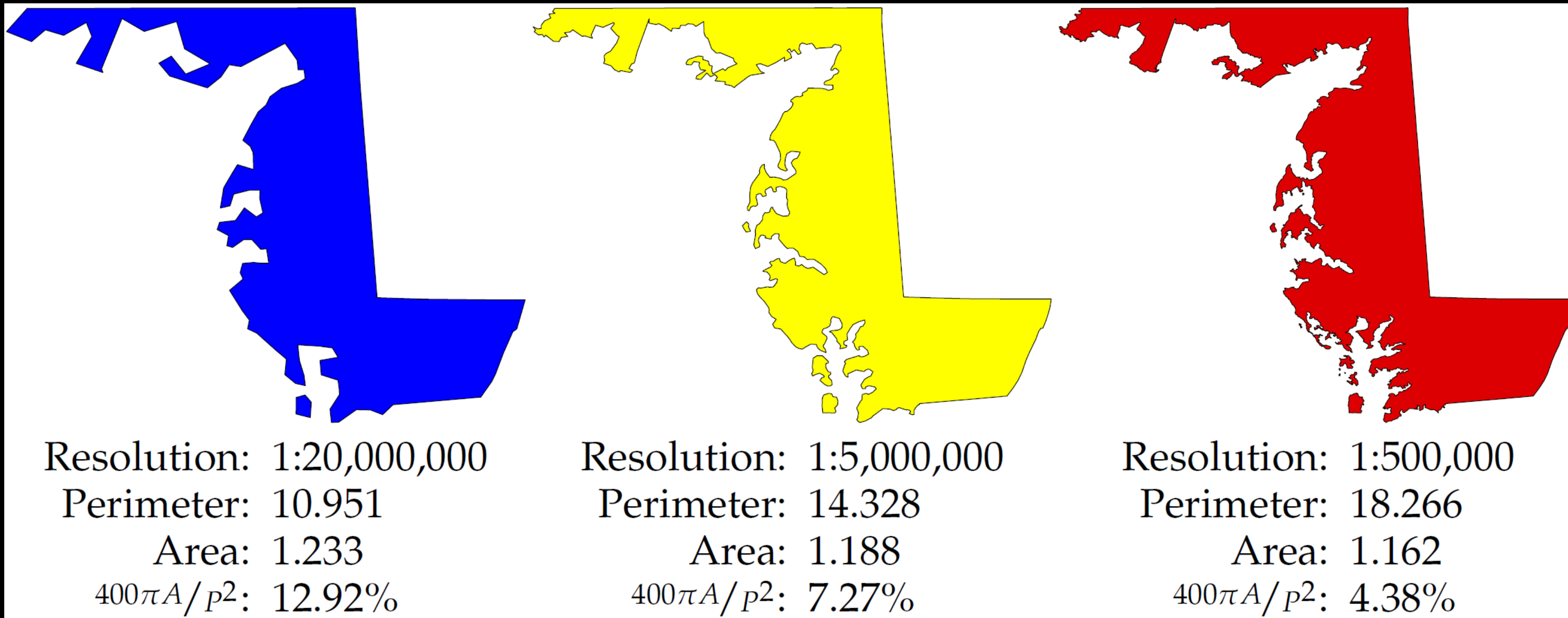


Counterexamples
in Political Redistricting

Justin Solomon
MIT EECS

MIT EECS

GDP @ MIT

GDP@MIT Home Team Publications Courses Contact

Geometric Data Processing Group

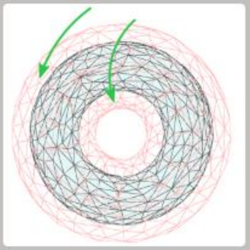
Principal investigator: Prof. Justin Solomon
Computer Science and Artificial Intelligence Laboratory (CSAIL)
Massachusetts Institute of Technology (MIT)

http://gdp.csail.mit.edu

Welcome!

The MIT **Geometric Data Processing Group** studies geometric problems in computer graphics, computer vision, machine learning, and other disciplines.

Our team includes students and researchers spanning a variety of disciplines, from theoretical mathematics to applications in engineering and software development. We enthusiastically welcome collaborators and staff at all levels and encourage interested parties to contact us with ideas, challenging problems, and avenues for joint research.

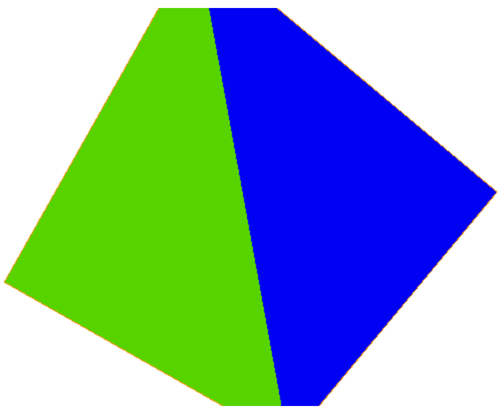


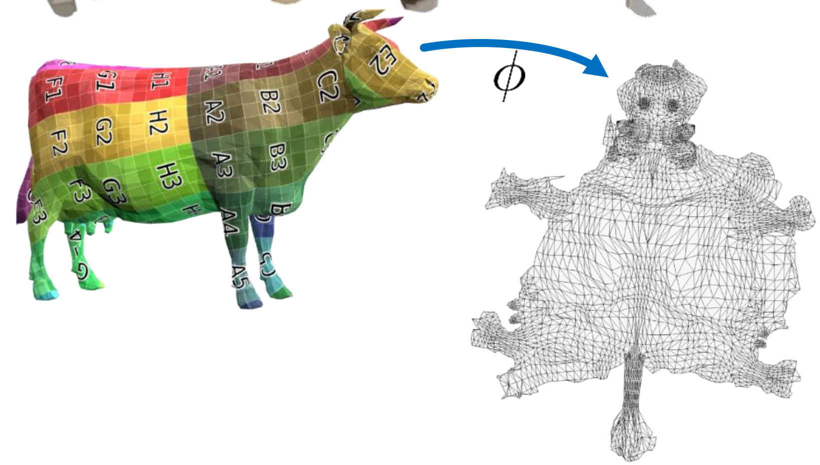
News

Affiliations: EECS, CSAIL, Metric Geometry & Gerrymandering Group, MIT Center for Computational Engineering, MIT-IBM Watson AI Lab

New website

Examples

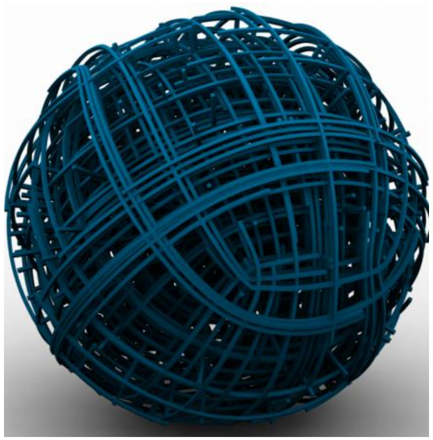


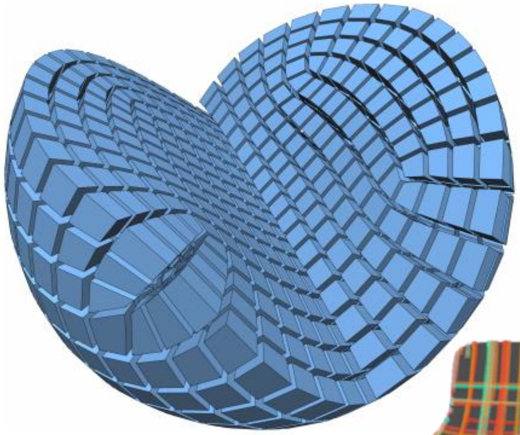


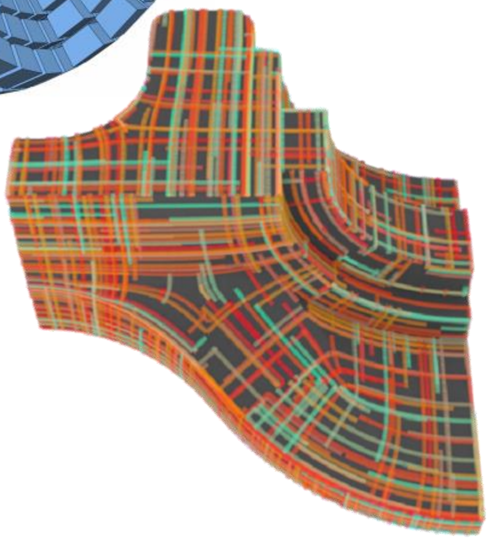
How do you embed domains into one another
efficiently and with low distortion?

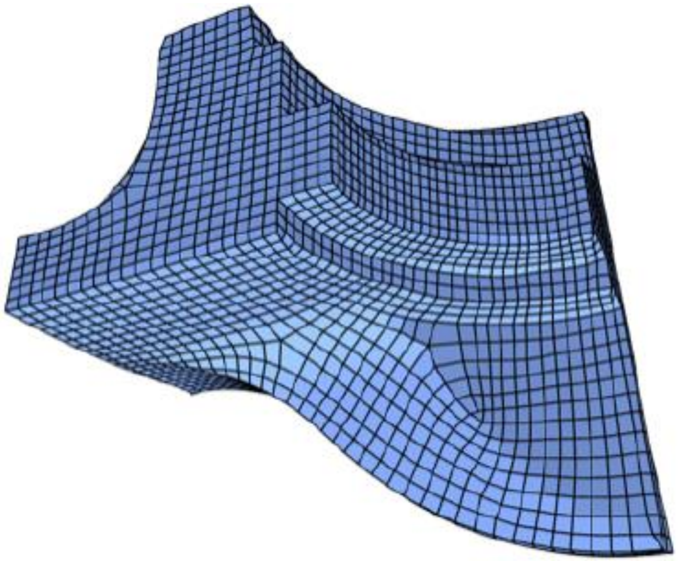
Claici et al. "Isometry-Aware Preconditioning for Mesh Parameterization." SGP 2017, London.
Li et al. "OptCuts: Joint Optimization of Surface Cuts and Parameterization." SIGGRAPH Asia 2018, Tokyo.
Gehre et al. "Interactive Curve Constrained Functional Maps." SGP 2018, Paris.

Examples





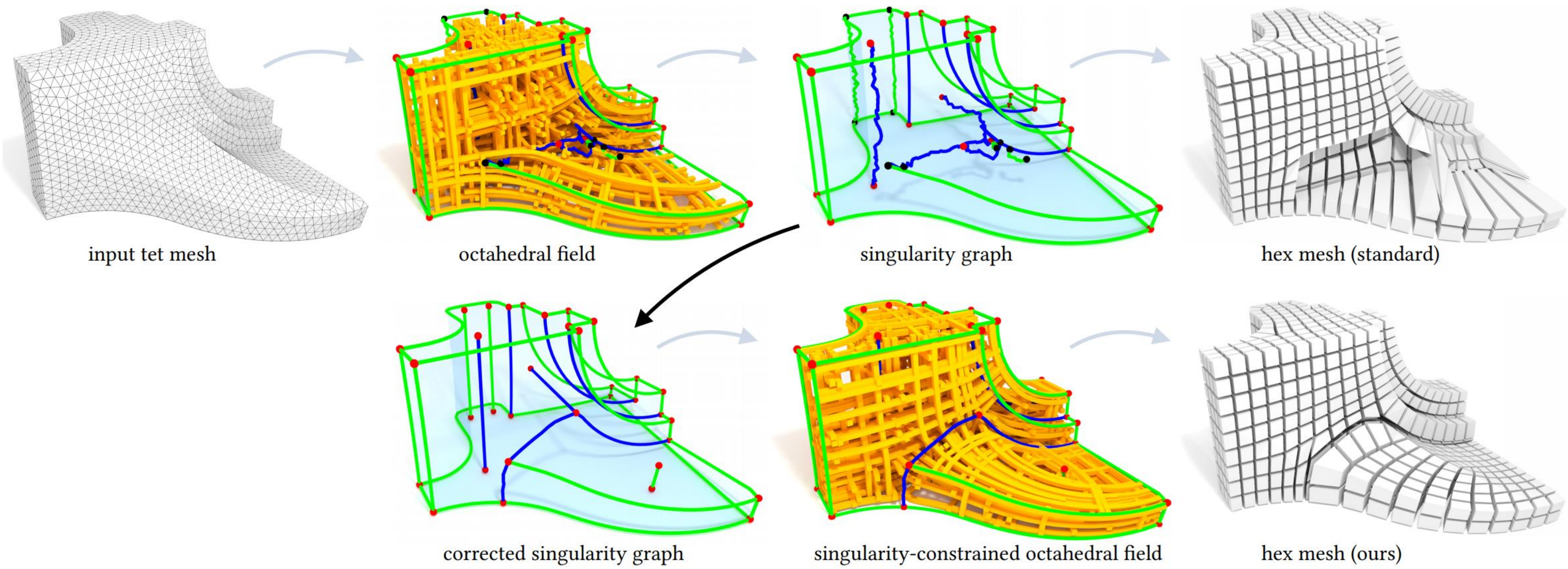




How can we tile a shape with simpler elements?

Solomon, Vaxman, and Bommes. "Boundary Element Octahedral Fields in Volumes." TOG 2018.
Zhang et al. "Spherical Harmonic Frames for Feature-Aligned Cross-Fields." Submitted.

Examples



How do we optimize in exotic spaces with topological constraints?

Liu, Zhang, Chien, Solomon, and Bommes.
"Singularity-Constrained Octahedral Fields for Hexahedral Meshing." SIGGRAPH 2018.

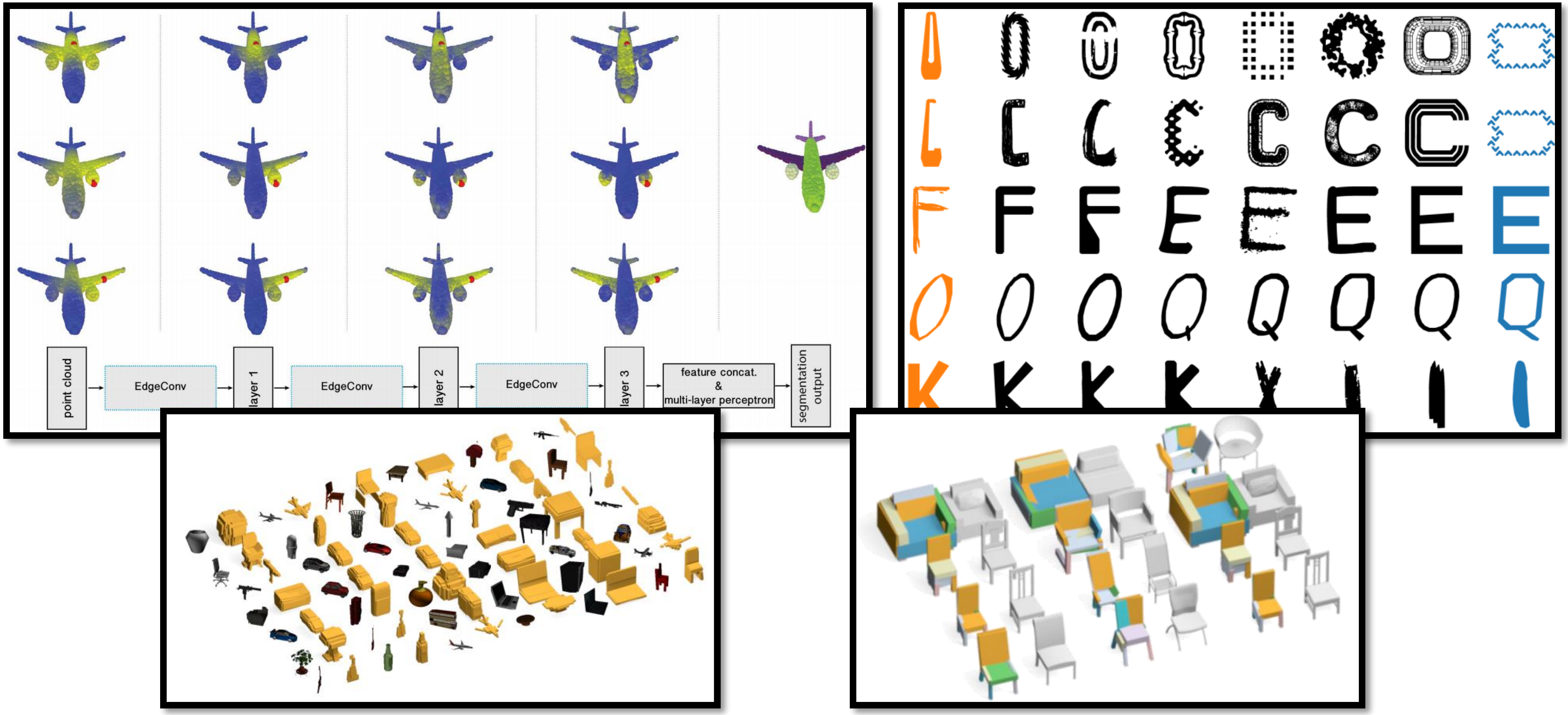
Examples

$$I_\Omega^{\mathrm{TV}}(t) := \begin{cases} \min_{f \in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\,dx = t \\ & 0 \leq f \leq \mathbb{1}_\Omega \end{cases}$$

How do we stabilize classical geometric measurements?

DeFord, Lavenant, Schutzman, and Solomon.
"Total Variation Isoperimetric Profiles." SIAM SIAGA 2019.

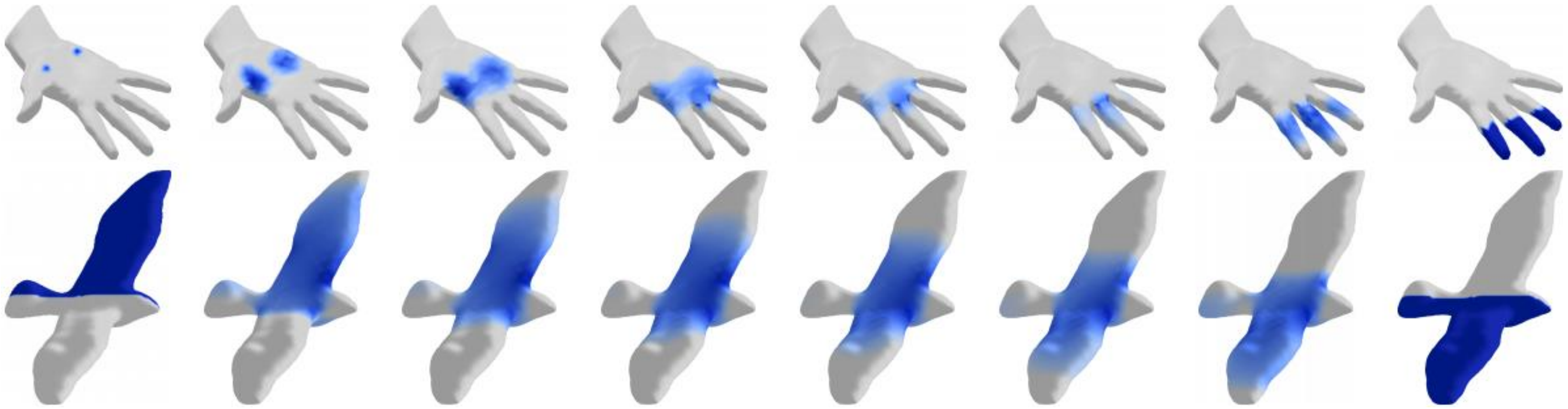
Examples



How do we learn from geometrically-structured data?

