Endogenous Social Networks and Inequality in an Intergenerational Setting

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Working paper version

Abstract

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In a world where individuals interact in myriads of ways, one wonders how the benefits of one's connections with others compare with those conferred by individual characteristics when it comes to acquisition of human capital. It is particularly interesting to be able to distinguish between connections that are the outcome of deliberate decisions by individuals and connections being given exogenously and beyond an individual's control. The paper explores the consequences of the joint evolution of social connections and human capital investments. It thus allows one to study a broad range of possibilities in which social connections may influence inequality in consumption, human capital investment and welfare across the members of the economy, cross-sectionally and intertemporally. It embeds inequality analysis in models of endogenous social network formation. The novelty of the model lies in its joint treatment of human capital investment and social network formation in intergenerational settings, while distinguishing between the case of impact on human capital from endogenous as opposed to exogenous social networking. Among several results in the case of exogenous connections, we demonstrate conditions under which the limit distribution of human capital has a Pareto upper tail.

One of the dynamic models we develop allow for intergenerational transfers in a dynastic version of the infinite horizon Ramsey-Cass-Koopmans model. The models share the property that human capital accumulation, transfers and social connections, when all are optimized, are, along steady states, proportional to cognitive skills. Thus, intergenerational transfers of both human capital endowments and social networking endowments are jointly determined. Interestingly, the consequences for inequality of the endogeneity of social connections are underscored by examining the models when they are assumed to be exogenous. When social connections are not optimized, individuals' human capital reflect a much more general dependence on social connections. The dependence does not reduce to aggregate statistics of social connections. We show that the dynamics of demographically increasingly

complex models, as expressed by a sequence of models with increasing number of overlapping-generations, depend on the product of the adjacency matrices associated with each of the overlapping generations.

1 Introduction

In a world where individuals interact in myriads of ways, one wonders how the benefits of one's connections with others compare with those conferred by individual characteristics when it comes to acquisition of human capital. It is particularly interesting to be able to distinguish between connections that are the outcome of deliberate decisions by individuals and connections being given exogenously and beyond individuals' control. Such a distinction matters macroeconomically as well, if individuals stand to benefit from social connections in ways that affect consumption and investment. Individuals may seek to form social links with others, as an objective in its own right, in order to enrich their social lives and avoid social isolation. Social links provide conduits through which benefits from interpersonal exchange can be realized. Social isolation excludes them. The paper explores the consequences of the joint evolution of social connections and human capital investments. It thus allows one to study the full extent in which social connections may influence inequality in consumption, human capital investment and welfare across the members of the economy. It embeds inequality analysis in models of endogenous social networks formation. The novelty of the model lies in its joint treatment of human capital investment and social network formation, while distinguishing between the case of impact on human capital from endogenous as opposed to exogenous social networking.

The last few years have generated new research on social networks at a torrential rate, including books, most recently Goyal (2007), Jackson (2008), and Vega-Redondo (2007), and hundreds of papers. While social networks research was booming within econophysics for more than twenty years while being hardly noticed by economists, such research is increasingly spreading to virtually all economics fields, including notably experimental economics, too. Yet, As Jackson (2014), p. 14, points out, studying endogenous network formation continues to be an important priority.[1] The present paper aims at a deeper understanding of the consequences of social network formation for inequality. Such an emphasis has an intuitive appeal, that is whether social networking increases or decrease inequality, and how the process might be influenced by suitable policy.

It is straightforward to assess the difficulty of modeling social networking. For a given number of individuals I, there are $2^{\frac{I(I-1)}{2}}$ different possible networks connecting them. Thus, to a typical social group of $I = 100$ there correspond $2^{50\times49} \approx 10^{1500}$ network configurations, some of which might not be topologically distinct. To be able to conduct specific analyses that link differences in individual characteristics to differences in outcomes after individuals have formed social networks and have been influenced by those they end up being in social contact with one needs to be specific. It is for this reason that we start with a fairly tractable model of social network formation, which is due to Cabrales, Calvó-Armengol and Zenou (2011), which we extend into a dynamic model.[2]

The Cabrales, Calvó-Armengol, and Zenou framework originally starts from a familiar linear-quadratic model of individual decision making, about connecting with others, in a multi-person group context, with social links seen as outcomes of individual decisions, which are associated with a noncooperative Nash equilibrium.[3] A connection between any two individuals is associated with a connection weight, whose magnitude depends on inputs of effort by the two respective individuals, which can be either exogenous or functions of inputs decided upon by the respective parties. In a number of alternative simple settings, the model separates out the contribution of individual characteristics from the aggregate effects of population groups. Furthermore, because of equilibrium multiplicity that results entirely from social link formation, rich dynamic effects are possible whose consequences bear upon long-run income and wealth inequality. The results are obtained in a framework where links are symmetric (i.e., the underlying graph is undirected but weighted) and thus the benefits are mutual. The formation of symmetric links, as modelled here, presumes a certain degree of social coordination. That is, individuals recognize that even though their decisions are made in a non-cooperative context, they nonetheless result in creating social group formation. Asymmetric links, as where my being influenced by others (as by looking up to others) does not presume that those other individuals I am linked to are in turn influenced by me, provide avenues of social influence but do not connote social relations as such.

