is done in several steps. First, we use Lemma 3.2 and Lemma 3.5 to show $(p, \eta_p) \notin \mathrm{WF}_A(f)$ for all $p \in S_{xy}$ and all $\eta_p$ cotangent to $\ell_{xy}$ at $p$. Then, we perturb the picture to conclude that $(p, \eta_p) \notin \mathrm{WF}_A(f)$ for all $p \in L_{xy}$.

Since $S_{xy}$ is dense in $L_{xy}$, there is a point $p_0 \in S_{xy}$ within $\varepsilon$ units of $q_0$. Let $p \in S_{xy}$. By Lemma 3.5, there are points $p_1, \ldots, p_k = p$ in $S_{xy}$ and geodesic segments of length $a$ and $b$, $J_1, \ldots, J_k$, $J_i = \overline{p_{i-1} p_i}$ for $i = 1, \ldots, k$, and these segments join $p_0$ to $p = p_k$.

Since $f$ is zero near $p_0$, $(p_0, \eta_{p_0}) \notin \mathrm{WF}_A(f)$ for any $\eta_{p_0}$ cotangent to $\ell_{xy}$ at $p_0$. By construction, $J_1$ has length either $a$ or $b$. Assume its length is $a$. So, $J_1$ is the diameter of a disk of radius $a/2$ centered at a point of $L_a$. Furthermore, $f$ has zero integrals over all spheres of radius $a/2$ centered near $L_a$ and $(p_0, \eta_{p_0}) \notin \mathrm{WF}_A(f)$. Therefore, Lemma 3.2 and (3.1) imply that $(p_1, \eta_{p_1}) \notin \mathrm{WF}_A(f)$ for any $\eta_{p_1}$ cotangent to $\ell_{xy}$ at $p_1$. We can proceed in the same way if $J_1$ has length $b$ and then for $i > 1$ to conclude for each $i = 2, \dots, k$, that $(p_i, \eta_{p_i}) \notin \mathrm{WF}_A(f)$ for any $\eta_{p_i}$ cotangent to $\ell_{xy}$ at $p_i$. Since $p = p_k$ was arbitrary, this shows that

$$\forall p \in S_{xy}, \quad \forall \eta_p \text{ cotangent to } \ell_{xy} \text{ at } p, \quad (p, \eta_p) \notin \mathrm{WF}_A(f)\,. \tag{3.6}$$

Now we prove (3.6) for all $p \in L_{xy}$. Since $f$ is zero in a full neighborhood of $q_0$, the result (3.6) can be translated, and it is valid for all sufficiently small $\delta > 0$ and for all $p \in S^{\delta}_{xy} = g_{xy}(S + \delta)$. Since $S_{xy}$ is dense in $L_{xy}$, this proves (3.6) for all $p \in L_{xy}$. □

Proof of Theorem 2.1. The proof uses Lemma 3.1 and Lemma 3.6. We use the notation of Definition 3.4. Let $\overline{B}(y, R_0)$ be the closed disk of radius $R_0$ centered at $y$. We consider two cases.

In the first case, we assume $\overline{B}(y, R_0) \cap \operatorname{supp} f \subset \{y\}$. Since $a < R_0$, $f$ is zero on a neighborhood of a closed disk of radius $a/2$ contained in $\overline{B}(y, R_0) \setminus \{y\}$. Now, the main theorem of [**20**] can be used to conclude $f = 0$ on $B(y, R_0 + \varepsilon)$ since the set of spheres of radius $a/2$ contained in $B(y, R_0 + \varepsilon)$ is connected.

Now, in the second case we assume $\overline{B}(y, R_0) \cap \operatorname{supp} f$ contains points besides $y$. In this case, since $f$ is zero in a neighborhood of $S(y, r)$, there is a radius $r_1 \in (0, R_0]$ and a point $x_1 \in S(y, r_1) \cap \operatorname{supp} f$ and a neighborhood $U$ of $x_1$ such that $S(y, r_1) \cap U$ separates $U$ into two open sets and on one of which $f$ is zero (we expand or contract $r$ until the sphere first touches $\overline{B}(y, R_0) \cap \operatorname{supp} f$). Let $x$ be one of the two points on $S(y, R_0)$ on the geodesic through $x_1$ and $y$ in $B(y, I_y)$. So, $\ell_{xy}$ contains $x_1$. Lemma 3.1 implies that $(x_1, \eta_{x_1}) \in \mathrm{WF}_A(f)$ where $\eta_{x_1}$ is any covector cotangent to the geodesic $\ell_{xy}$ at $x_1$. However, because $f$ is zero on a neighborhood of $S(y, r)$, we can apply Lemma 3.6 to $x_1$ and $\ell_{xy}$, to conclude $(x_1, \eta_{x_1}) \notin \mathrm{WF}_A(f)$. This contradiction shows $x_1 \notin \operatorname{supp} f$ and $f$ is zero on $\overline{B}(y, R_0)$. Finally, we apply the first case to complete the proof. □

Proof of Theorem 2.2. Let $y$ be the center of the sphere near which $f$ is zero. Theorem 2.1 implies $f$ is zero on $B(y, R_0 + \varepsilon)$ for some $\varepsilon > 0$ (here we use that $\mathcal{A}$ is open). Let $A = B(\mathcal{A}, R_0)$. If $f \neq 0$ on $A$, since $\mathcal{A}$ is connected, there is some $y_1 \in \mathcal{A}$ such that $f = 0$ near $y_1$ but $f \neq 0$ on $B(y_1, R_0)$. Another use of Theorem 2.1 shows that $f = 0$ on $B(y_1, R_0)$, and this provides a contradiction. □

Proof of Theorem 2.3. If $r$ is the radius of the sphere near which $f$ is zero, choose $R_0 \in (\max(a + b, r), I_M)$ and apply Theorem 2.2 to the connected set $\mathcal{A} = M$. □

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