Wang et al. "Dynamic Graph CNN for Learning on Point Clouds." TOG 2019.
Smirnov et al. "Deep Parametric Shape Predictions using Distance Fields." Submitted.

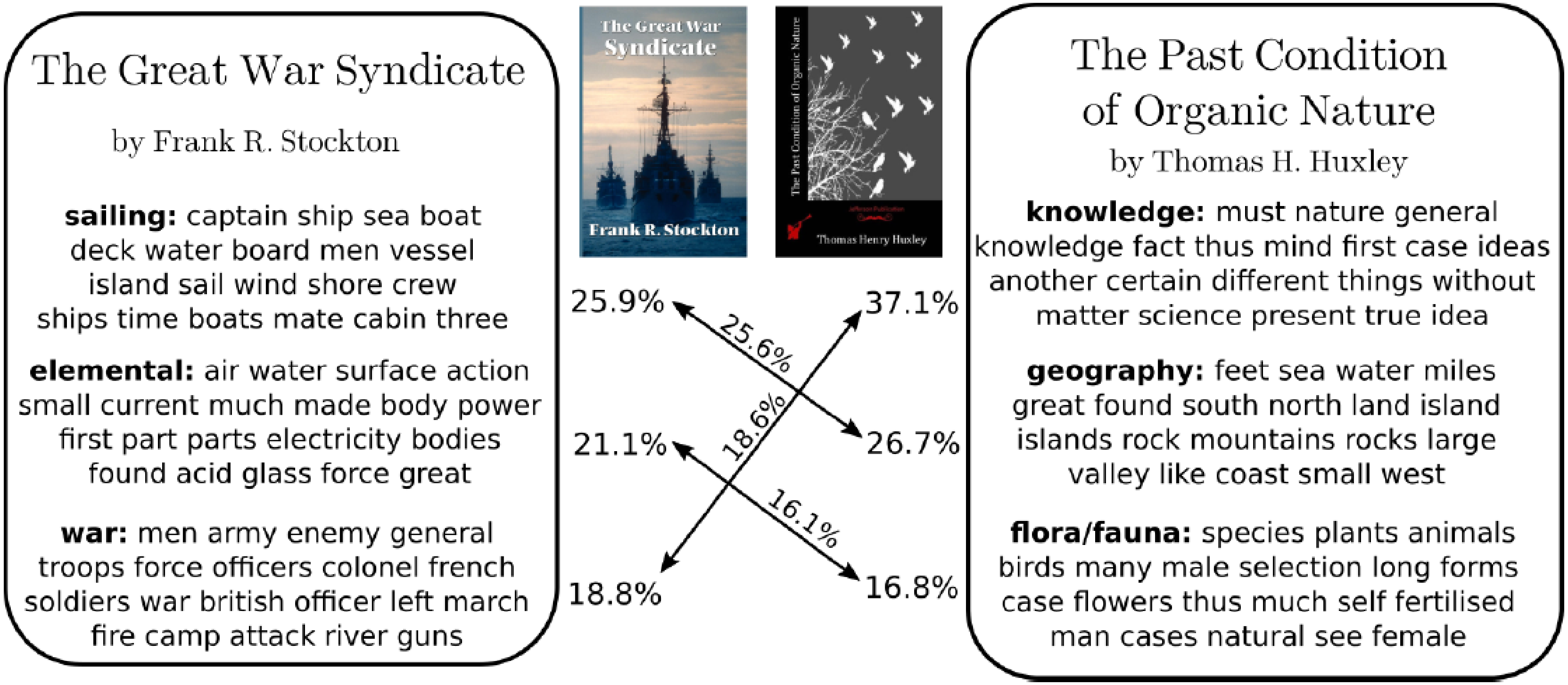
Examples



How do we interpolate along geometric domains?

Lavenant et al. "Dynamical Optimal Transport on Discrete Surfaces." SIGGRAPH Asia 2018.
Solomon & Vaxman. "Optimal Transport-Based Polar Interpolation of Directional Fields." SIGGRAPH 2019.

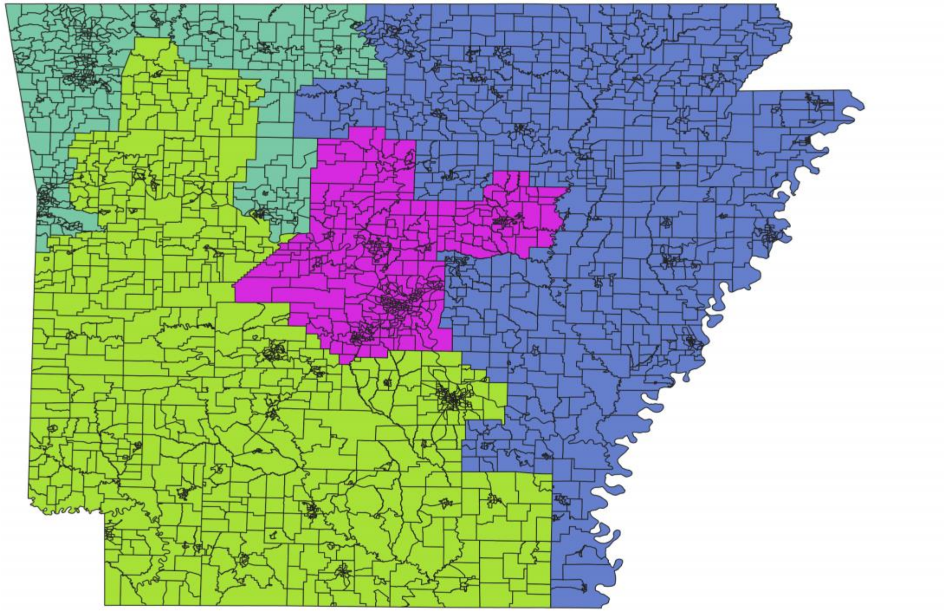
Examples



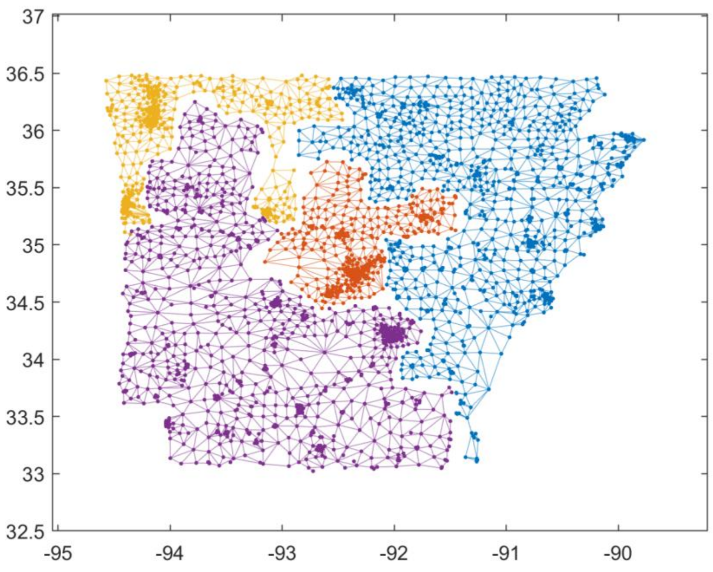
Can we find geometry in data?

Yurochkin et al. "Lightspeed Document Distance Computation via Hierarchical Optimal Transport." Submitted.

Today: Redistricting

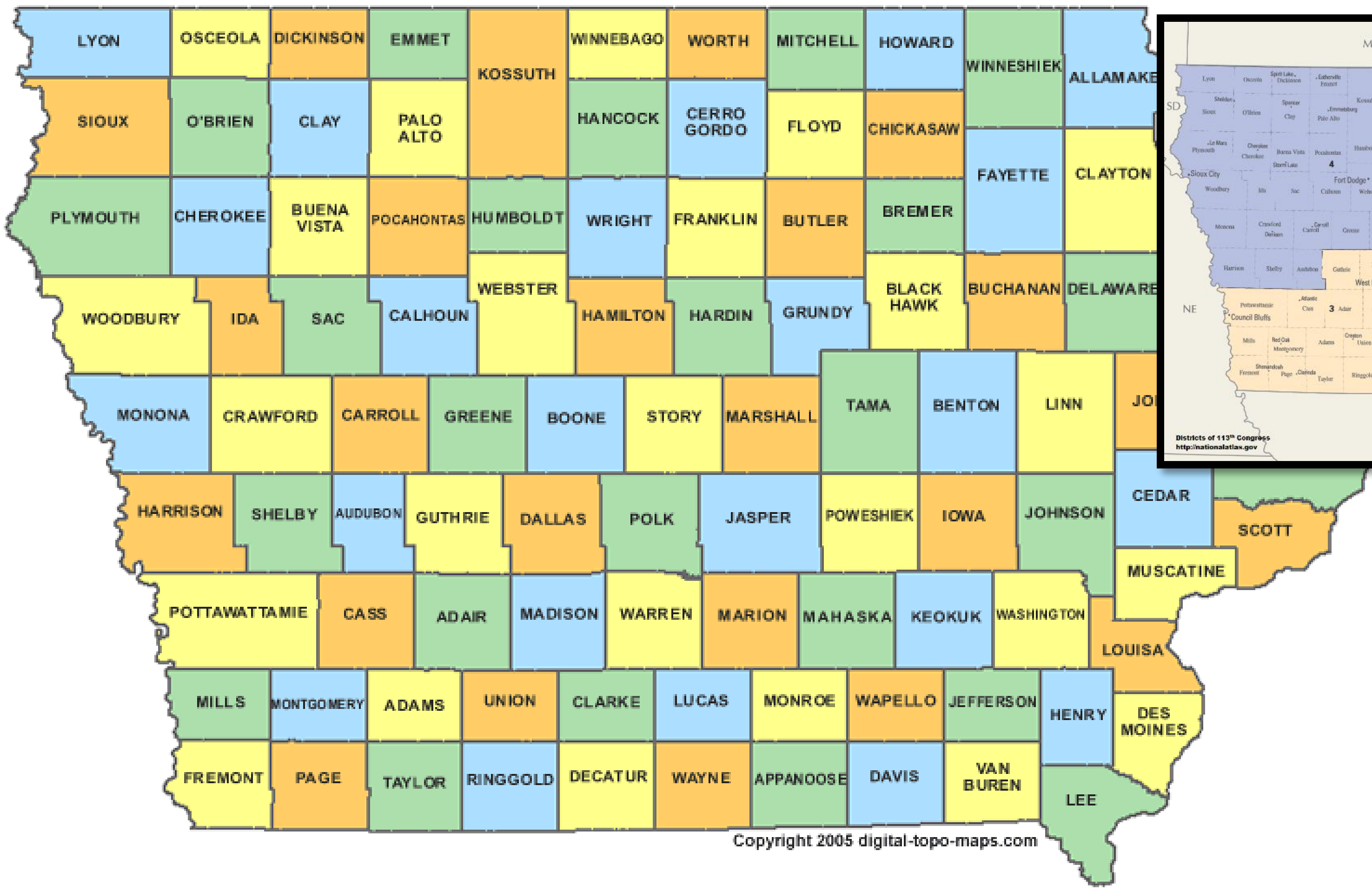


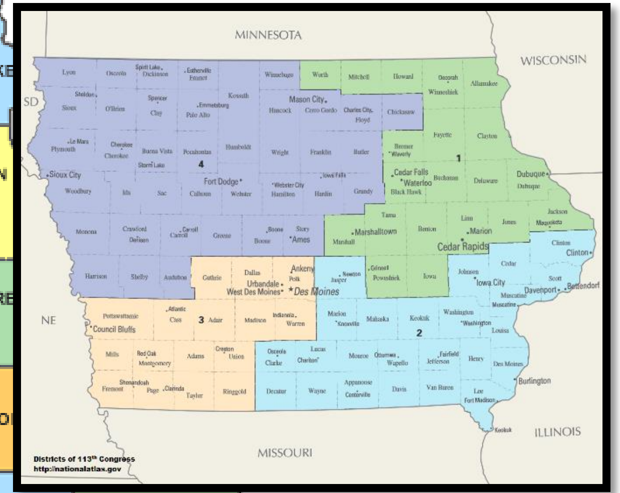
(a) Geography



(b) Dual Graph

Huge Landscape of Possibilities





Iowa: 99 counties, 4 districts, quintillions of possible plans

Reality Check

Likely no single “best” plan.

Typical criteria:

- Contiguity
- Population balance
- Compactness
- Communities of interest
- Municipal boundaries
- Competitiveness
- Incumbency
- …

Reality Check 2.0

THE COMPUTATIONAL COMPLEXITY OF AUTOMATED REDISTRICTING: IS AUTOMATION THE ANSWER?

MICAH ALTMAN*

There is only one way to do reapportionment—feed into the computer all the factors except political registration.

—Ronald Reagan[1]

The rapid advances in computer technology and education during the last two decades make it relatively simple to draw contiguous districts of equal population [and] at the same time to further whatever secondary goals the State has.

—Justice William Brennan[2]

I. REDISTRICTING AND COMPUTERS

Ronald Reagan and Justice Brennan have both suggested that computers can remove the controversy and politics from redistricting.[3] In fact proponents of auton the "optimal" districting plan can be specified values. The Supreme Co sentiment by addressing such mecha and compactness in two recent redis

* Division of Humanities and Social tute of Technology, Pasadena, CA, 91125. Kousser, Scott Page, and Richard McKelvey helpful suggestions.

1. Tom Goff, *Reinecke Denounces Co tion*, L.A. TIMES, Jan. 19, 1972, at A24.

2. Karcher v. Daggett, 462 U.S. 725, 733 (1983).

Even if we could agree on a single measure…

D. *Redistricting is a Computationally Hard Problem*

Redistricting is deeply connected to mathematical partitioning problems. Many researchers in computer science have examined partition problems and have reached some conclusions about their computational complexity. The redistricting problem in general, and even many simpler redistricting sub-problems, are likely to be intractable.

Reality Check 3.0

And even if P=NP…

"The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State **by the Legislature thereof**"
US Constitution (Article I, Section 2)

"…the **legislature shall by law reapportion** the state senatorial districts and representative districts…"
Kansas Constitution (Article 10, Section 1)

"…the **legislature shall enact** a redistricting plan for congressional districts apportioned to Michigan."
Michigan Congressional Redistricting Act of 1999, Section 3.62

"…the **legislature shall apportion and district** anew the members of the senate and assembly, according to the number of inhabitants."
Wisconsin state constitution, Section 3

"The **independent redistricting commission** … shall prepare a redistricting plan to establish senate, assembly, and congressional districts every ten years commencing in two thousand twenty-one…"
New York State Constitution, Article III, Section 4(b)

Humans draw districts.

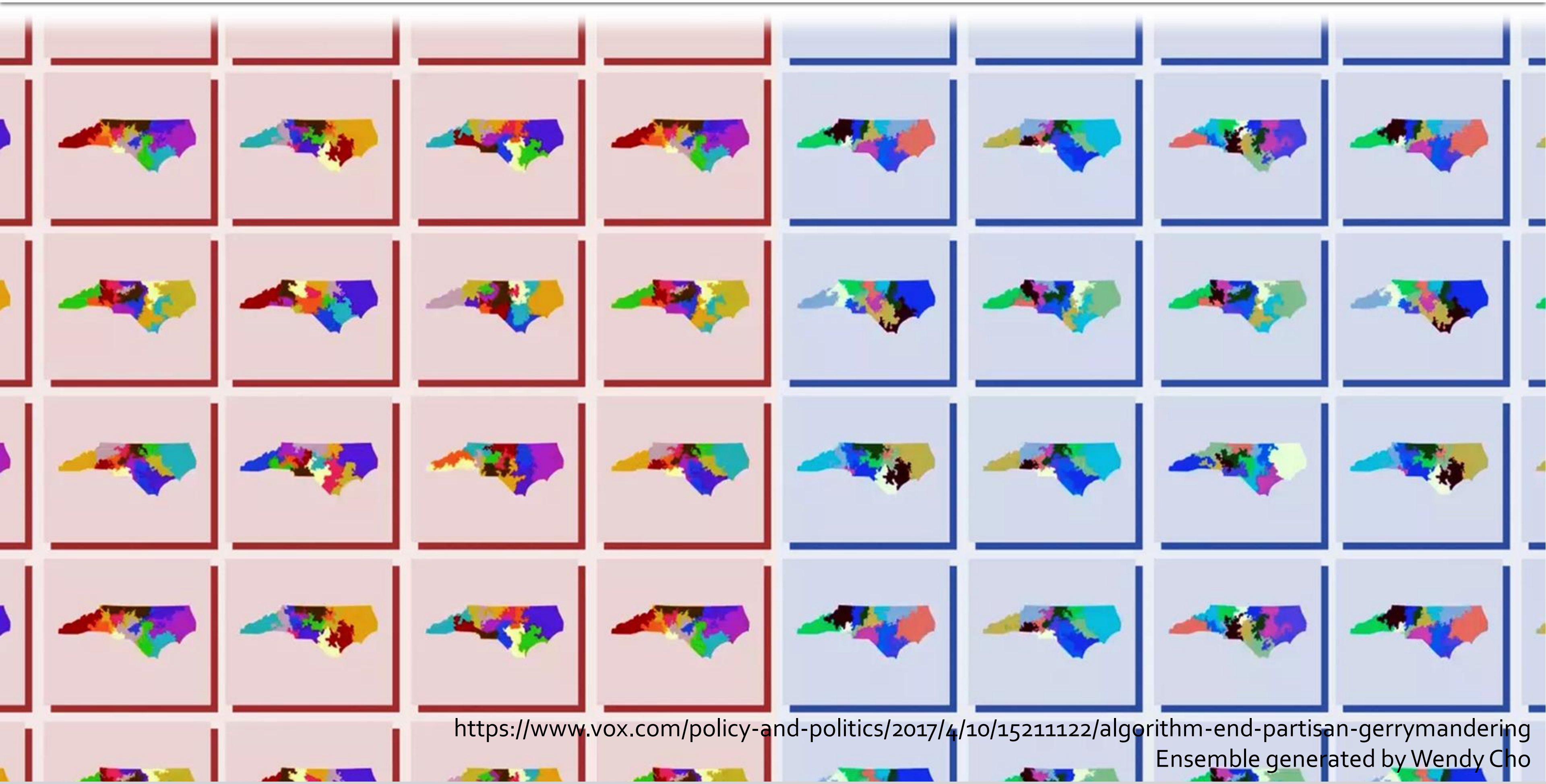
Aside:

Computational Redistricting is Valuable

10^9 computations/second
No legal understanding
No sympathy

?? computations/second
Strong legal understanding
Potentially sympathetic

Recent Focus



https://www.vox.com/policy-and-politics/2017/4/10/15211122/algorithm-end-partisan-gerrymandering
Ensemble generated by Wendy Cho

Analysis of districting plans

Trustworthiness

$$\text{Quantitative} \neq \text{Fair}$$

Critical Challenges

- **Disingenuous analysis**

  Incentive to make your proposed plan looked good

- **Mistaken analysis**

  Many objectives and a huge space of possible plans

Today: Two Examples

- *Single measurement:*

  Measuring compactness

  Challenge: Instability

  (Partial) solution: Isoperimetric profile

- *Aggregate measurement:*

  Ensemble analysis

  Challenge: Mixing time

  (Even more partial) solution: Recombination

Today: Two Examples

- *Single measurement:*

  ## Measuring compactness

  Challenge: Instability

  (Partial) solution: Isoperimetric profile

- *Aggregate measurement:*

  ## Ensemble analysis

  Challenge: Mixing time

  (Even more partial) solution: Recombination

Compactness as a Proxy for Fairness?