The paper is extended by means of a number of dynamic models of human capital investment and social network formation in order to allow for intergenerational transfers of wealth and of social connections. First, we interpret the dynamic model as one with the representative individual being infinitely lived. A variation of that model is to take social connections as given exogenously and not subject to optimization. This variation allows us to highlight the importance of endogenous setting of social connections for the cross-sectional distribution of human capital and explore conditions under which the social connections help magnify or reduce the impact of the dispersion in cognitive skills. When social connections are endogenous, the distribution of human capital mirrors that of the cognitive skills. Next, we follow a long tradition in economics that links life cycle savings, human capital investment and intergenerational transfers. Starting from Loury (1981), but also Becker and Tomes (1979; 1986) [see also Goldberger (1989), Ioannides (1986) and Ioannides and Sato (1987)], a number of papers have linked intergenerational transfers and the cross-section distributions of income and of wealth. In a recent paper, Lee and Seshadri (2014) model human capital accumulation in the presence of intergenerational transfers, while allowing for multiple stages of investment over the life cycle, such as investment during childhood, college decision and on-the-job human capital accumulation. It is one of very few papers that take Heckman's forceful suggestion [see Cunha and Heckman (2007); Heckman and Mosso (2014)] seriously, namely to allow for complementarity between early and later child investments, *inter alia,* by means of a model of 78-overlapping generations (and thus many more than the commonly used two overlapping generations) with infinitely lived altruistic dynasties. Their model shows, using numerical simulation methods, that investment in children and parents' human capital have a large impact on the equilibrium intergenerational elasticities of lifetime earnings, education, poverty and wealth, while staying consistent with cross-sectional inequality. They also show that education subsidies and progressive taxation can significantly reduce the persistence in economic status across generations. But they do not model social connections.

This paper presents a sequence of models, with parents making decisions about how much wealth to transfers to the children and about social connections along with investment in human capital. Parents recognize that due to the timing of implementing their social networking decisions their children stand to benefit from them, as they themselves have benefited from the decisions of their own parents. By moving to a model with two overlapping generations, we can determine how the pattern of dynamics reflects the demographic structure of the economy. Specifically, as the number of overlapping generations increases, the matrix characterizing the dynamic evolution of the state variable has a multiplicative factorial structure: each additional overlapping generation included contributes a factor to the product. Finally, the paper develops a variation of the two overlapping generations model with two subperiods which makes it possible for individuals to invest in augmenting the cognitive skills of their children. The impact of availability of such investments on the dynamics of evolution of human capital investments and social connections is considerably more complicated, but a factorial structure is still evident.

The remainder of this document is organized as follows. Section 2 introduces the basic model in a static setting, which takes off from the Cabrales, Calvó-Armengol and Zenou Model [Cabrales, Calvó-Armengol and Zenou (2011)]. This model allows us to explore the empirical implications of endogeneity of social connections by allowing for different assumptions about the effects of interactions. While the value of interactions and their consequences for income inequality have been explored before, notably by Benabou (1996) and Durlauf (1996), those earlier analyses were not conducted so as to allow for social network formation. Next we use the model to explore a number of alternative assumptions about the impact of interactions of each individual with her social contacts, such as allowing for individual characteristics to influence weights in a great variety of ways, for homogeneity of degree other than in interactions weights, and for CES-type of interactions. Section 3 introduces cognitive and socialization shocks to individuals' cognitive parameters and to their propensity to network, respectively. Section 4 presents an infinite-horizon model of an evolving economy consisting of many agents who build connections among each other. Section 5 assesses some consequences for cross-sectional inequality. Section 6 interprets the model in an infinite-horizon dynastic life cycle context, and section 6.1 extends the model first to an overlapping generations context, ultimately with two-overlapping generations. Subsection 6.1.2 examines, in particular, the effects of social networking on intertemporal wealth transfer elasticities. The solution allows one to conjecture about the properties of models with more than two overlapping generations. These extensions allow for parents' circumstances to influences their children's wealth endowments via transfers, social networking, as well as possibly persistent cognitive skills.

2 Endogenous Social Structure: The Cabrales, Calvó-Armengol and Zenou Model

In commonly employed formulations of models of individuals' actions subject to social interactions and in the definition of the group choice problem each individual is typically assumed to be affected by group averages of contextual effects and of decisions [Ioannides 2013, Ch. 2]. It is easy to contemplate that individuals may deliberately seek social interactions that are not necessarily uniform across their social contacts and to examine their determinants. I use Cabrales, Calvó-Armengol and Zenou (2011) as a starting point and briefly develop their key results, with individuals' engaging in networking efforts (socialization, in their terminology) that determine the probabilities of contacting others simultaneously with deciding on their own actions. Further below, I will interpret individuals' actions as human capital investments.