Note:

Falling out of favor.

Polsby-Popper Score

Theorem (Isoperimetric inequality). Let Ω be be a bounded open subset of the plane $\mathbb{R}^2$ with perimeter $P < \infty$ and area A. Then, $4\pi A \leq P^2$, with equality if and only if Ω is a circle.

Rigorous proof by Weierstrass, 1870; dates back to ~800 BC

$$\mathrm{PP}(\Omega) := \frac{4\pi A}{P^2}$$

Polsby & Popper, 1991 🙄

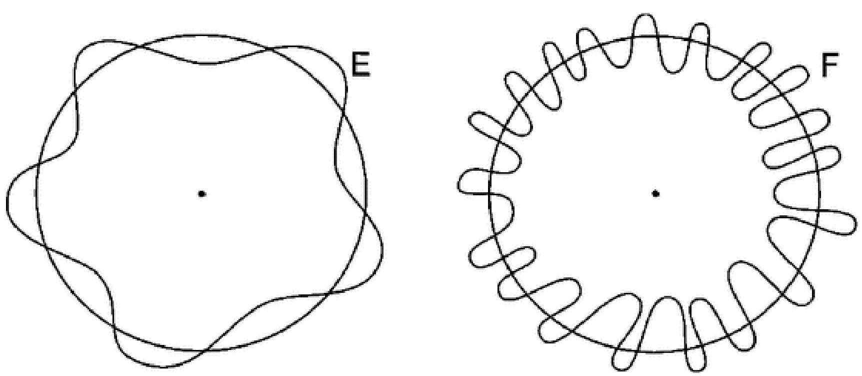
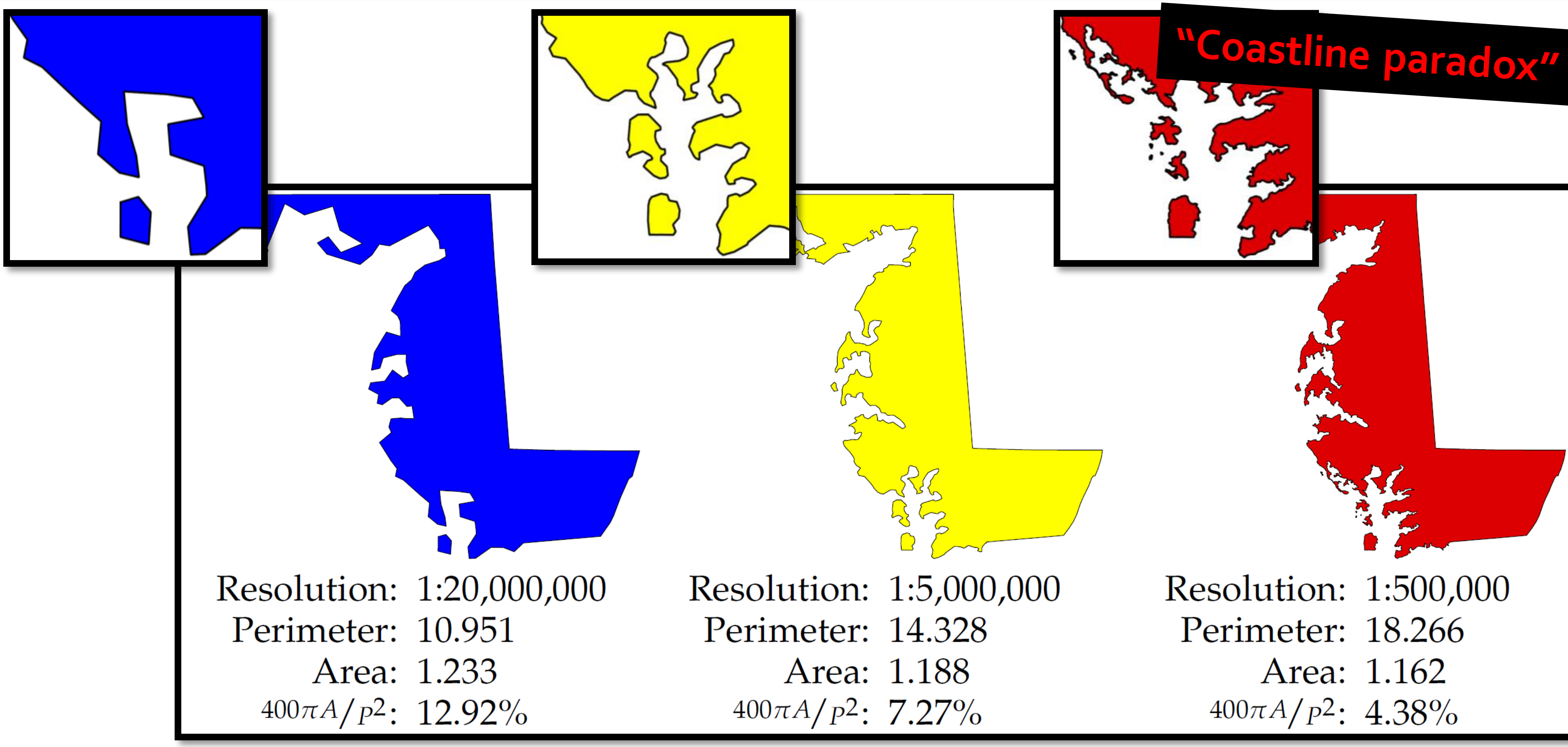


Image from Fusco 2015

Issue with Polsby-Popper



Example courtesy Mira Bernstein and Assaf Bar-Natan

Maryland district 1

More Issues

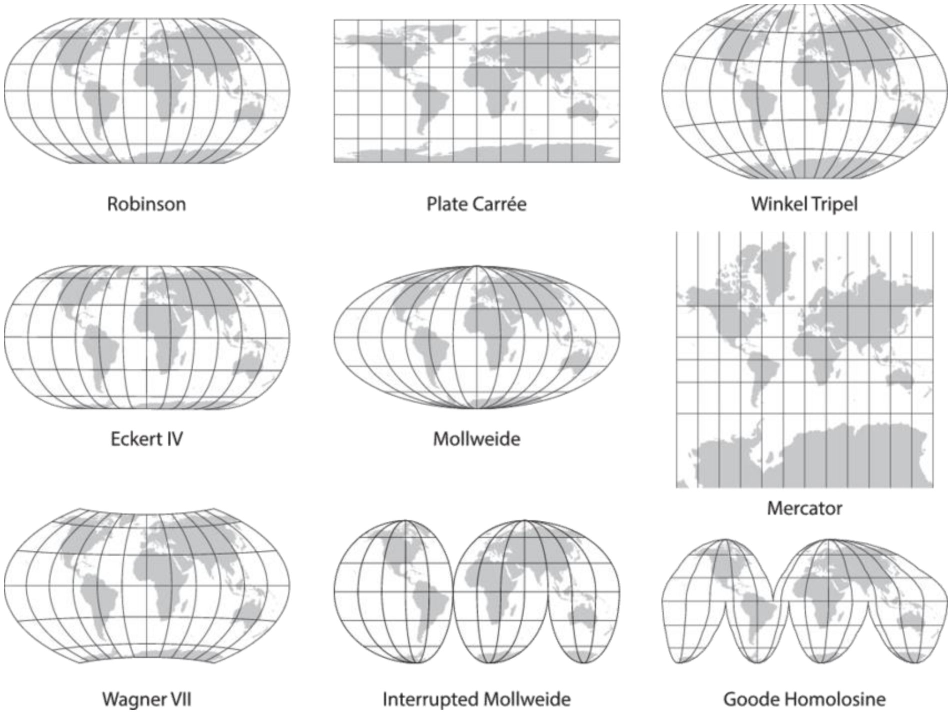
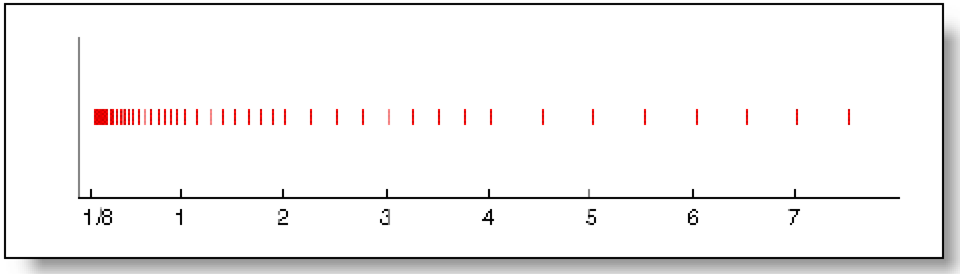


Image from "User preferences for world map projections" (Šavrič et al. 2015)

Map projections?

More Issues



https://blogs.mathworks.com/simulink/2009/12/02/floating-point-numbers/

Floating point?

Adversarial Problem

Input:

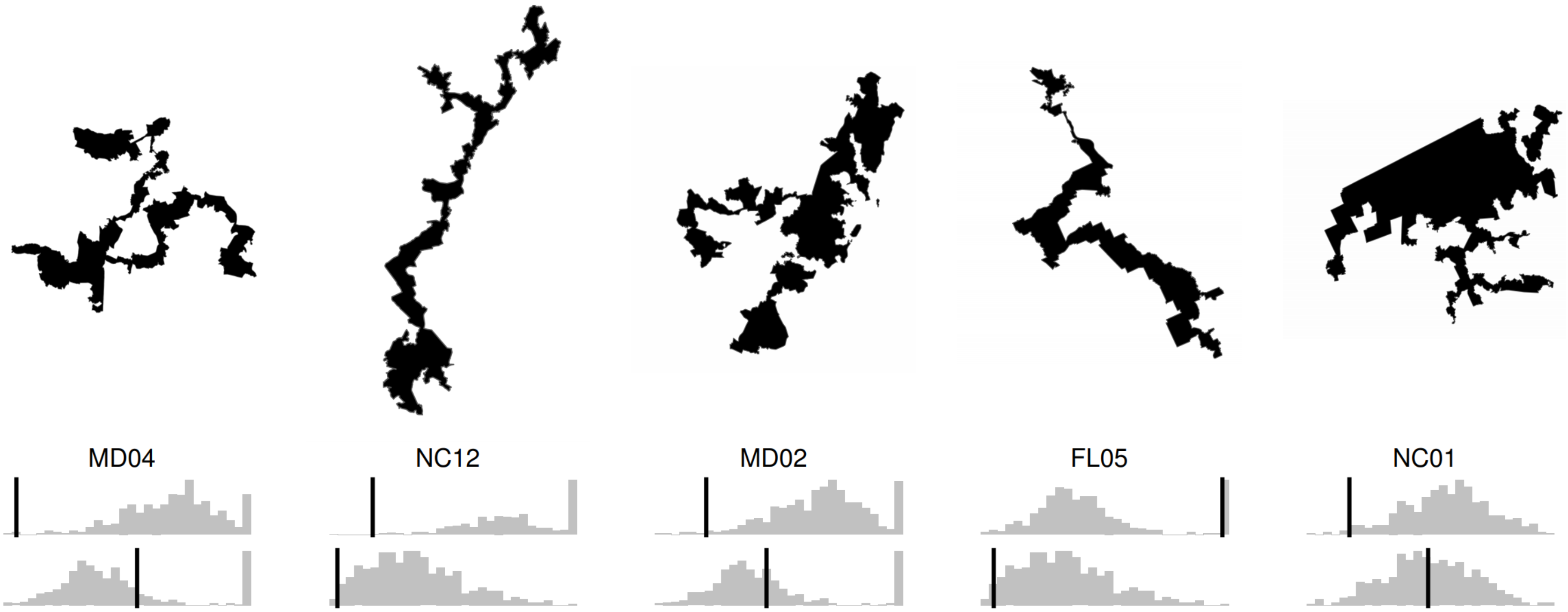
- List of compactness scores
- Set of districts
- Desired percentile

Output:

- Score that achieves percentile

"Gerrymandering and Compactness: Implementation Flexibility and Abuse"
Barnes & Solomon, *Political Analysis* (pending revision)

Frightening Results

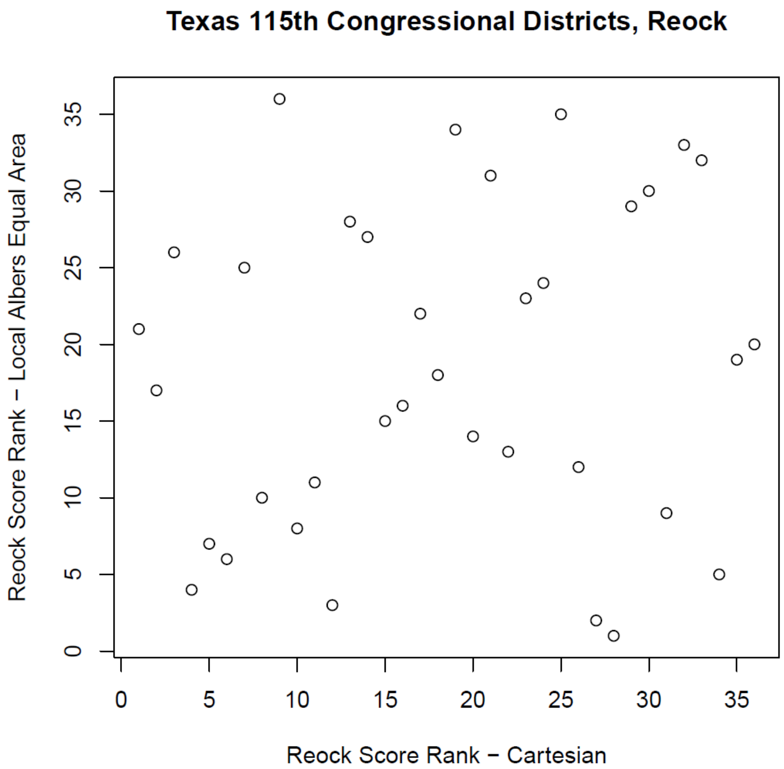


You can engineer your percentile!

Variables: Score, map resolution, map projection

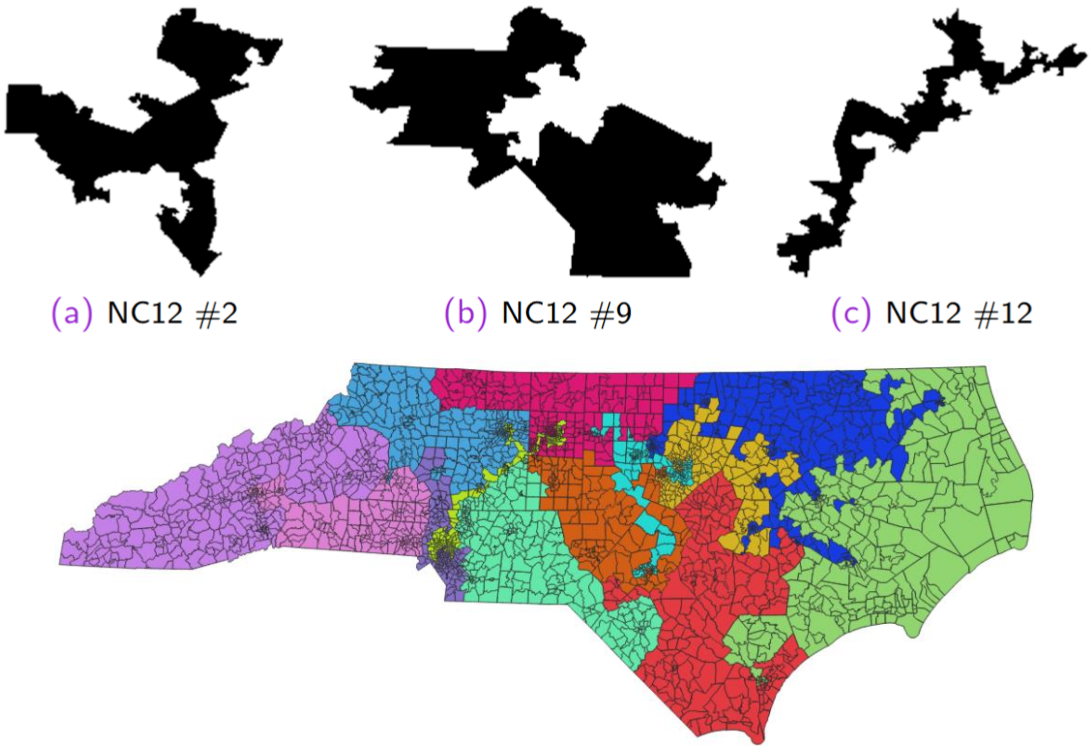
Recent Theoretical Result

"we … demonstrate that **for any choice of map projection**, there are two regions, A and B, such that A is more compact than B on the sphere but B is more compact than A when projected to the plane."



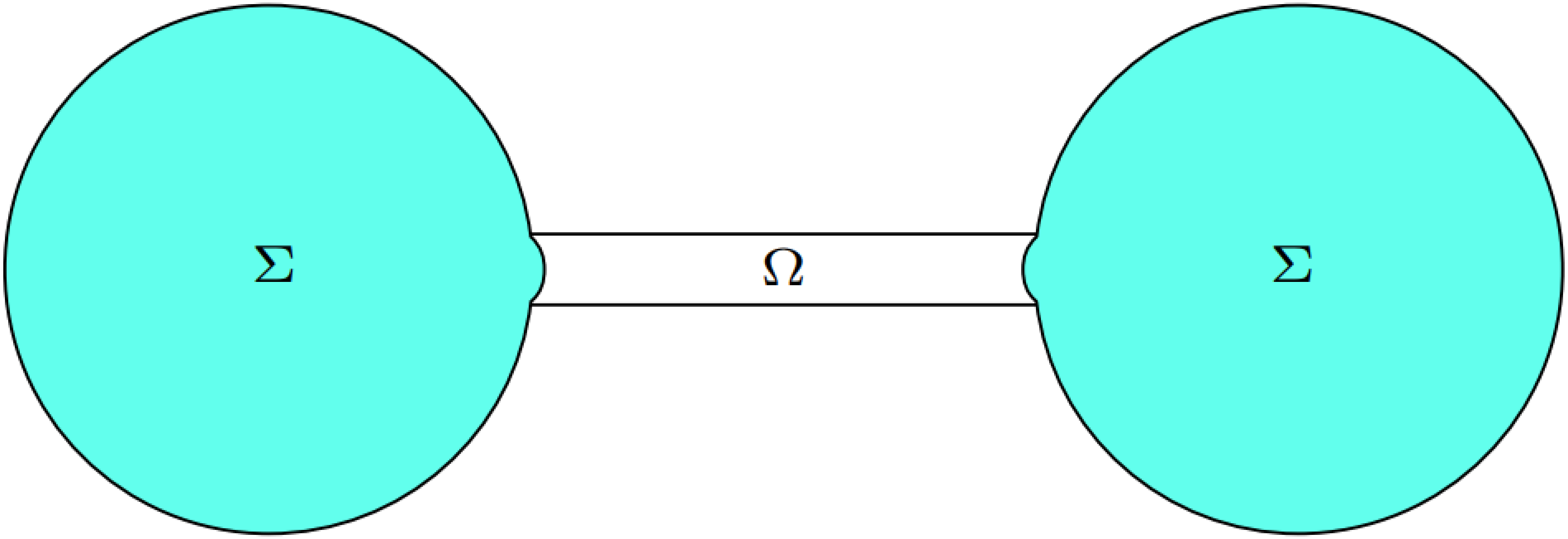
"The Gerrymandering Jumble: Map Projections Permute Districts' Compactness Scores."
Bar-Natan, Najt, & Schutzman; Arxiv 1905.03173.

Additional Observation



Multiple versions of non-compactness

Potentially Intractable Solution



$$I_\Omega(t) := \min\{\text{area}(\partial\Sigma) : \Sigma \subseteq \Omega \text{ and } \text{vol}(\Sigma) = t\}$$

Isoperimetric profile

Perimeter as Total Variation

$$\begin{aligned}\mathrm{TV}[f] &:= \sup_{\|\phi\|_\infty \le 1} \int [f(x) \nabla \cdot \phi(x)]\, dx \\ &= \int_0^\infty \mathrm{area}(\partial\{f \ge s\})\, ds \text{``} = \text{''} \int \|\nabla f(x)\|_2\, dx\end{aligned}$$

$$\mathbb{1}_\Sigma(x) := \begin{cases} 1 & \text{if } x \in \Sigma \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{area}[\partial\Sigma] = \mathrm{TV}[\mathbb{1}_\Sigma]$$

Convex Relaxation: TV Profile

$$I_\Omega(t) := \min\{\text{area}(\partial\Sigma) : \Sigma \subseteq \Omega \text{ and } \text{vol}(\Sigma) = t\}$$

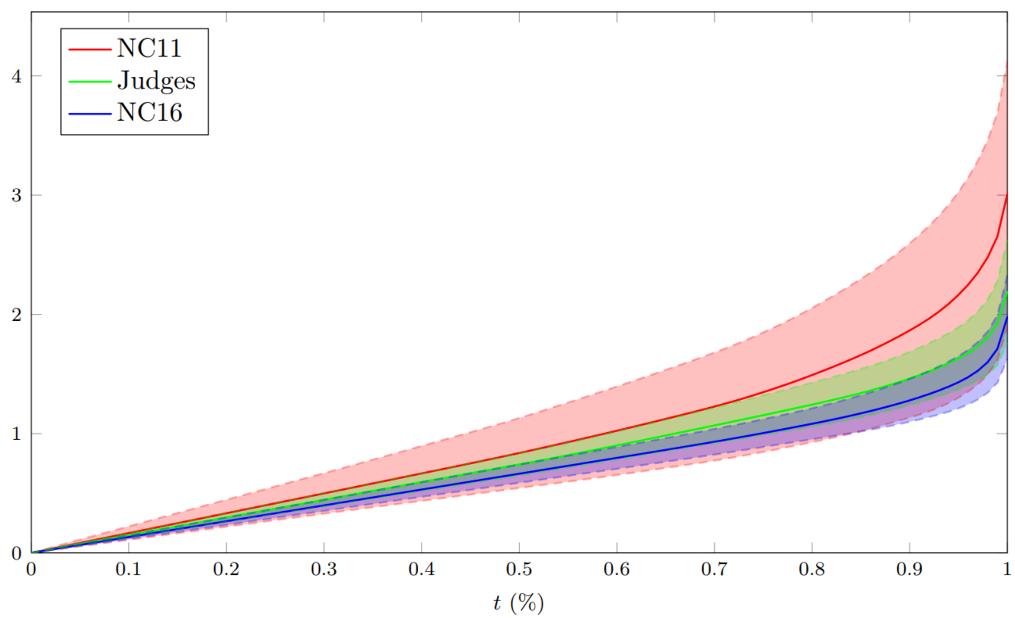
$$I_\Omega^{\text{TV}}(t) := \begin{cases} \min_{f \in L^1(\mathbb{R}^n)} & \text{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\,dx = t \\ & 0 \leq f \leq \mathbb{1}_\Omega \end{cases}$$

Theoretical properties:

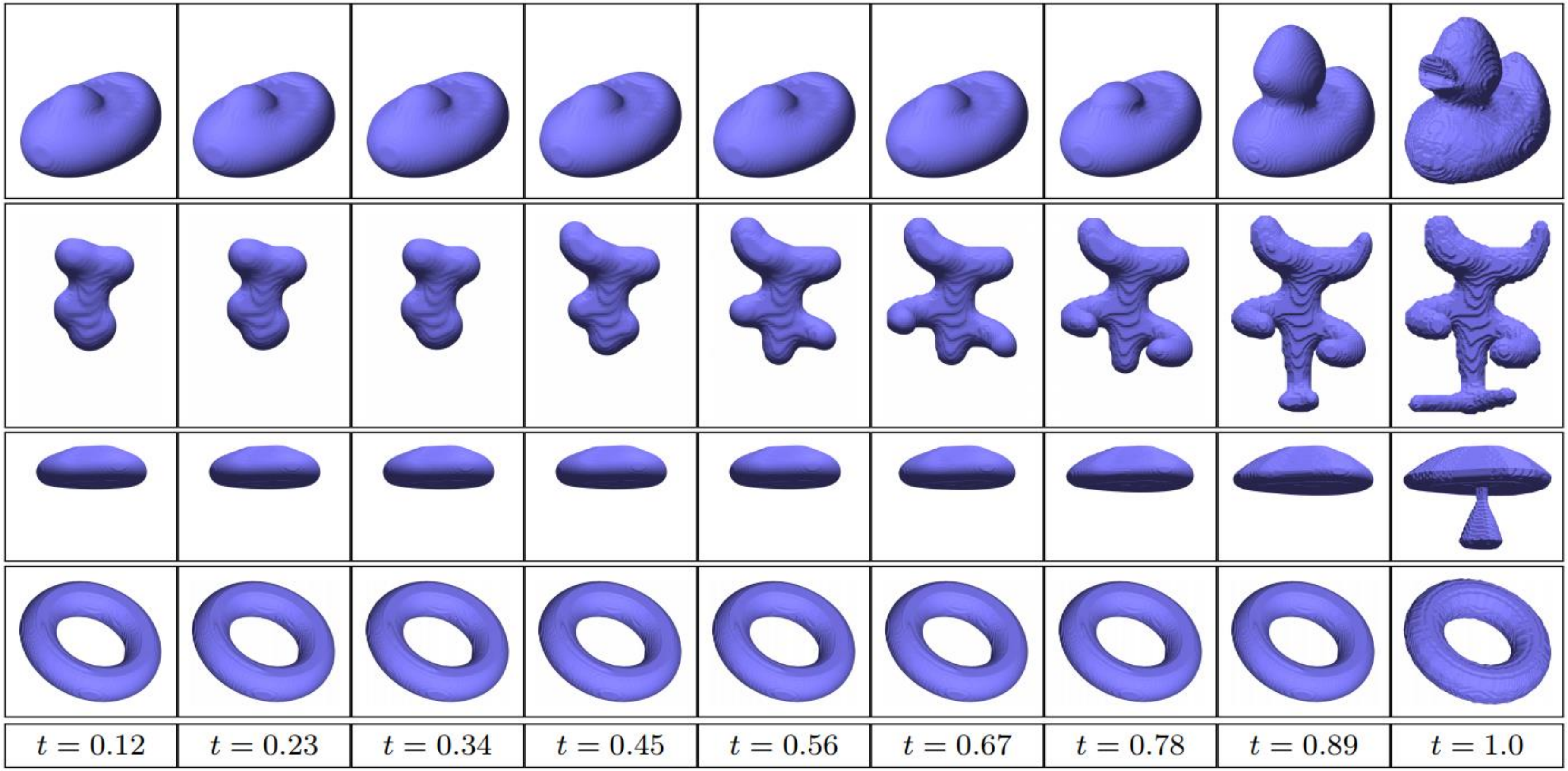
- **Convex function of *t***
- **Minimized at any *t* for a circle**
- **(Surprising) optimal *f* takes on at most 3 values: {0, *c*, 1}**

DeFord et al. *Total Variation Isoperimetric Profiles*. SIAM SIAGA, pending revision.

Examples

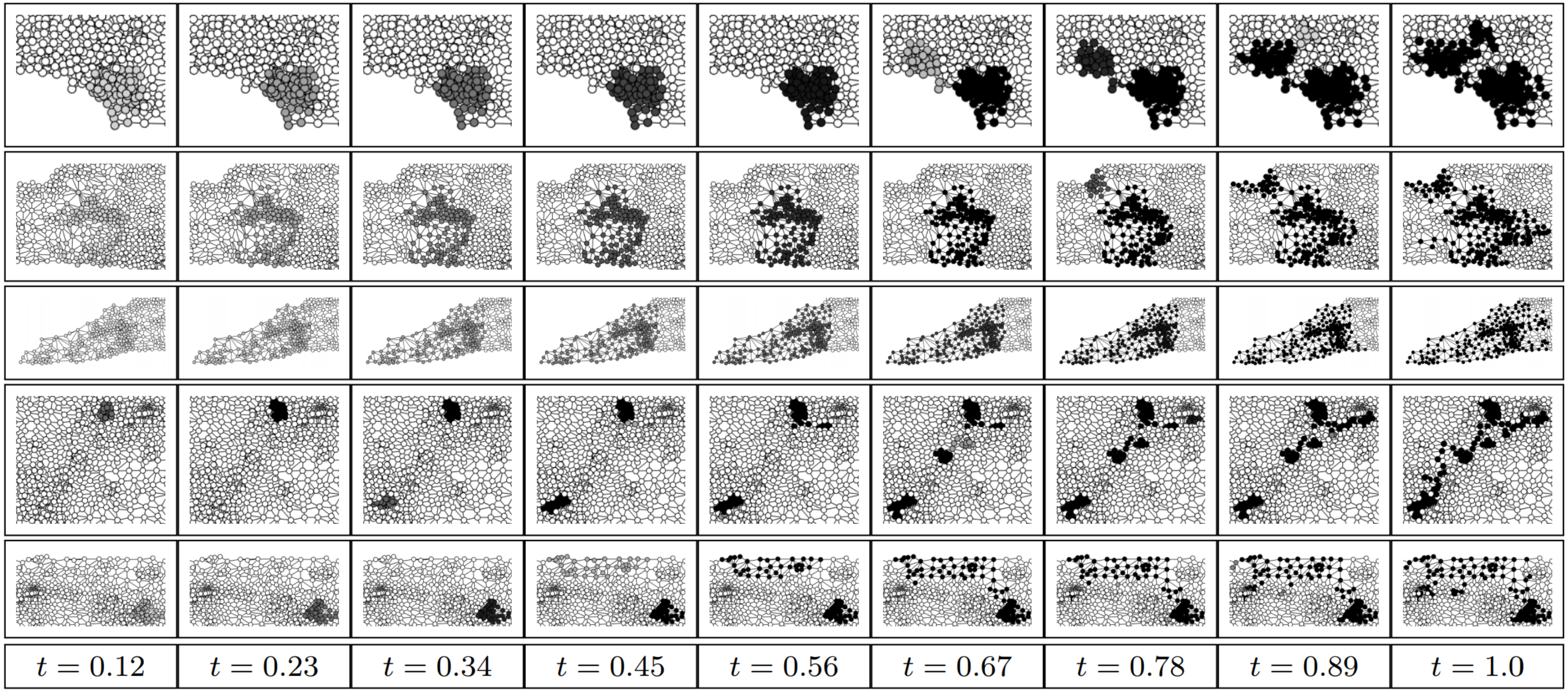


In Case You're Wondering



Works in 3D (Why bother? Why not!)

Graph Analog

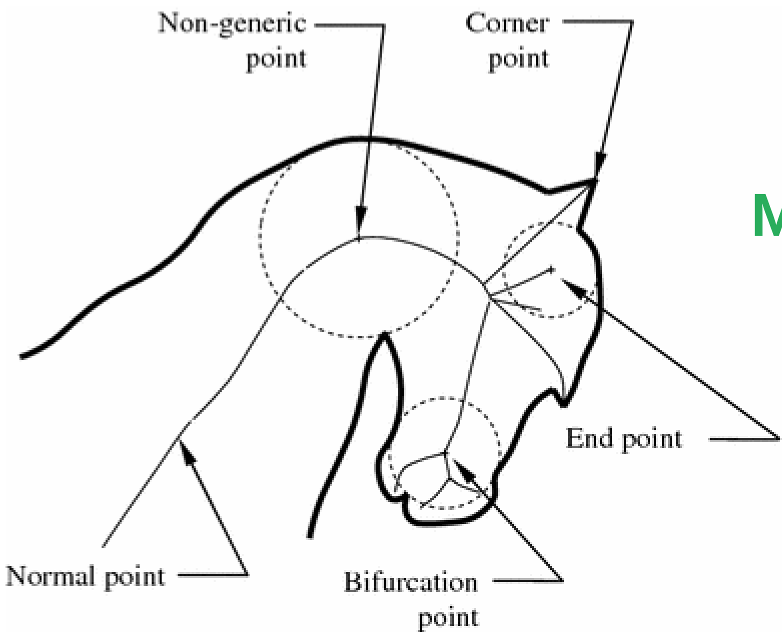


$$I_{V_0}^{\mathrm{TV}}(t) := \begin{cases} \min_{f \in \mathbb{R}^V} & \sum_{(v,w)\in E} |f(v) - f(w)| \\ \text{subject to} & \sum_{v \in V_0} f(v) = t|V_0| \\ & f(v) = 0 \ \forall v \notin V_0 \\ & f(v) \in [0,1] \ \forall v \in V \end{cases}$$

Open Problem

Problem:

Compute isoperimetric profile without TV relaxation.



Medial axis?

Image from [Au 2012]

Trade-Off

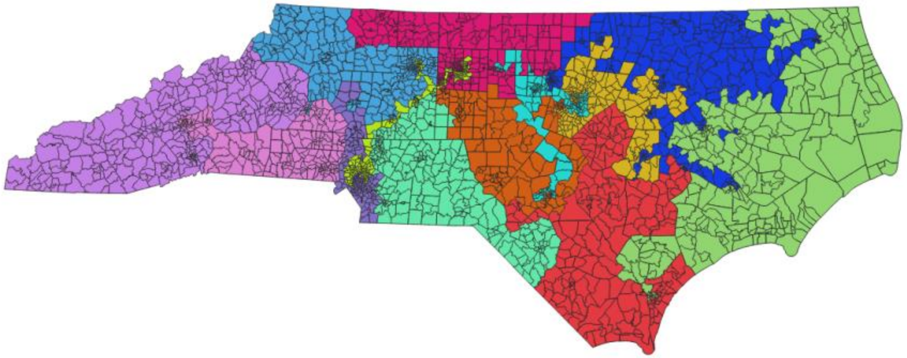
Positive:

- Stable
- Computable
- Nuanced/multiscale

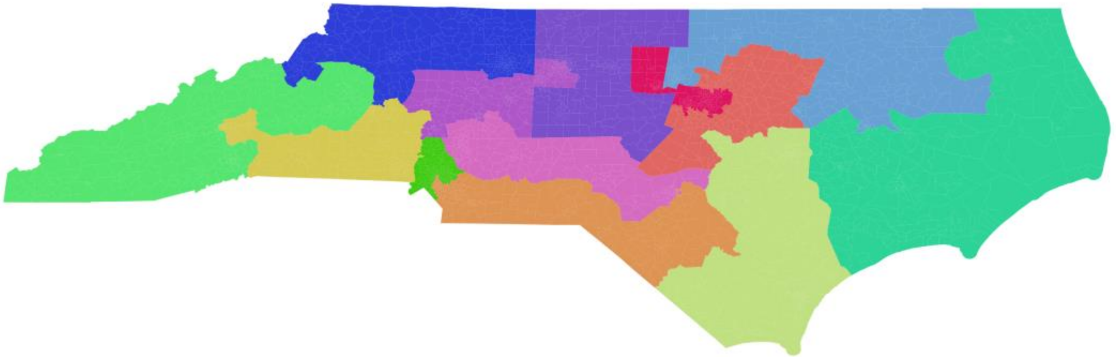
Negative:

- Not a single score
- Not a great proxy for fairness

Fundamental Issue



(a) NC12



(b) NC16

Thematic Take-Away

Stability is subtle and can be leveraged by an adversary.

Provably stable measurements are hard to design.