Individual i chooses action k_i and socialization effort s_i, taking as given actions and socialization efforts by all other individuals, $i, j \in \mathcal{I}$, so as to maximize:

$$U_{i,\tau(i)}(\mathbf{s}, \mathbf{k}) \equiv b_{\tau(i)} k_i + a \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}) k_i k_j - c\frac{1}{2}k_i^2 - \frac{1}{2}s_i^2, \tag{1}$$

where $\tau(i)$ denotes the individual type[4] individual i belongs to. I will simplify this notation for clarity, when it is not necessary, by using i instead of $\tau(i)$. The terms $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_I)$ denote the full vector of networking efforts, and $\mathbf{k} = (k_1, \ldots, k_i, \ldots, k_I)$, those of actions. The weights of social interaction g_{ij} may be defined in terms of socialization efforts in a number of alternative ways. In the simplest possible case, let the weights, which are obtained axiomatically by Cabrales *et al.*, be defined as:

$$g_{ij}(\mathbf{s}) = \frac{1}{\sum_{j=1}^{I} s_j} s_i s_j, \text{ if } \forall s_i \neq 0; \quad g_{ij}(\mathbf{s}) = 0, \text{otherwise.} \tag{2}$$

The coefficient of the interactive term in definition (1) is a key parameter in the determination of $\mathbf{s}$, the vector of connection intensities. Individual i chooses (s_i, k_i) so as to maximize (1).

I follow Cabrales *et al.* (2011) and define, for later use, an auxiliary variable

$$\tilde{a}(\mathbf{b}) = a\frac{\sum_{\tau\in\mathcal{T}} b_\tau^2}{\sum_{\tau\in\mathcal{T}} b_\tau}, \tag{3}$$

where $\mathcal{T}$ denotes the set of agent types, with generic element τ, as distinct from the set of individuals, $\mathcal{I}, I = |\mathcal{I}|$, and the functions $\bar{x}(\mathbf{x}), \overline{x^2}(\mathbf{x})$ are defined as follows:

$$\bar{x}(\mathbf{x}) \equiv \frac{\sum_{\tau\in\mathcal{T}} x_\tau}{|\mathcal{T}|},\ \overline{x^2}(\mathbf{x}) \equiv \frac{\sum_{\tau\in\mathcal{T}} x_\tau^2}{|\mathcal{T}|}. \tag{4}$$

The normalized sums in this definition reflect relative frequencies of individual types.

The first-order conditions are, with respect to k_i, s_i, as follows:

$$b_{\tau(i)} + a\sum_{j=1,j\neq i}^{I} g_{ij}(\mathbf{s})k_j - ck_i = 0; \tag{5}$$

$$a\sum_{j=1,j\neq i}^{I} k_i k_j \frac{\partial g_{ij}(\mathbf{s})}{\partial s_i} - s_i = 0. \tag{6}$$

With $g_{ij}(\mathbf{s})$ given by (2),

$$\frac{\partial g_{ij}(\mathbf{s})}{\partial s_i} = \frac{1}{\sum_{j=1}^{I} s_j}s_j - \frac{1}{(\sum_{j=1}^{I} s_j)^2}s_i s_j.$$

Following Ballester *et al.* (2006) and Cabrales *et al.* (2011), it is convenient to rewrite the first-order conditions, respectively, as follows:

$$\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]\cdot c\mathbf{k} + a\ \mathrm{diag}\left(\mathbf{G}(\mathbf{s})\right)\cdot\mathbf{k} = \mathbf{b}. \tag{7}$$

As they note, the matrix $[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})]$ is invertible and has a particularly simple form, using which (7) becomes:

$$c\mathbf{k} + a[\mathbf{I} + \lambda_{a/c}(\mathbf{s})\mathbf{G}(\mathbf{s})]\cdot \mathrm{diag}\left(\mathbf{G}(\mathbf{s})\right)\cdot\mathbf{k} = [\mathbf{I} + \lambda_{a/c}(\mathbf{s})\mathbf{G}(\mathbf{s})]\cdot\mathbf{b}, \tag{8}$$

where $\lambda_{a/c} \equiv \frac{a}{c}\frac{\bar{x}(\mathbf{s})}{\bar{x}(\mathbf{s}) - \frac{a}{c}\overline{x^2}(\mathbf{s})}$. Rewriting (6), the first-order conditions for the s_i's, yields:

$$s_i = ak_i\frac{\mathbf{s}\cdot\mathbf{k}}{I\bar{x}(\mathbf{s})} - as_i k_i\frac{\mathbf{s}\cdot\mathbf{k}}{(I\overline{x^2}(\mathbf{s}))} - a\frac{s_i k_i}{I\bar{x}(\mathbf{s})} + a\frac{(s_i k_i)^2}{I\overline{x^2}(\mathbf{s})}, \tag{9}$$

where $\mathbf{s}\cdot\mathbf{k} = \sum_{i=1}^{I} s_j k_j$.

2.1 Solving with a Large Number of Agents

As $I \to \infty$, the last three terms on the RHS of (9) vanish, yielding:

$$s_i = ak_i \frac{\mathbf{s} \cdot \mathbf{k}}{I\bar{x}(\mathbf{s})}. \tag{10}$$

Similarly, since $g_{ii}(\mathbf{s}) = \frac{s_i^2}{\sum_{j=1}^{I} s_j}$, diag $(\mathbf{G}(\mathbf{s}))$ vanishes at the limit, as I becomes large, and the respective first-order conditions become:

$$c\mathbf{k} = [\mathbf{I} + \lambda_{a/c}(\mathbf{s})\mathbf{G}(\mathbf{s})] \cdot \mathbf{b}. \tag{11}$$

It readily follows from (10) that the necessary conditions imply that $\frac{s_i}{k_i}$ is independent of i. Let the common ratio be

$$\frac{s_i}{k_i} = \varpi. \tag{12}$$

With the notation introduced in (4) above, the auxiliary term $\lambda_{a/c}$ in (11) becomes, using (10):

$$\lambda_{a/c} = \frac{a}{c} \frac{\bar{x}(\mathbf{s})}{\bar{x}(\mathbf{s}) - \frac{a}{c}\overline{x^2}(\mathbf{s})} = \frac{a}{c - \varpi^2}.$$

In view of these results, (11) is simplified as follows:

$$ck_i = b_i + \frac{a}{c - \varpi^2} s_i \frac{\sum_{j=1}^{I} b_j s_j}{I\bar{\mathbf{s}}}. \tag{13}$$

Using the previous results with the equation, it follows that k_i/b_i is constant,

$$\frac{k_i}{b_i} = \vartheta. \tag{14}$$

Thus,

$$c\vartheta = 1 + \vartheta \frac{a}{c - \varpi^2} \frac{\overline{x^2}(\mathbf{b})}{\overline{x(\mathbf{b})}},$$

Recalling the definition of $\tilde{a}$ in (3) above, (10) becomes:

$$\varpi = \tilde{a}\vartheta. \tag{15}$$

This allows us to write the above condition as:

$$\vartheta = \frac{1}{c - \varpi^2}. \tag{16}$$

The system of equations (15–16) define the solution (ϖ^*, ϑ^*), to the multi-person game. The solutions for (k_i, s_i) follow:

$$k_i^* = \vartheta^* b_{\tau(i)}, \quad s_i = \varpi^* \vartheta^* b_{\tau(i)}. \tag{17}$$

If agents do not value connections, $a = 0$, and from (3) and (15), $\varpi = 0$. In that case, we refer to the respective values as autarkic: $k_{i,\text{aut}} = \frac{1}{c} b_{\tau(i)}$. If agents are connected, $\varpi = 0 > 0$, and $\vartheta^* > \frac{1}{c}$, and the k_i's exceed their autarkic values. The feasibility condition for a non-autarkic solution, which Cabrales *et al.* obtain,[5] readily follows:

$$2 \left(\frac{c}{3} \right)^{\frac{3}{2}} \geq \tilde{a}. \tag{18}$$

For at least one solution to exist, the magnitude of the social interactions coefficient adjusted by the excess dispersion of the individual productivities, measured by $\tilde{a}$, must not exceed a function of the marginal cost of action coefficient. If the above condition is satisfied with inequality, then two solutions exist.

Numerous alternative formulations for the interaction structure are possible. In view of the applications emphasized by this paper, it would be interesting to allow for more general formulations of the social interaction structure. Next, I start by examining a number of alternative formulations which are proposed in an unpublished earlier version of Cabrales *et al.* (2011) and proceed with a CES-type structure for the social interactions effects which, in a computable sense, nest a number of other specifications.