Today: Two Examples

- *Single measurement:*

  Measuring compactness

  Challenge: Instability

  (Partial) solution: Isoperimetric profile

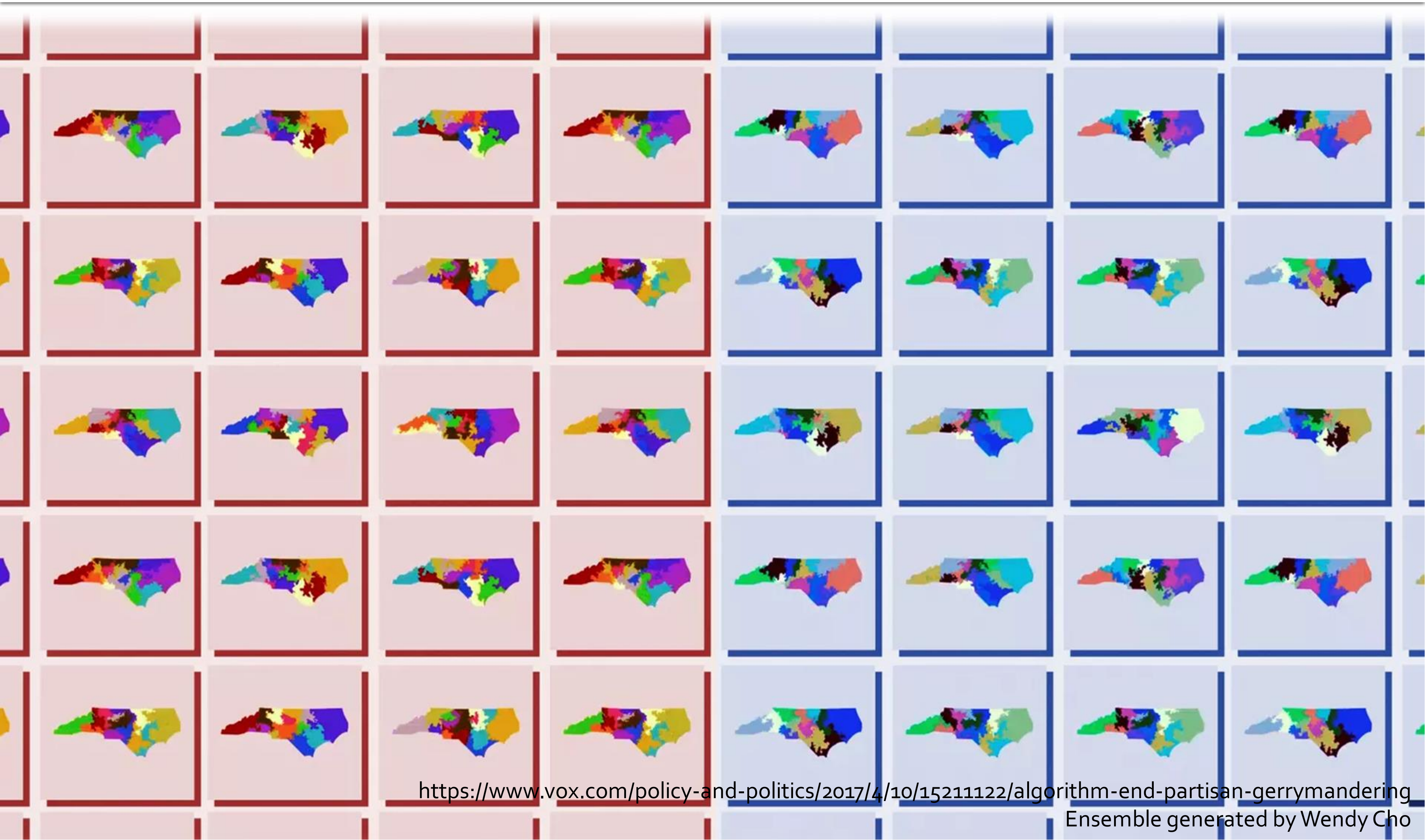
- *Aggregate measurement:*

  Ensemble analysis

  Challenge: Mixing time

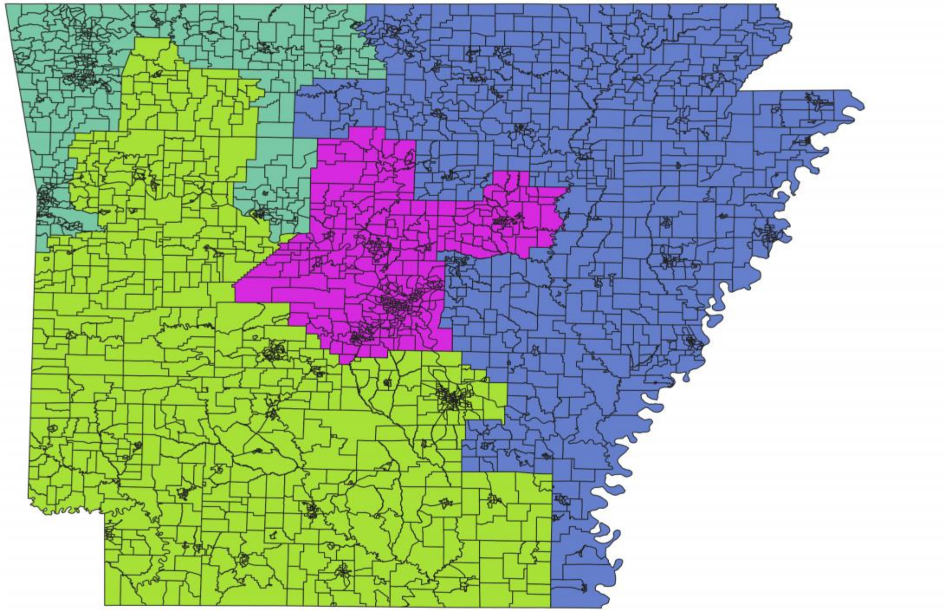
  (Even more partial) solution: Recombination

Ensembles: Redistricting in Context

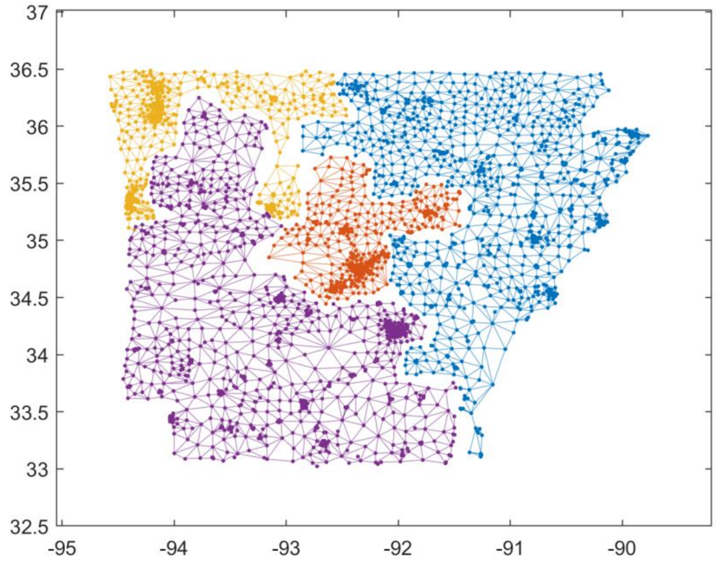


https://www.vox.com/policy-and-politics/2017/4/10/15211122/algorithm-end-partisan-gerrymandering
Ensemble generated by Wendy Cho

Discrete Problem



(a) Geography



(b) Dual Graph

$$p : V \to \{a_1, \ldots, a_n\}$$

Language Matters

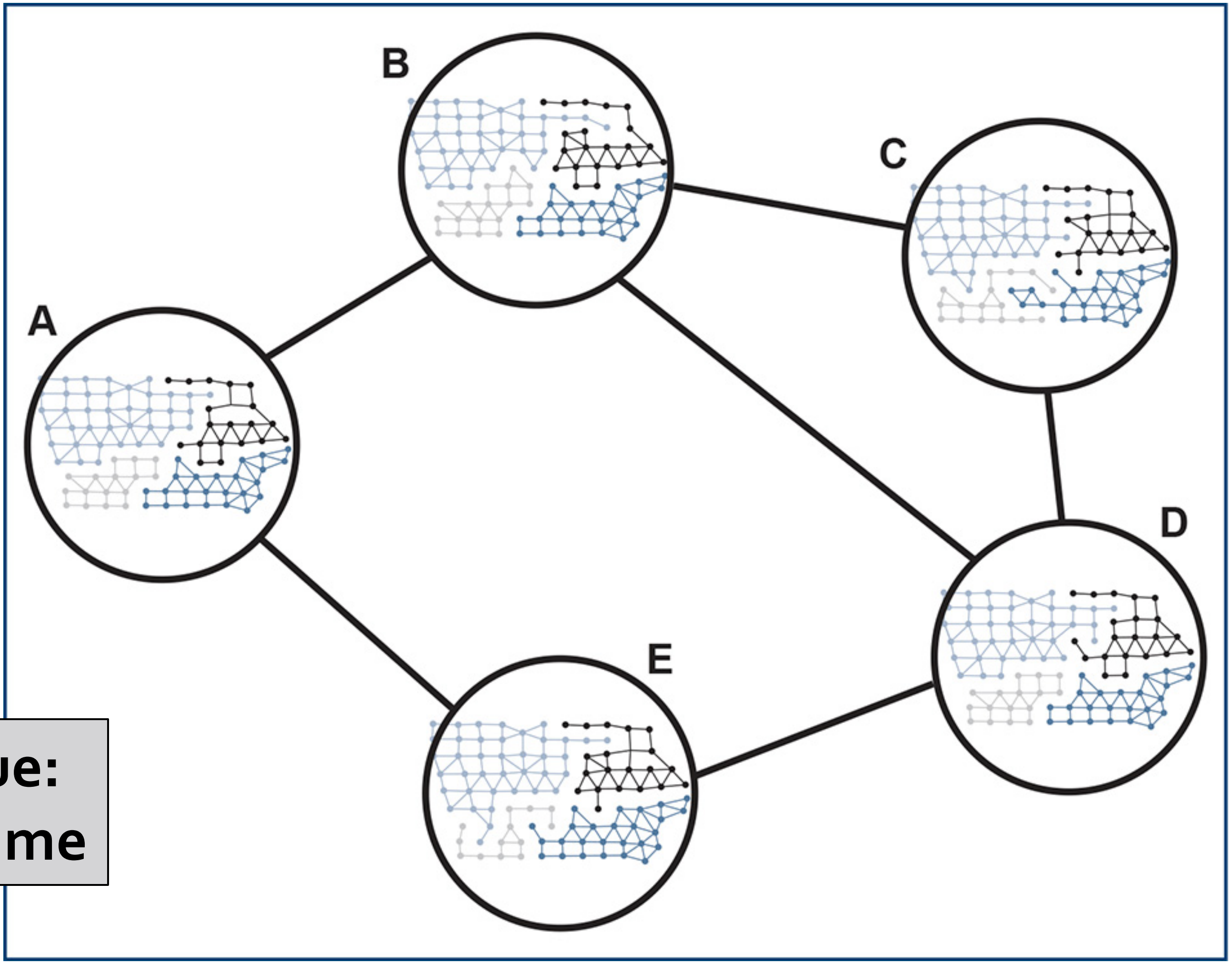
OK:

"We were able to generate k plans with favorable property P."

Not (necessarily) OK:

"Our plan scores better/worse than p% of reasonable plans."

Random Walk Approach



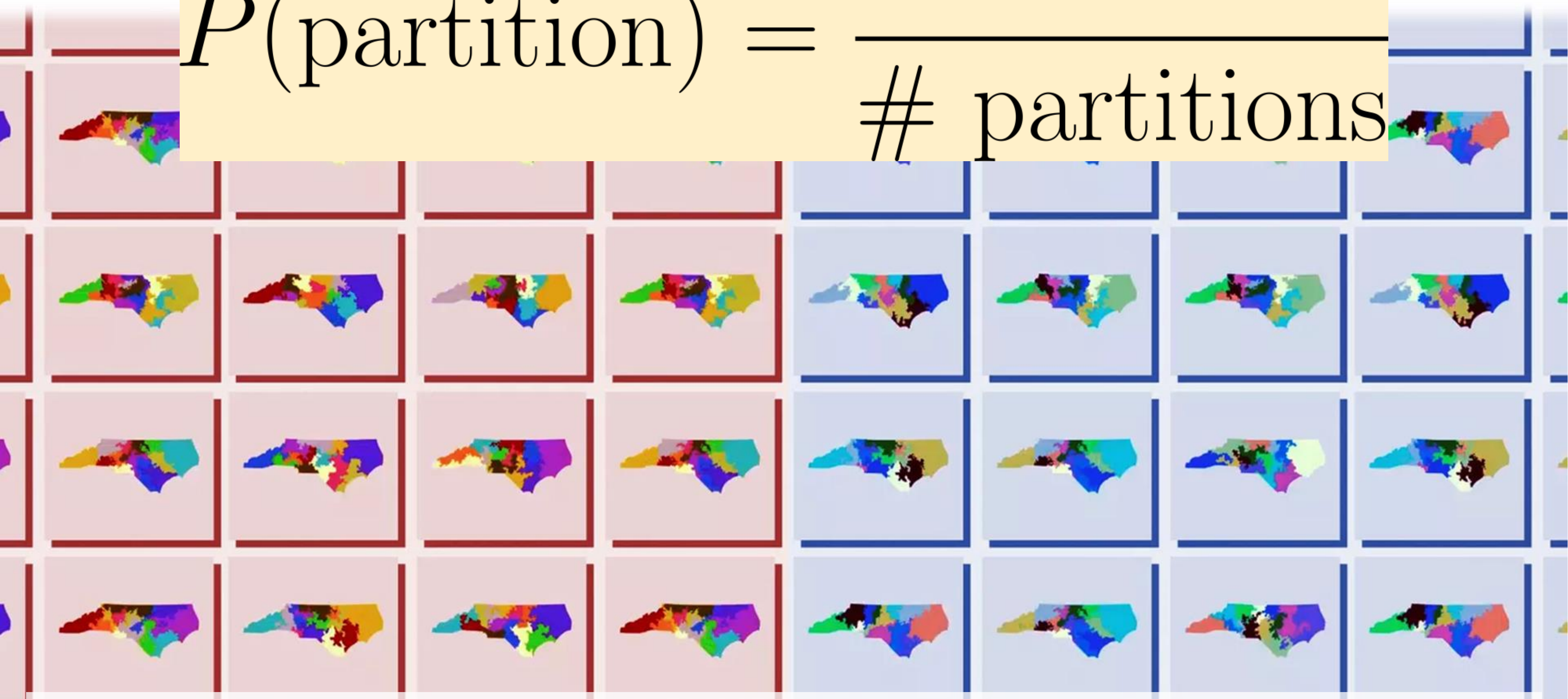
Key issue: Mixing time

https://www.amacad.org/news/redistricting-and-representation

Sampling Problem

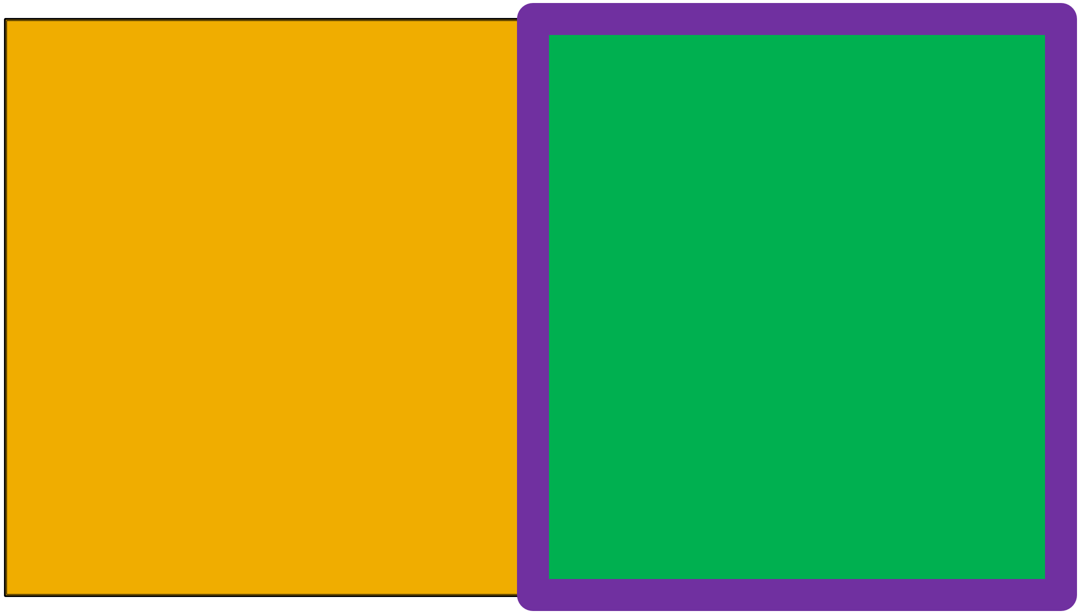
Uniform distribution:

$$P(\text{partition}) = \frac{1}{\#\ \text{partitions}}$$



Najt, DeFord, & Solomon. “Complexity of Sampling from Connected Graph Partitions.” In preparation.

From 2-Partitions to Cycles



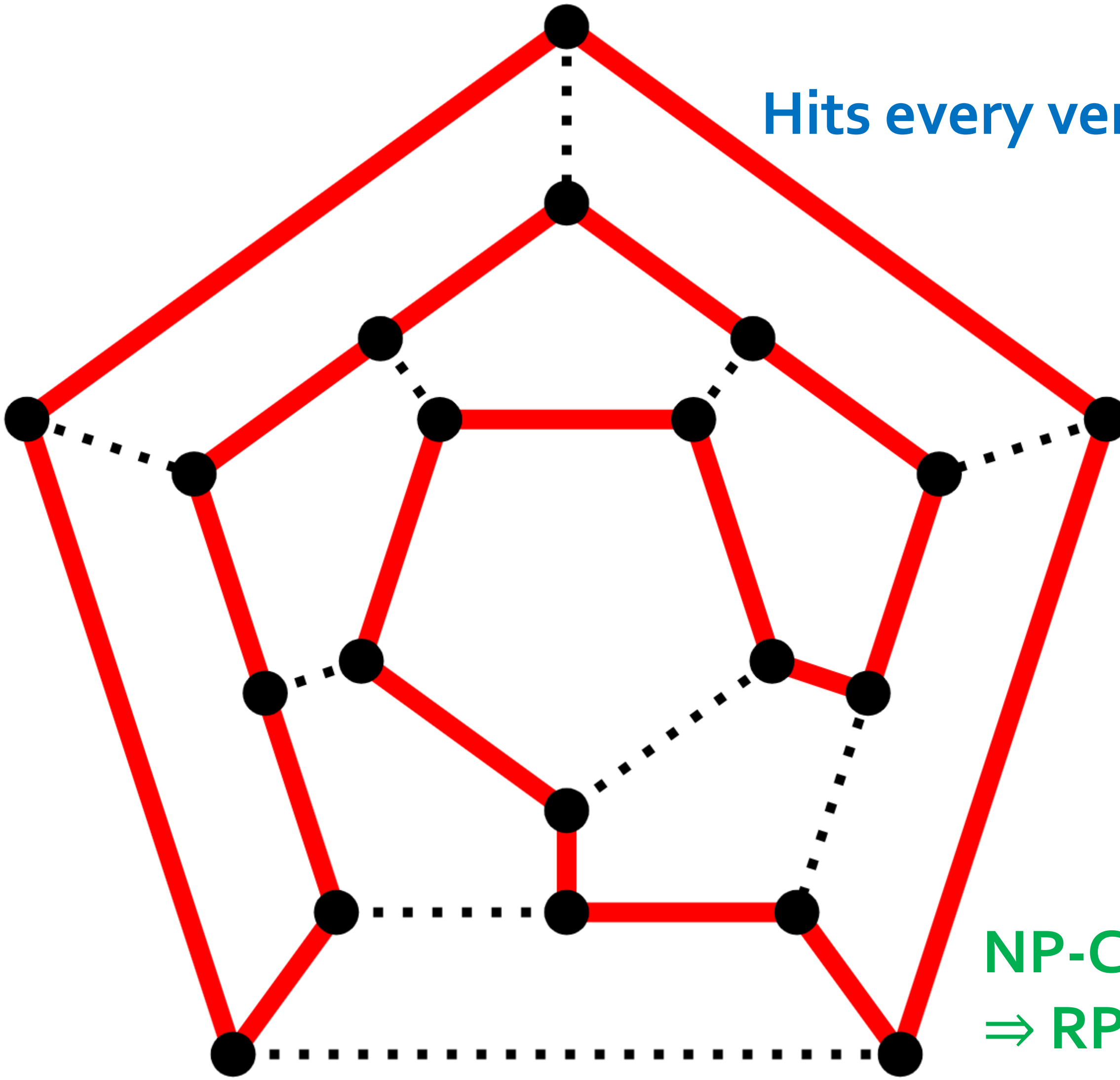
RP Completeness

Randomized polynomial time (RP):

Exists a probabilistic Turing machine that

- Runs in polynomial time
- Always correctly returns **NO**
- If the correct answer is **YES**, returns **YES** with probability $\geq {}^{1}/_{2}$

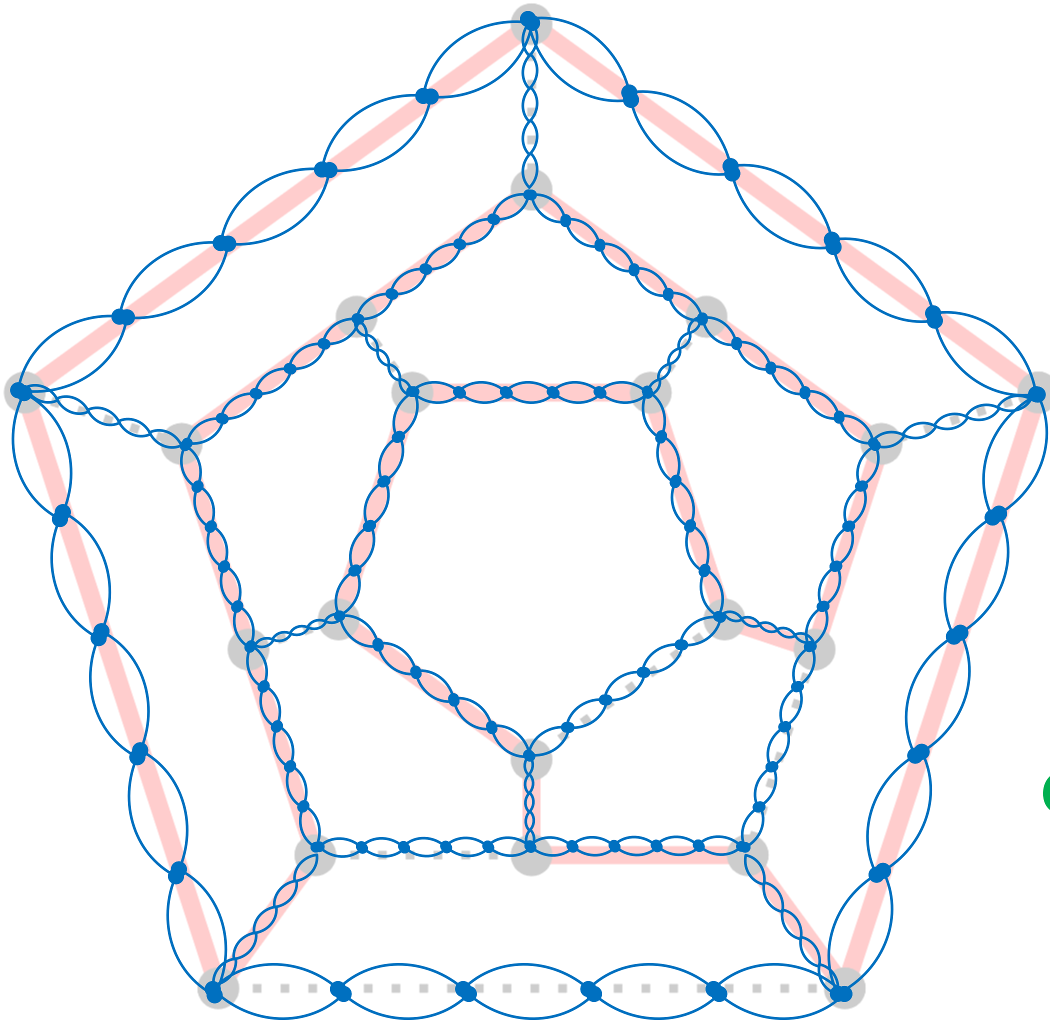
Hamiltonian Cycle



Hits every vertex once

NP-Complete
⇒ RP-Complete

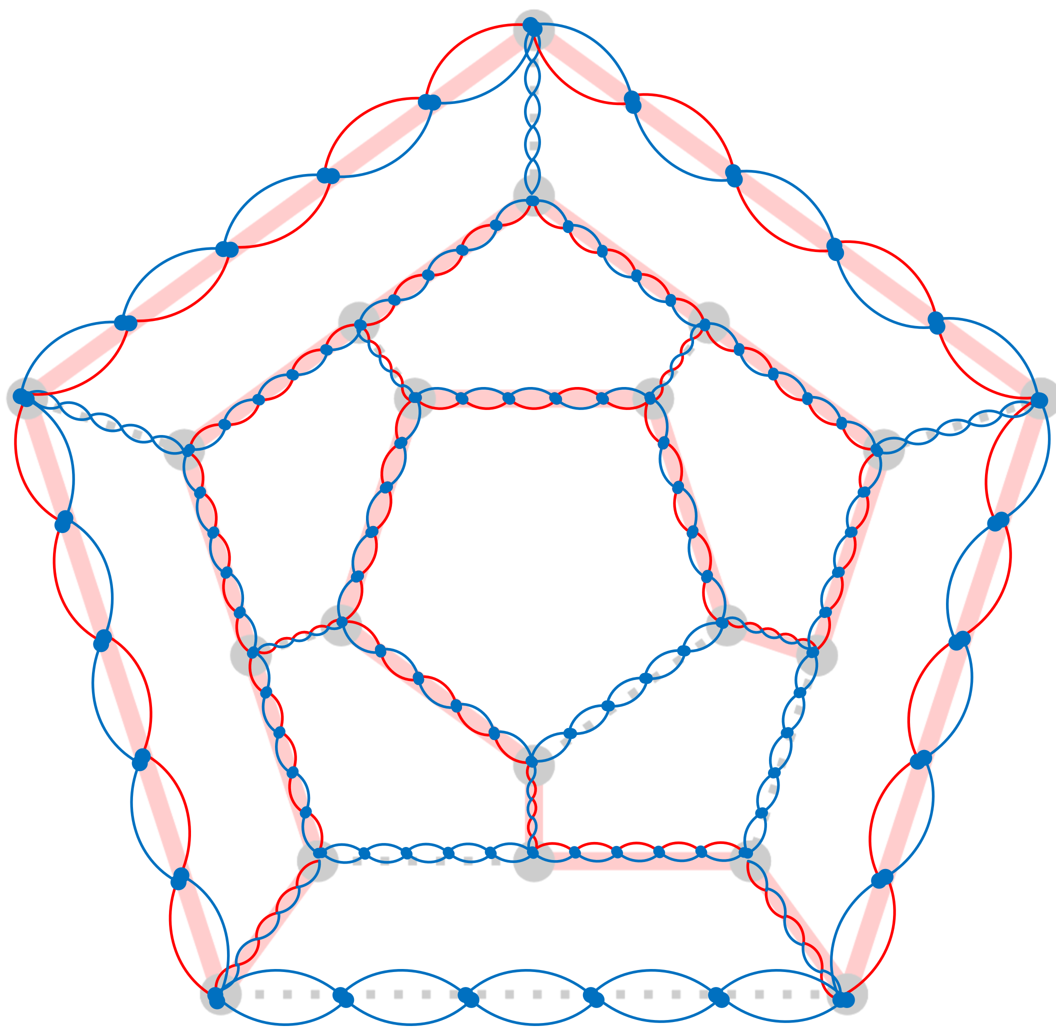
A Simple Counterexample



Chain of bigons:
Linear number of edges in |E|

Proof follows [Jerrum, Valiant, and Vazirani 1986]

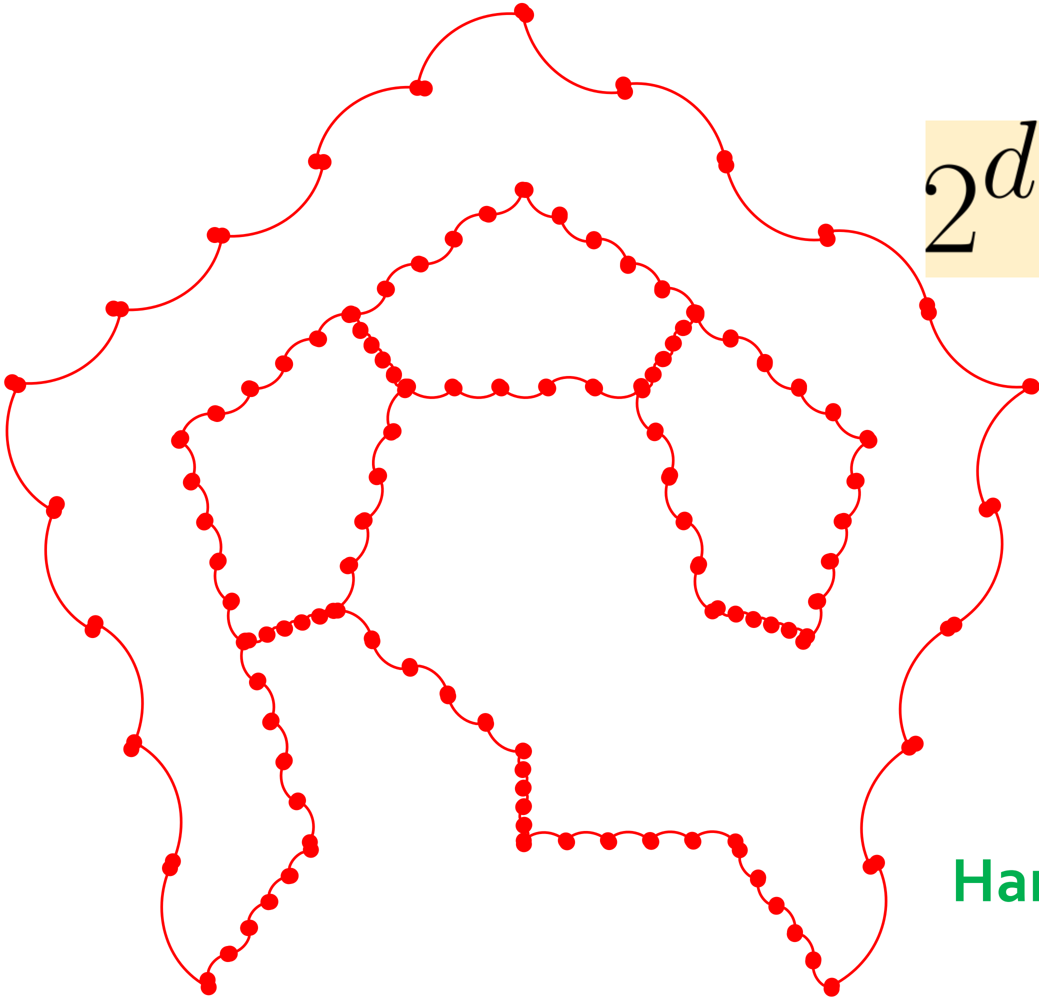
A Simple Counterexample



Proof follows [Jerrum, Valiant, and Vazirani 1986]

A Simple Counterexample

RP-Hard!



2^{dL} copies

Hamiltonian cycle

Proof follows [Jerrum, Valiant, and Vazirani 1986]

Tougher Proof, Same Result

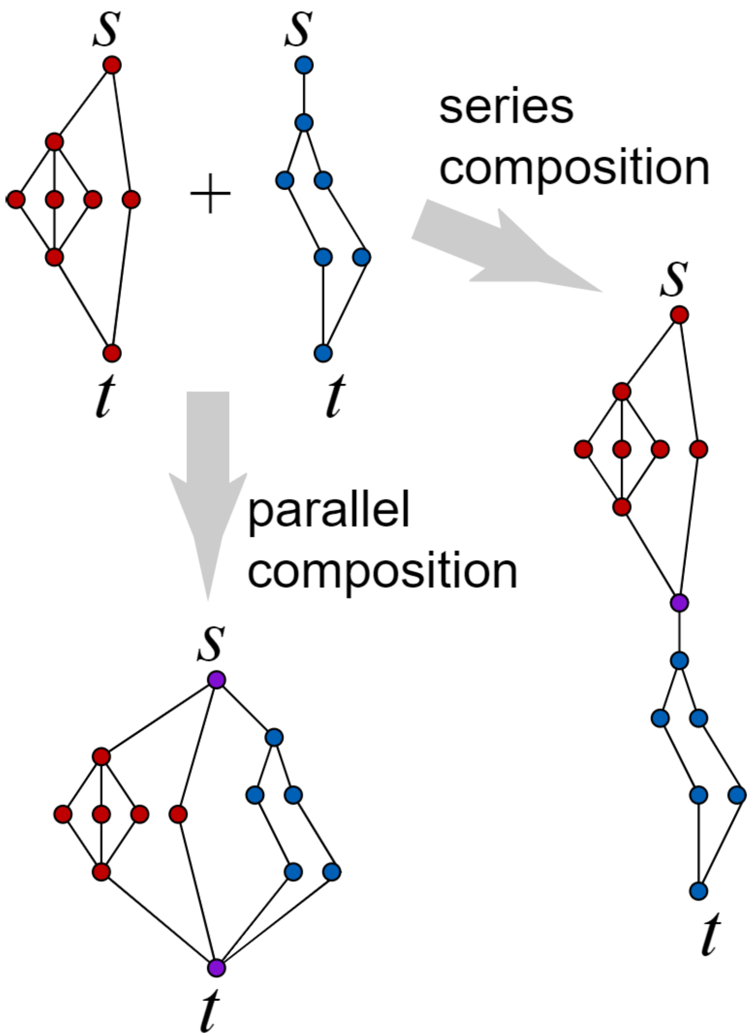
Remains hard with extra assumptions:

- **Maximal planar graph**
- **Bounded vertex degree**
- **Balanced partition**

Relationship to Mixing

Fast mixing would imply polynomial time (near)-uniform sampling!

Series Parallel Graphs

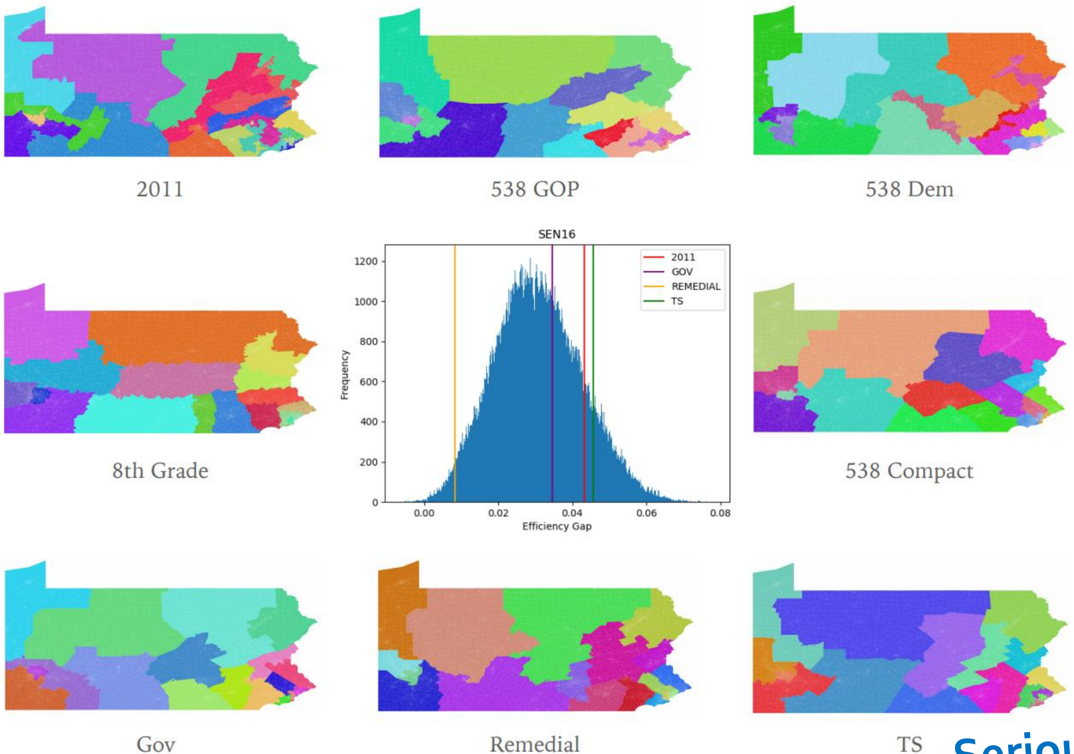


Polynomial-time sampler

Exponentially slow mixing

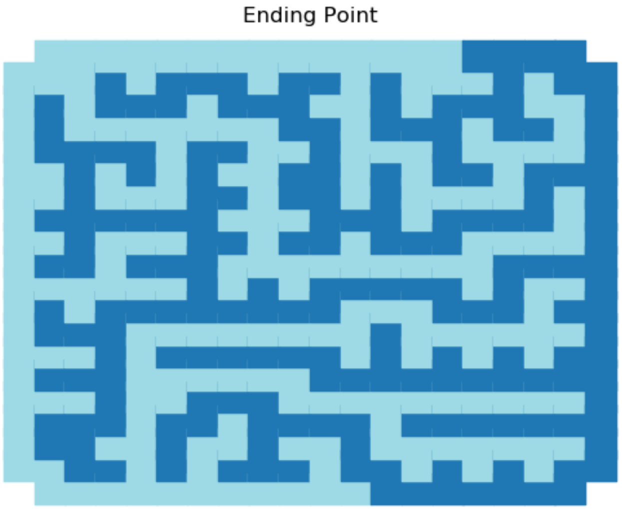
Implication

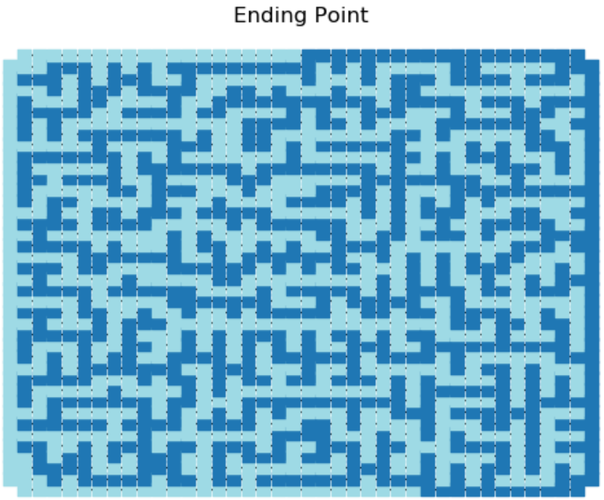
Popular sampling tools are unlikely to see a significant or representative sample of plans.