2.2 Generalizing the value of interactions

The above solution clearly depends on the specific assumptions made about the interaction terms. But in view of the applications emphasized by this paper, it would be interesting to allow for more general formulations of the social interaction weights. They may be generalized in two directions. One is about the interaction weights, i.e. the terms $g_{ij}(\mathbf{s})$, which may easily be generalized to allow for dependence on the $\mathbf{b}$'s that is, $g_{ij}(\mathbf{s}; \mathbf{b})$. I turn to this immediately below, where I explore in turn some alternative formulations proposed in an earlier unpublished version of Cabrales *et al.* (2011). A second involves the k_i's. Although the

k_i's and the s_i's enter in the original formulation in part symmetrically, they have distinctly different interpretations. For example, it is helpful to interpret the term $k_i k_j$ in i's utility function as learning. In fact, the underlying assumption of complementarity between human capitals of different individuals may be set in more general terms, for which the assumption of a CES social interactions structure is particularly useful. I take that up further below.[6]

2.2.1 Homogeneity of degree greater or less than one in connection weights

Combining connection intensities as in the original definition (1) but in a homogeneous of degree *greater than one* fashion,

$$g_{ij}(\mathbf{s};\mathbf{b}) = s_i s_j, \text{ if } \mathbf{s} \neq \mathbf{0}; g_{ij}(\mathbf{s};\mathbf{b}) = 0, \text{if } \mathbf{s} = \mathbf{0}, \tag{19}$$

yields the following first-order conditions:

$$\frac{s_i}{k_i} = a \sum_{j=1, j\neq i}^{I} s_j k_j,$$

$$ck_i = b_i + \frac{\frac{a}{c}\sum_{j=1}^{I} s_i s_j b_j}{1 - \frac{a}{c} I \sum_{j=1}^{I} s_j^2},$$

provided that $1-\frac{a}{c}I\sum_{j=1}^{I} s_j^2$. This particular case demonstrates the consequences of increasing returns to scale in the interactions structure. The feasibility condition is unlikely to be satisfied in the asymptotic case. There is no congestion in the synergies, and when I is large, the quantity $1-\frac{a}{c}I\sum_{j=1}^{I} s_j^2$ becomes negative, and the steps necessary to obtain asymptotic results are violated. There are in effect too many synergies. Therefore, homogeneity greater than one, at least in this formulation, is associated with infeasibility.[7]

Combining connection intensities as in the original definition (**??**) but in a homogeneous of *degree less than one* fashion,

$$g_{ij}(\mathbf{s};\mathbf{b}) = \frac{(s_i s_j)^{\alpha}}{\sum_{j=1}^{I} s_j^{\alpha}}, \text{ if } \mathbf{s} \neq \mathbf{0}; g_{ij}(\mathbf{s};\mathbf{b}) = 0, \text{if } \mathbf{s} = \mathbf{0}, \tag{20}$$

and slightly modifying the individual's objective

$$U_{i,\tau(i)}(\mathbf{s},\mathbf{k}) \equiv b_{\tau(i)} k_i + a \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s})(k_i k_j)^{2-\alpha} - c\frac{1}{2}k_i^2 - \frac{1}{2}s_i^2, 0 < \alpha < 2, \tag{21}$$

leads to first-order conditions as follows:

$$\left(\frac{s_i}{k_i}\right)^{2-\alpha} = \alpha a \frac{\sum_{j=1}^{I} k_j^2 \left(\frac{s_j}{k_j}\right)^{\alpha}}{\sum_{j=1}^{I} s_j^{\alpha}};$$

$$c\frac{k_i}{b_i} = 1 + \left(\frac{s_i}{k_i}\right)^{\alpha}\left(\frac{k_i}{b_i}\right) \frac{\frac{a}{c}\sum_{j=1}^{I} s_j^{\alpha}}{\sum_{j=1}^{I} s_j^{\alpha} - \frac{a}{c}\sum_{j=1}^{I} k_j^2 \left(\frac{s_j}{k_j}\right)^{2\alpha}} \frac{\sum_{j=1}^{I} b_j k_j \left(\frac{s_j}{k_j}\right)^{\alpha}}{\sum_{j=1}^{I} s_j^{\alpha}}.$$

Rewriting these conditions, we have:

$$\varpi^2 = \alpha\tilde{a}\varpi^{\alpha}\vartheta^{2-\alpha}\frac{\bar{b}}{\bar{b}_{\alpha}}; \tag{22}$$

$$c\vartheta = 1 + \frac{\varpi^{\alpha}\vartheta^{2-\alpha}\tilde{a}\bar{b}}{c\bar{b}_{\alpha} - \varpi^{\alpha}\vartheta^{2-\alpha}\tilde{a}\bar{b}}, \tag{23}$$

where $\bar{b}_{\alpha} \equiv \sum_{j=1}^{I} s_j^{\alpha}/|\mathcal{T}|$. This can be solved in the same fashion as with the simple weights of section **??**, that is, we may solve for values for $\varpi = \frac{s_i}{k_i}, \vartheta = \frac{k_i}{b_i}$. The equilibrium values depend on aggregates of the primitive parameters.