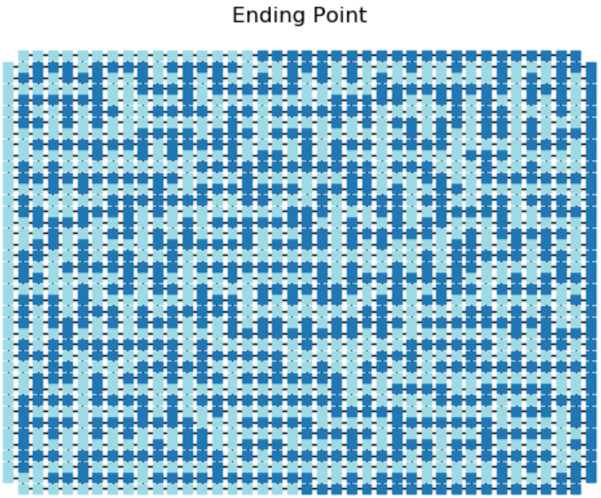


Serious challenge for "outlier analysis."

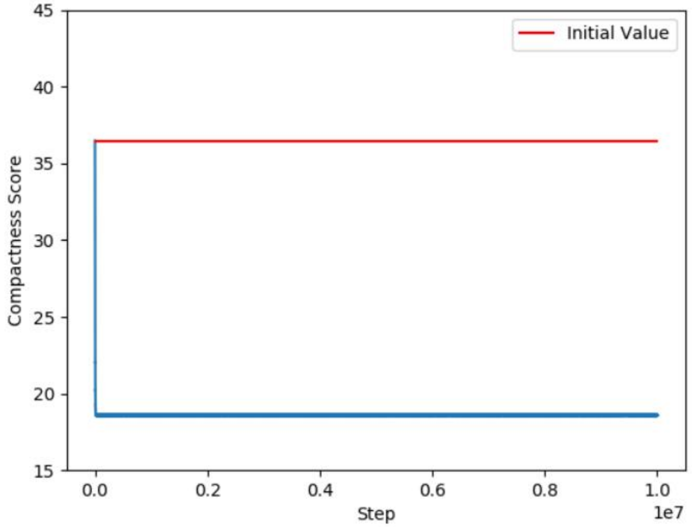
Is Uniform Even Desirable?



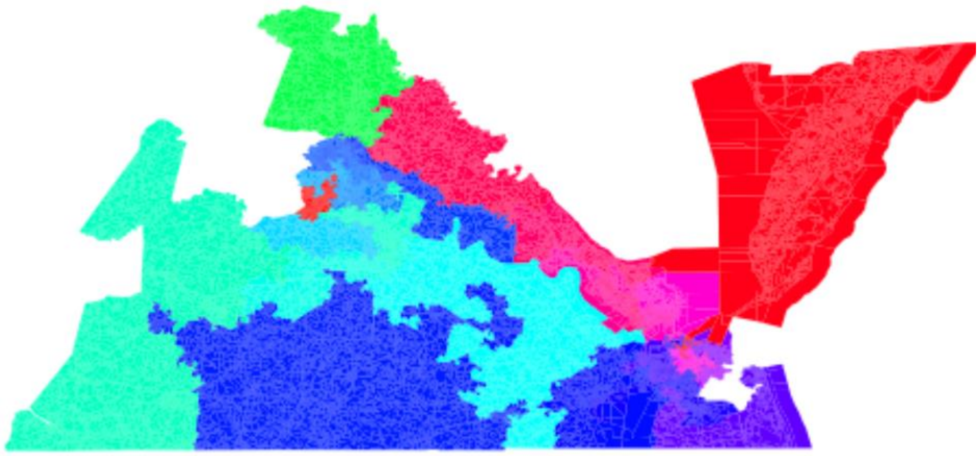




Saturation of Compactness

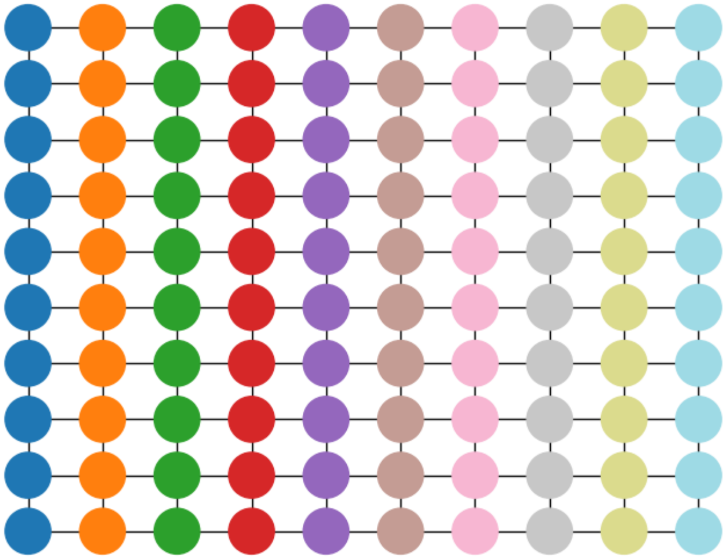


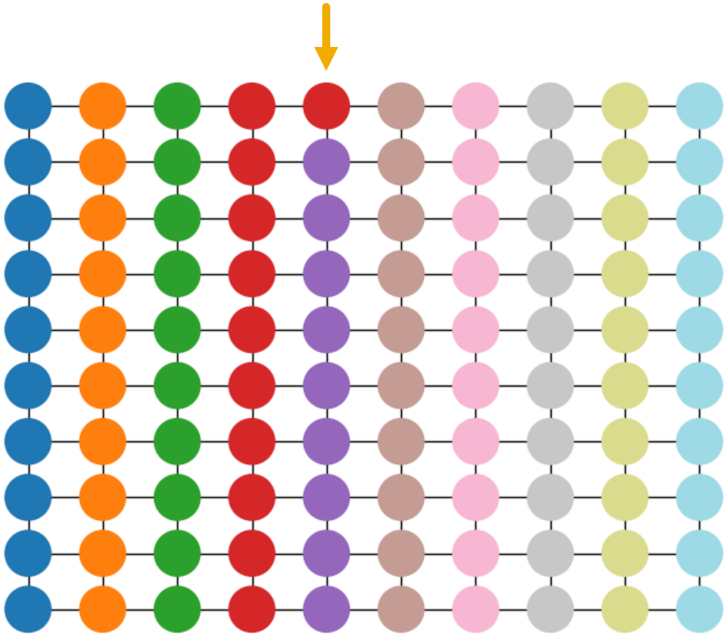
(a) Compactness



(b) 11996 cut edges

Recall: Flip Proposal

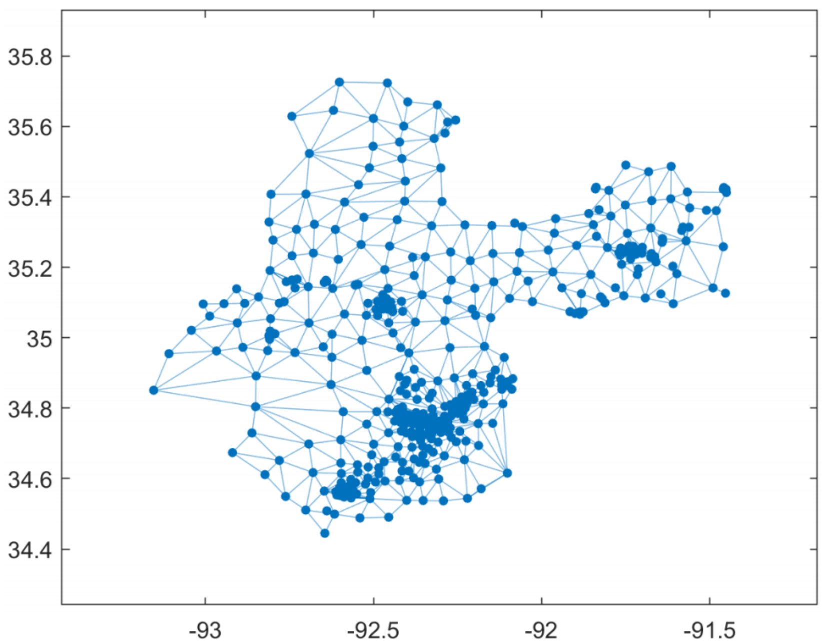




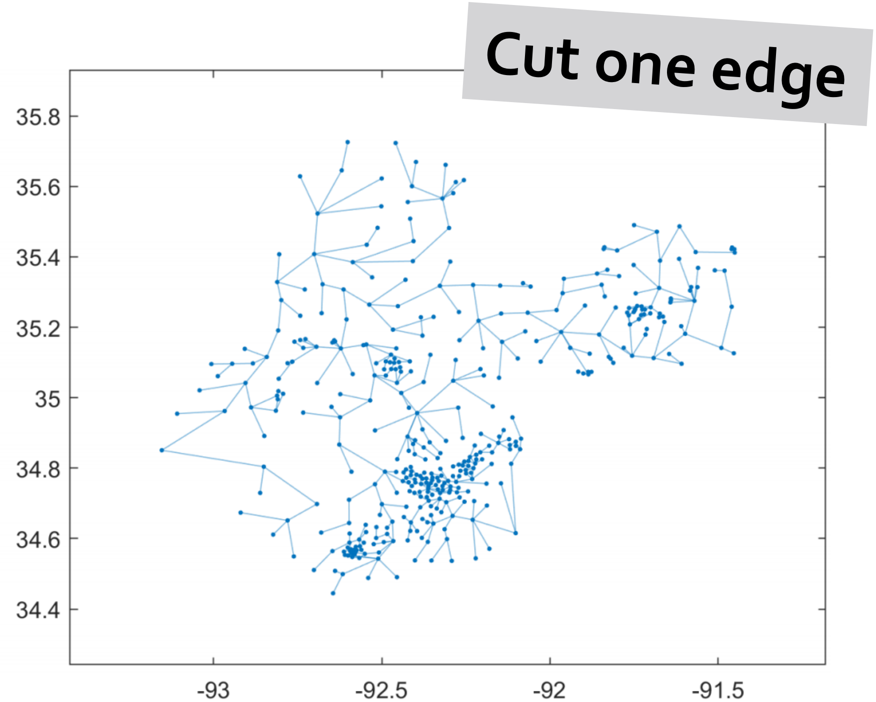
1. Uniformly choose a cut edge
2. Change label of an incident node

[Mattingly et al. 2017, 2018; Pegden et al. 2017]

Potential Way Out



(a) District



(b) Spanning Tree

"Recombination: A Markov Chain for Redistricting"
DeFord, Duchin, & Solomon, in preparation

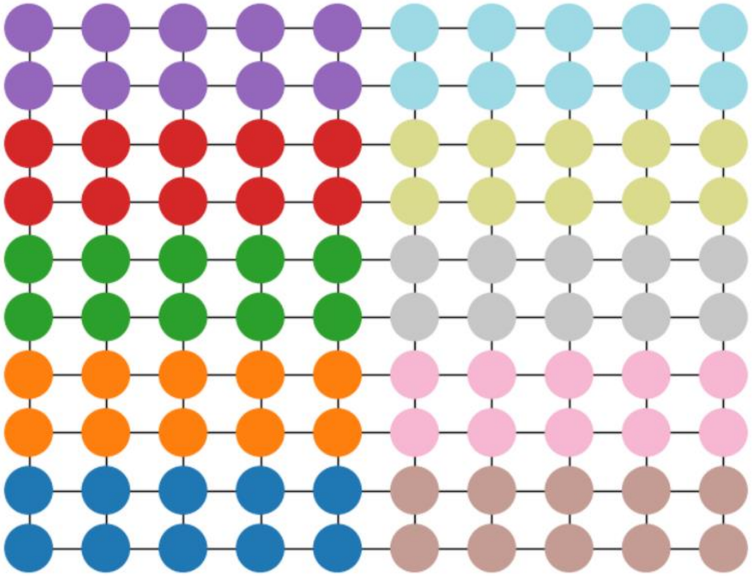
Recombination Step

- Select two adjacent districts
- Merge them together
- Draw a random spanning tree
- Delete an edge (can maintain balance)

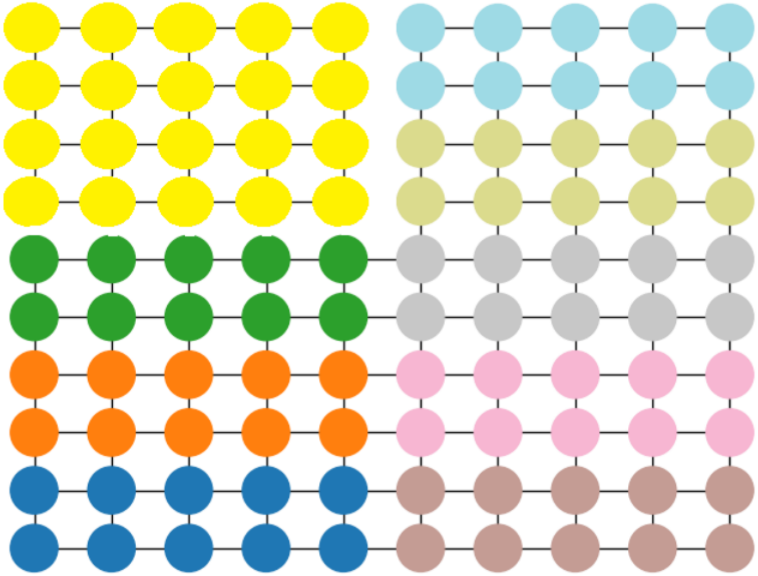
Conjecture. Mixing time is proportional to the number of districts.

Other
"tree proposals"
possible

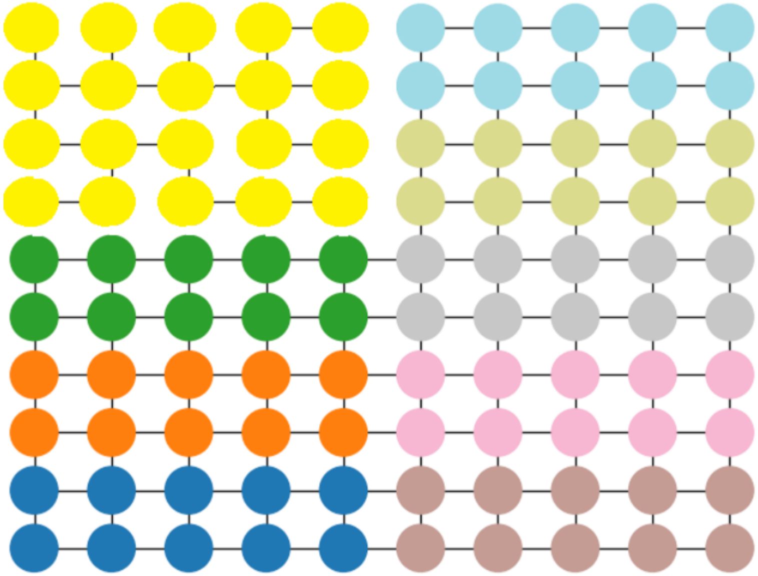
Example



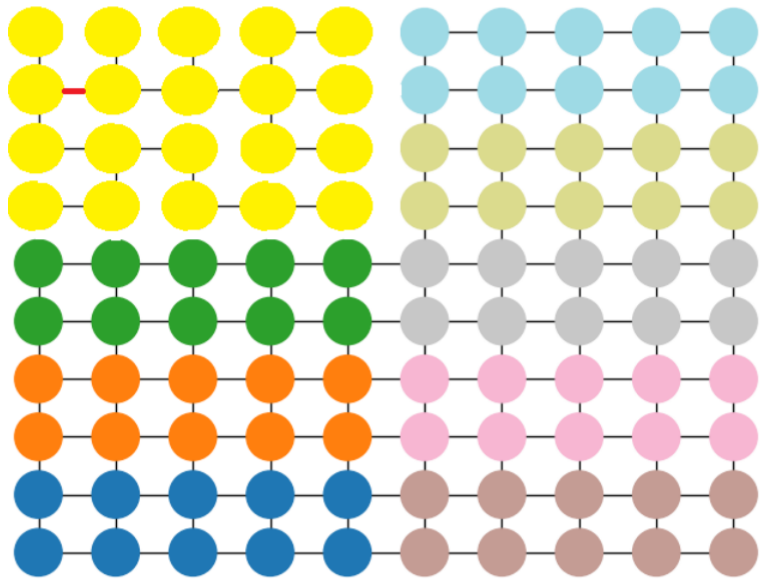
Example



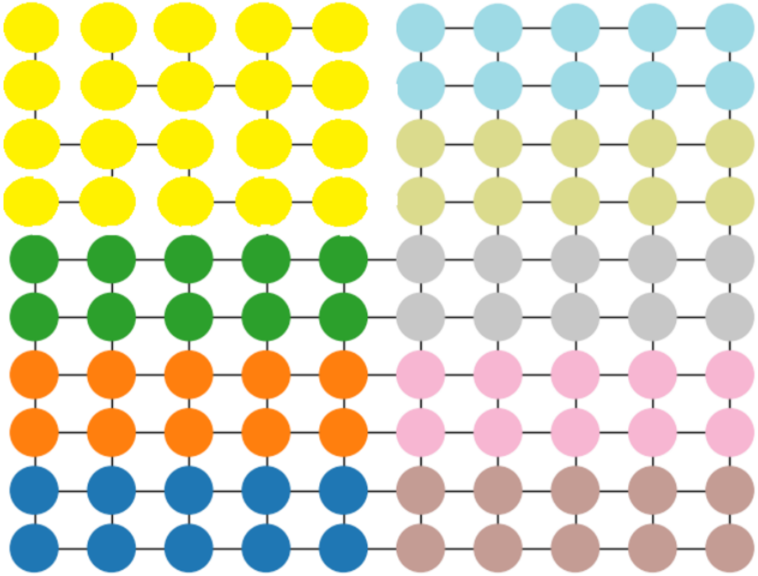
Example



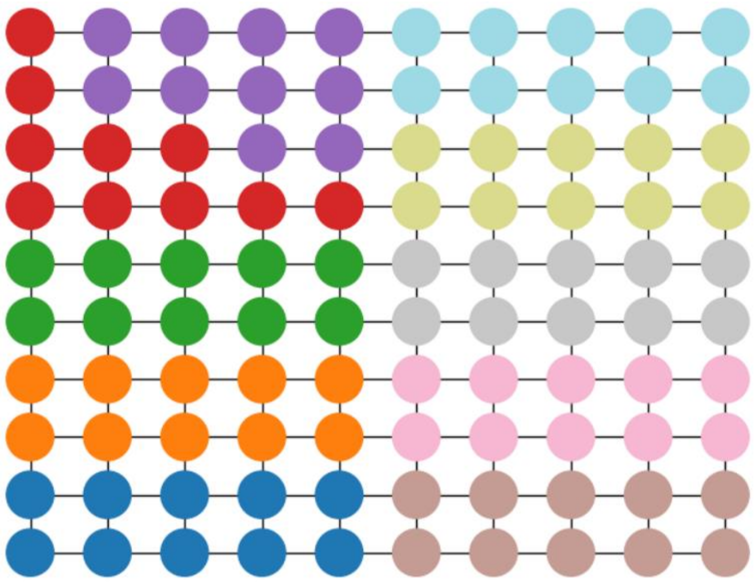
Example



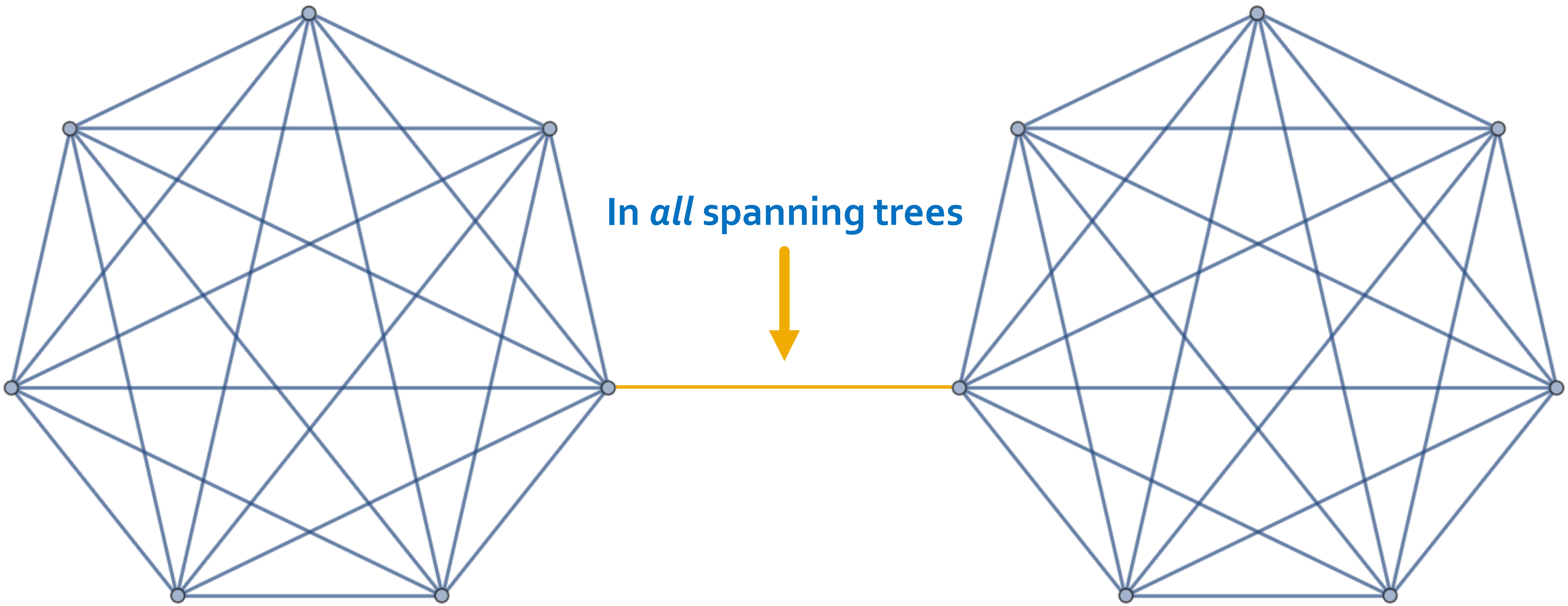
Example



Example



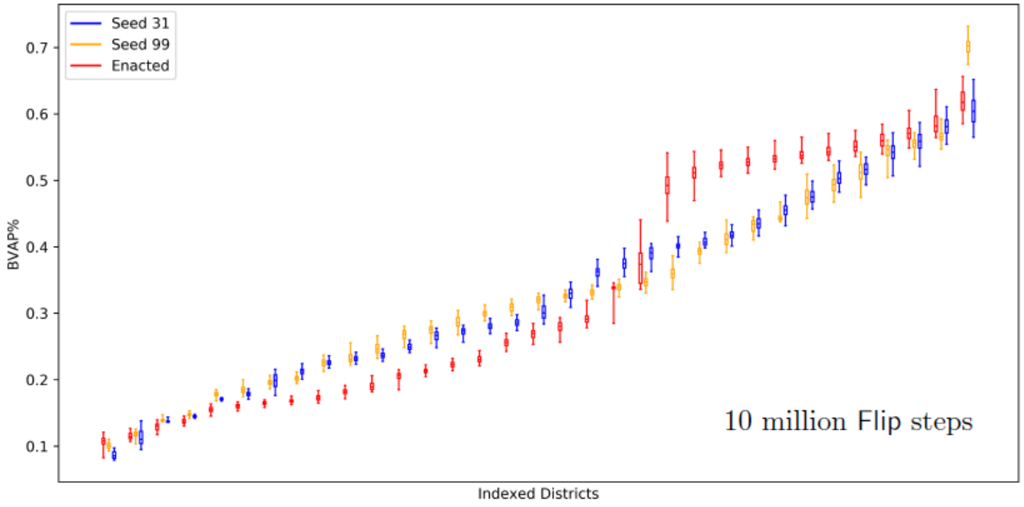
Distributional Bias

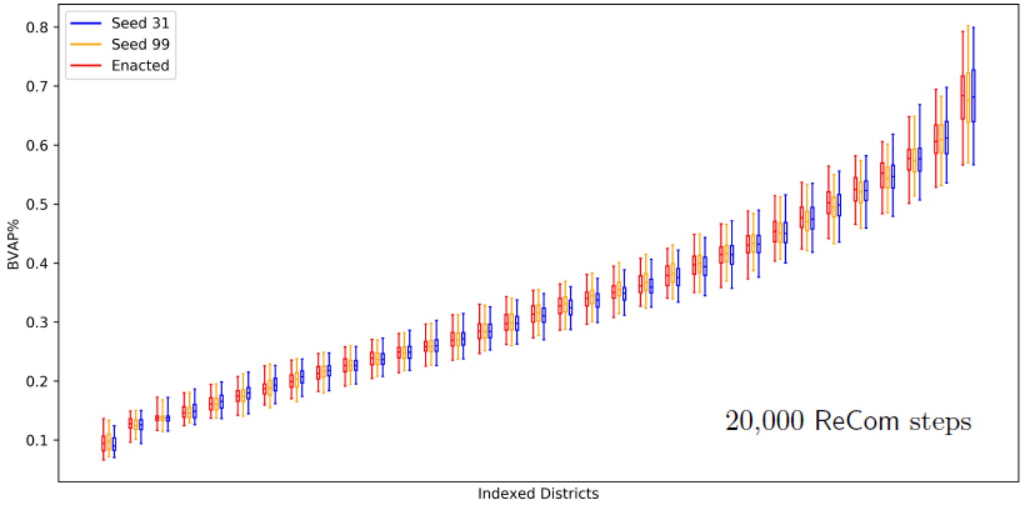


$$P(A, B) \propto [\mathrm{trees}(A)] \cdot [\mathrm{trees}(B)] \cdot [\mathrm{cut}(A, B)]$$

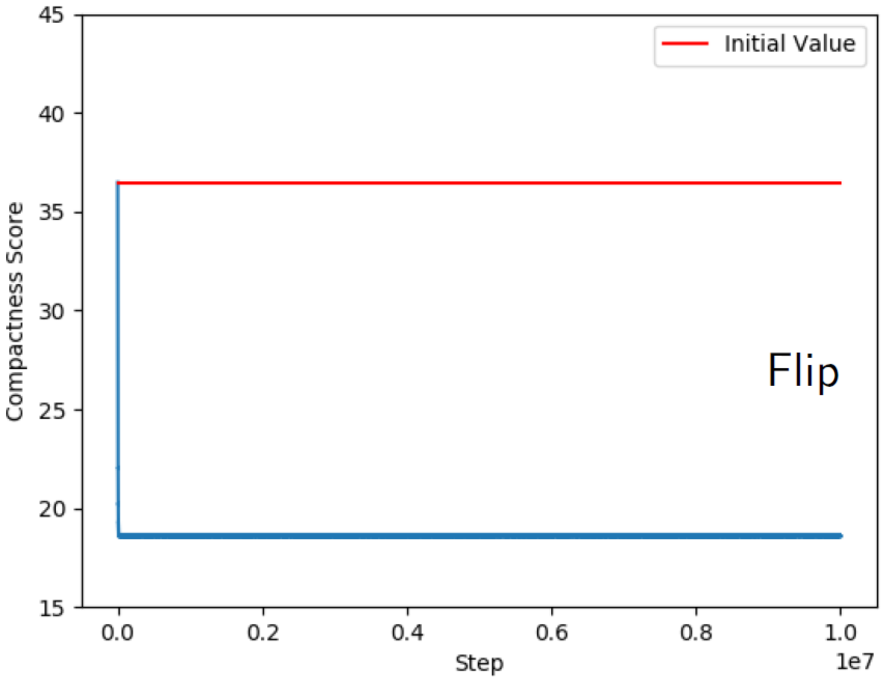
$$\text{Kirchoff: } \mathrm{trees}(G) = \frac{1}{n} \prod_{k=2}^{n} \lambda_k$$

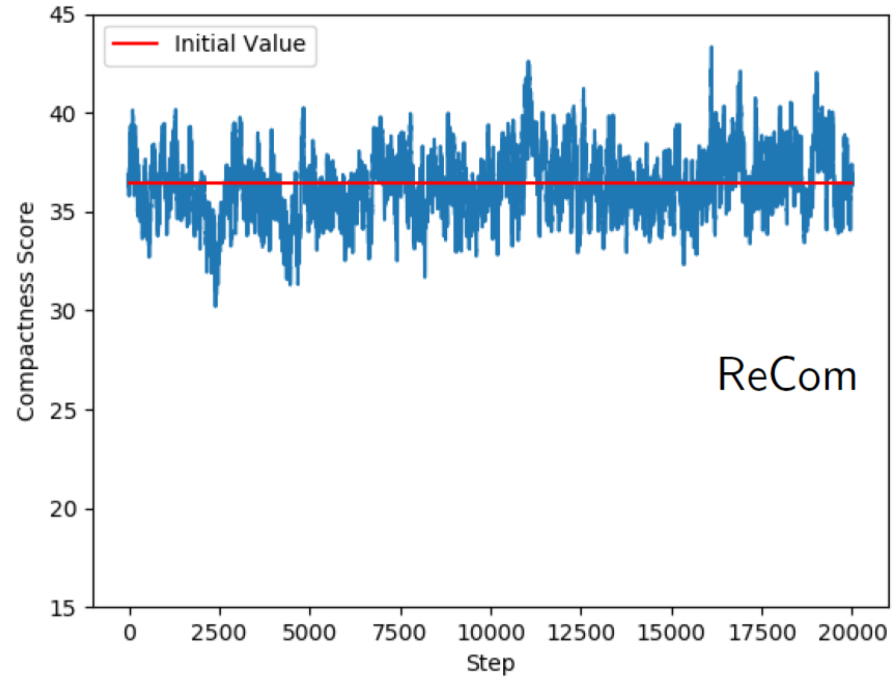
Empirical Evidence



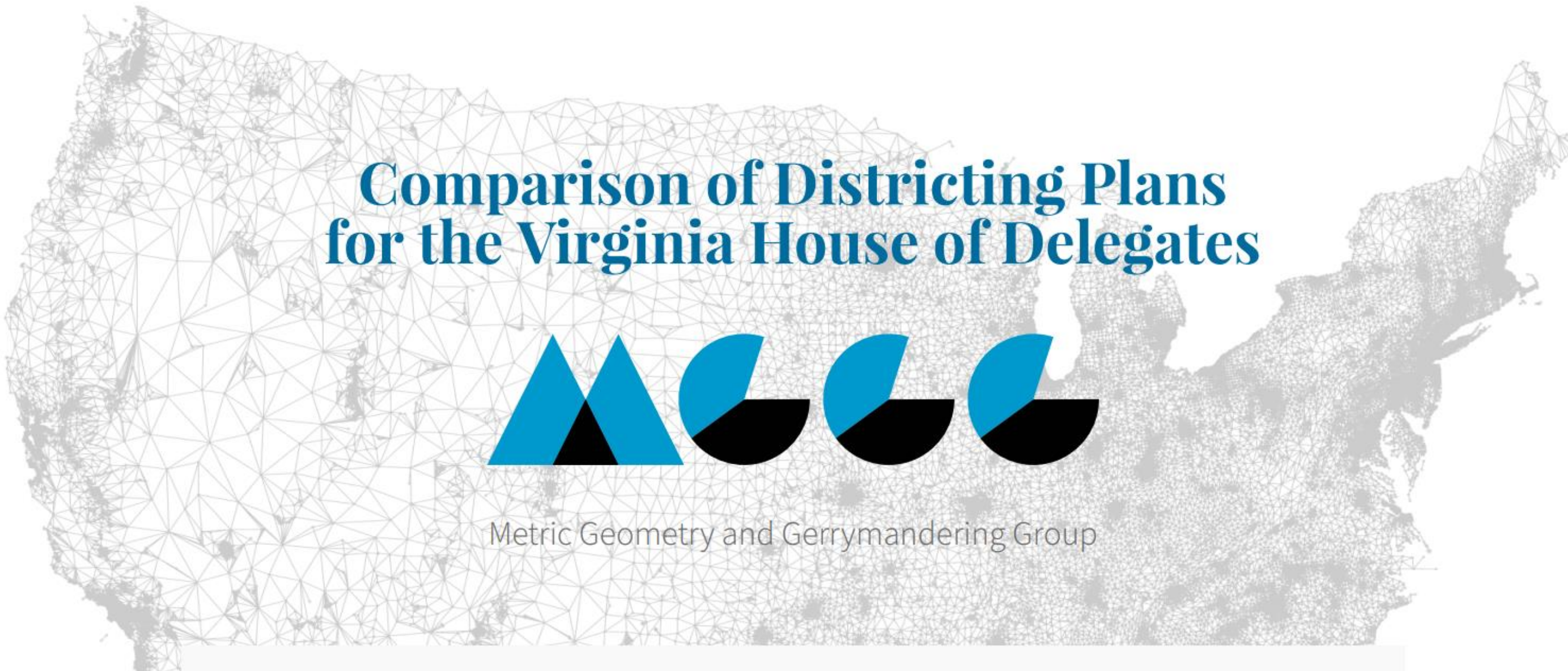


Compactness





Example Application



Abstract

At the time of writing, Virginia is in the process of replacing its House of Delegates districting plan after eleven of the districts were ruled unconstitutional by a District Court in June 2018. This report presents a large ensemble of alternative valid districting plans, which we propose to use as a baseline for comparison in the evaluation of newly proposed plans. Our method highlights and quantifies the dilutive effects of packing Black Voting Age Population.

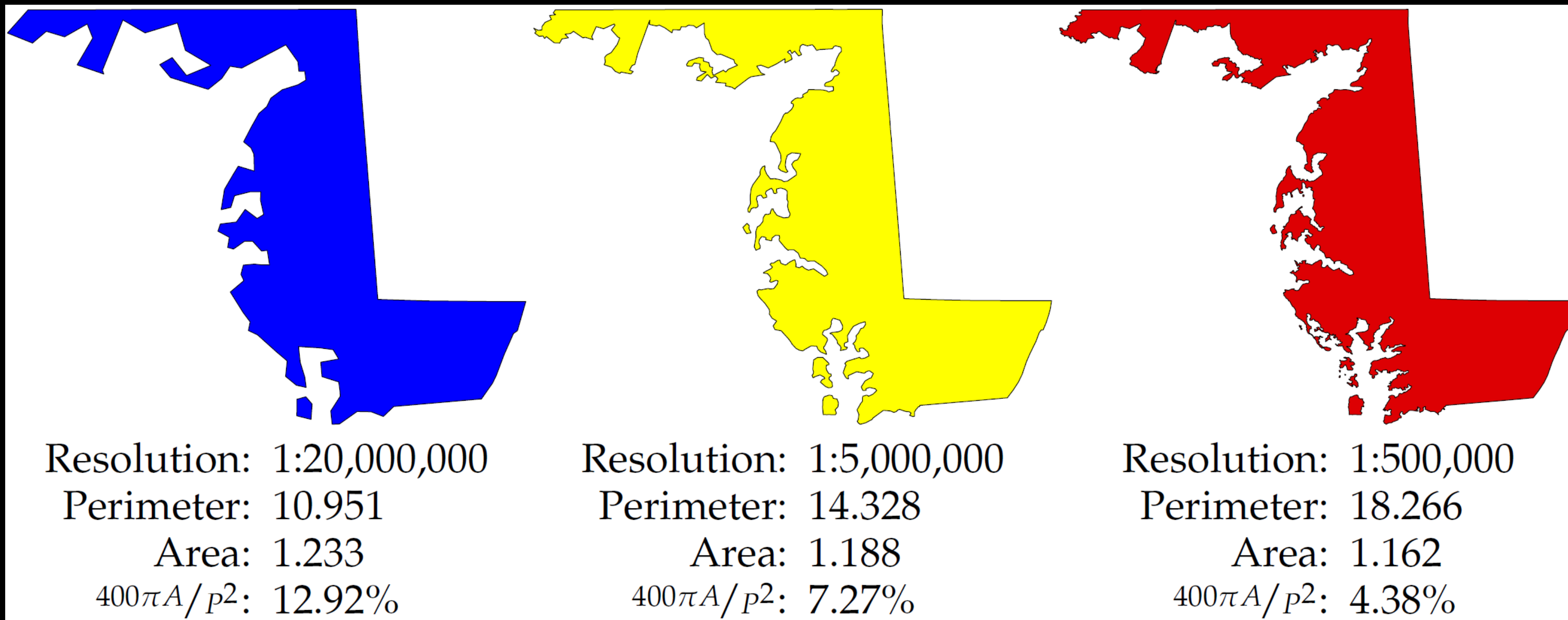
This is a novel application to racial gerrymandering of industry-standard techniques from statistics and computational science.

Take-Away

Quantitative analysis of districting plans is subtle.

Computational redistricting is not a solved problem.

with apologies to D. DeFord



Counterexamples
in Redistricting

Questions?