2.2.2 Individualized interaction weights

In order to go beyond networking intensities that depend only on functions of networking efforts, we may assume connection weights that are weighted by individual "productivities," the b_i's, in a homogeneous of degree one fashion as follows:

$$g_{ij}(\mathbf{s};\mathbf{b}) = \frac{(b_i b_j)^{1-\alpha}(s_i s_j)^{\alpha}}{\sum_{j=1}^{I} b_j^{1-\alpha} s_j^{\alpha}}, \text{ if } \forall \mathbf{s} \neq \mathbf{0}; g_{ij}(\mathbf{s};\mathbf{b}) = 0, \text{if } \mathbf{s} = \mathbf{0}. \tag{24}$$

The respective first-order conditions become:

$$\left(\frac{s_i}{k_i}\right)^{2-\alpha} = \alpha a \left(\frac{b_i}{k_i}\right)^{1-\alpha} \frac{\sum_{j=1}^{I} k_j b_j^{1-\alpha} s_j^{\alpha}}{\sum_{j=1}^{I} b_j^{1-\alpha} s_j^{\alpha}};$$

$$c\frac{k_i}{b_i} = 1 + \left(\frac{s_i}{b_i}\right)^{\alpha} \frac{\frac{a}{c}\sum_{j=1}^{I} b_j^{1-\alpha} s_j^{\alpha}}{\sum_{j=1}^{I} b_j^{1-\alpha} s_j^{\alpha} - \frac{a}{c}\sum_{j=1}^{I}\left(b_j^{1-\alpha} s_j^{\alpha}\right)^2} \frac{\sum_{j=1}^{I} b_j b_j^{1-\alpha} s_j^{\alpha}}{\sum_{j=1}^{I} b_j^{1-\alpha} s_j^{\alpha}}$$

They may be rewritten so as to make it clearer that at the steady state, they imply values for $\frac{s_i}{k_i}, \frac{b_i}{k_i}$, which are independent across i's.

$$\left(\frac{s_i}{k_i}\right)^{2-\alpha} = \alpha a \left(\frac{b_i}{k_i}\right)^{1-\alpha} \frac{\sum_{j=1}^{I} k_j b_j \left(\frac{s_j}{b_j}\right)^{\alpha}}{\sum_{j=1}^{I} b_j \left(\frac{s_j}{b_j}\right)^{\alpha}};$$

$$c\frac{k_i}{b_i} = 1 + \left(\frac{s_i}{b_i}\right)^{\alpha} \frac{\frac{a}{c}\sum_{j=1}^{I} b_j \left(\frac{s_j}{b_j}\right)^{\alpha}}{\sum_{j=1}^{I} b_j \left(\frac{s_j}{b_j}\right)^{\alpha} - \frac{a}{c}\sum_{j=1}^{I} b_j^2 \left(\frac{s_j}{b_j}\right)^{2\alpha}} \frac{\sum_{j=1}^{I} b_j^2 \left(\frac{s_j}{b_j}\right)^{\alpha}}{\sum_{j=1}^{I} b_j \left(\frac{s_j}{b_j}\right)^{\alpha}}$$

Rewriting these conditions, we have:

$$\varpi^{2-\alpha} = \alpha\vartheta\tilde{a}; \tag{25}$$

$$c\vartheta = 1 + \frac{(\varpi\vartheta)^{\alpha}}{c\bar{b} - a(\varpi\vartheta)^{\alpha}\overline{\overline{x^2(\mathbf{b})}}}\tilde{a}. \tag{26}$$

All these examples, which were proposed in an earlier version of Cabrales *et al.* (2011), share the property that social networking efforts are proportional to the respective $\mathbf{b}$'s, $s_i = \varpi\vartheta b_i$. Differences across models in the factors of proportionality demonstrate the rich set of possibilities afforded by the model.

2.3 CES Interactions Structure

The previous specifications of the interactions weights characterize qualitatively similar complementarity effects. They simply involve auxiliary variables that are more complicated functions of $\mathbf{b}$ than $\tilde{a}(\mathbf{b})$. For example, the terms $g_{ij}(\mathbf{s};\mathbf{b})k_ik_j$, express synergy weights between agents i and j. The marginal utility of human capital k_i depends positively on those of other agents via a convex structure. It is straightforward to generalize this assumption, such as by means of a CES structure,[8] which may be convex or concave in the inputs. It is well known that in the limit, such a structure allows for an individual to benefit from the *maximum* or the *minimum*, respectively, among all other individuals he interacts with [Benabou (1996); Polya *et al.* (1952), p. 15, Theorem 4].

That is, if the interaction term in (2) may be assumed to be instead of the form:

$$k_i s_i \left[\sum_{j\neq i} \frac{s_j}{\sum_i s_i} k_j^{1-\frac{1}{\xi}}\right]^{\frac{\xi}{\xi-1}}, \tag{27}$$

then it admits as a special case the original assumption (2), as well as a number of commonly used assumptions as additional special cases. That is, special cases of (27) are notable:

1. $\xi \to \infty : k_i\ s_i \frac{s_j}{\sum_i s_i} k_j$;

2. $\xi \to 1 : k_i s_i \prod_{j \neq i} k_j^{\left(\frac{s_j}{\sum_i s_i}\right)}$;

3. $\frac{1}{\xi} \to \infty : \min_j \{k_j\}$: *one bad apple spoils the bunch.*

4. $\frac{1}{\xi} \to -\infty : \max_j \{k_j\}$: *the best individual is the role model.*

Case 1 above coincides with the original specification in Section 1 above. Case 2 is the classic Cobb-Douglas function as special case of the CES structure; case 3 is the Leontieff case; case 4 is the extreme case of a convex interaction structure.[9] These new set of possibilities will be particularly helpful when we introduce uncertainty into the model. As it will be seen below, the nature of the solutions are qualitatively similar to the original model.

2.3.1 "One Bad Apple *Spoils* the Bunch."

For this case, the first-order conditions are:

$$b_i + as_i \min_{j \neq i}\{k_j\} - ck_i = 0; \tag{28}$$

$$ak_i \min_{j \neq i}\{k_j\} - s_i = 0. \tag{29}$$

By dividing both sides of the above equations by k_i we obtain a system of two equations in the same auxiliary variables as above:

$$\vartheta^{-1} + a\varpi \min_{j \neq i}\{k_j\} = c; \tag{30}$$

$$a \min_{j \neq i}\{k_j\} = \varpi. \tag{31}$$

Eliminating $\min_{j \neq i}\{k_j\}$ between these two equations yields (16), from which and one of the equations can solve for the equilibrium values of (ϖ, ϑ). Disregarding the imprecision that $\min_{j \neq i}\{k_j\} = \min_{j \in \mathcal{I}}\{k_j\}$ we have that $\min_{j \in \mathcal{I}}\{k_j\} = \vartheta \min_{j \in \mathcal{I}}\{b_j\} = b_{\min}$. Thus,

$$\varpi = ab_{\min}, \ \vartheta = \frac{1}{c - (ab_{\min})^2},$$

and $k_j = b_j \frac{1}{c-(ab_{\min})^2}, s_j = b_j \frac{ab_{\min}}{c-(ab_{\min})^2}$. Therefore, human capital and socialization effort are still proportional to the respective cognitive skill, but the social multiplier reflects the impact of the "one bad apple."

2.3.2 The Best Individual is *the* Role Model"

For this case, the first-order conditions are:

$$b_i + as_i \max_{j \neq i}\{k_j\} - ck_i = 0; \tag{32}$$

$$ak_i \max_{j \neq i}\{k_j\} - s_i = 0; \tag{33}$$

Since $\max_{j \neq i}\{k_j\}$ is a convex function of $\mathbf{k}$, the system of equations (28–29) has enough structure to allows us to characterize its solutions.[10] By substituting for s_i from (33) into (32), the resulting equations are defined solely in terms of $\mathbf{k}$ as fixed points of:

$$k_i = \frac{b_i}{c_i} \frac{1}{c - a^2 \left(\max_{j \neq i}\{k_j\}\right)^2}, i \in \mathcal{I}.$$

By working in like manner as above, we have that:

$$\varpi = ab_{\max}, \ \vartheta = \frac{1}{c - (ab_{\max})^2},$$

and $k_j = b_j \frac{1}{c-(ab_{\max})^2}$, $s_j = b_j \frac{ab_{\min}}{c-(ab_{\max})^2}$. Therefore, even though human capital investment and socialization effort are still proportional to the respective cognitive skill, the social multiplier is now larger than in the previous example of "one bad apple spoils the bunch." It reflects the fact that the social effects is associated with "best individual" as the role model.

3 Social Portfolio Analysis

I pursue further the role of interactions between individuals' actions and networking efforts by assuming that stochastic shocks affect the individuals' parameters. I consider first an additive shock, ψ_i, to b_i, which I refer to as a *cognitive* shock, which is known to individual i when she sets her own human capital decision and networking effort. Thus, both those decisions are function of one's own cognitive shock: $k_i(b_i + \psi_i), s_i(b_i + \psi_i)$. Another possible shock could be to affect the social competence parameter a, which weights the contribution to individual i's welfare from social interactions. I work first with cognitive shocks' being extreme-valued distributed, which allows me to utilize two of the limit results discussed above, that is, the extreme cases of $\frac{1}{\xi} \to \infty$, and $\frac{1}{\xi} \to -\infty$, which usher in the "one bad apple spoils the bunch," and the best individual is the role model" metaphors, respectively.

Alternatively, shocks to social networking outcomes may be interpreted as non-cognitive shocks. Here below I develop a number of examples of cognitive shocks, and leave the case of non-cognitive shocks to future research.

3.1 Cognitive Shocks

I model cognitive shocks as additive shocks to the b_i's. Individual i observes $b_i + \psi_i$ and sets (k_i, s_i), which as a result do depend on ψ_i, under the assumption that all other individuals do likewise under uncertainty about the cognitive shocks of others. It readily follows that the cognitive shocks are transmitted to the networking decisions. If the shocks are independent across individuals, then we can easily characterize the nature of the solution.

3.1.1 "One Bad Apple Spoils the Bunch."

For this case, the first-order conditions are:

$$b_i + \psi_i + as_i \; E_{\psi_{j|\psi_i}} \min_{j \neq i}\{k_j(b_j + \psi_j)\} - ck_i = 0; \tag{34}$$

$$ak_i \; E_{\psi_{j|\psi_i}} \min_{j \neq i}\{k_j(b_j + \psi_j)\} - s_i = 0; \tag{35}$$

By substituting for s_i from (35) into (34), we get:

$$\frac{k_i(b_i + \psi_i)}{b_i + \psi_i} = \frac{1}{c - a^2 \left[E_{\psi_{j|\psi_i}} \min_{j \neq i}\{k_j(b_j + \psi_j)\}\right]^2}. \tag{36}$$

Under our assumption that the ψ_i's are independent, the RHS of (36) does not depend on ψ_i and therefore, so should the LHS. This suggests that $\frac{k_i(b_i+\psi_i)}{b_i+\psi_i} = \nu_i$, where ν_i is independent of ψ_i but does depend on all parameters of the problem. That is, $k_i(b_i + \psi_i) = (b_i + \psi_i)\nu_i$. Since this holds for all i's, the previous condition may be rewritten as:

$$\nu_i = \frac{1}{c - a^2 \left[E_{\psi_{j|\psi_i}} \min_{j \neq i}\{\nu_j(b_j + \psi_j)\}\right]^2}, \; i \in \{\mathcal{I}\}. \tag{37}$$

To get a sense of the properties of the solution let us assume that the random variables ψ_j are Fréchet-distributed and conditionally independent. Let the cumulative distribution distribution of ψ_i be given by: $\exp\left[-\left(\frac{\psi}{\sigma_i}\right)^{-\chi}\right]$. It follows that the cumulative distribution function of $(b_i+\psi_i)\nu_i$ is given by:

$$\exp\left[-\left(\frac{\kappa-b_i\nu_i}{\nu_i\sigma_i}\right)^{-\chi}\right].$$

The corresponding cumulative distribution function of $\min_{j\neq i}\{(b_j+\psi_j)\nu_j\}$ is written in terms of

$$1-\Pi_{j\neq i}\left(1-\text{Prob}\left\{(b_j+\psi_j)\nu_j\leq\kappa\right\}\right).$$

It follows that the expectation of $\min_{j\neq i}\{(b_j+\psi_j)\nu_j\}$ is monotonically increasing in $\nu_j,\ j\neq i$. It thus follows that the RHS of Eq. (37) is monotonically increasing in $\nu_j,\ j\neq i$. The ν_i's follow as solutions to the system of equations (37), but in general the solution is not unique. Finally, the solutions for the networking efforts, the s_i's, follow from (35).

3.1.2 "The Best Individual is *the* Role Model"

For this case, the first-order conditions are:

$$b_i+\psi_i+as_i\ E_{\psi_{j|\psi_i}}\max_{j\neq i}\{k_j(b_j+\psi_j)\}-ck_i=0; \tag{38}$$

$$ak_i\ E_{\psi_{j|\psi_i}}\max_{j\neq i}\{k_j\}-s_i=0; \tag{39}$$

By working in like manner as in the previous section, we have:

$$k_i(b_i+\psi_i)=\frac{b_i+\psi_i}{c-a^2\left[E_{\psi_{j|\psi_i}}\max_{j\neq i}\{k_j(b_j+\psi_j)\}\right]^2}. \tag{40}$$

Under our assumptions, the RHS does not depend on ψ_i and therefore, so should the LHS. This suggests that $\frac{k_i(b_i+\psi_i)}{b_i+\psi_i}=\nu_i$, where ν_i is independent of ψ_i but does depend on all parameters of the problem. That is, $k_i(b_i+\psi_i)=(b_i+\psi_i)\nu_i$. Since this holds for all i's, the previous condition may be rewritten as:

$$\nu_i=\frac{1}{c-a^2\left[E_{\psi_{j|\psi_i}}\max_{j\neq i}\{\nu_j(b_j+\psi_j)\}\right]^2},i\in\{\mathcal{I}\}. \tag{41}$$

To get a sense of the properties of the solution let us again assume that the random variables ψ_j are Fréchet-distributed and conditionally independent. Let the cumulative distribution distribution of ψ_i be given by: $\exp\left[-\left(\frac{\psi}{\sigma_i}\right)^{-\chi}\right]$. It follows that the cumulative distribution function of $(b_i+\psi_i)\nu_i$ is given by:

$$\exp\left[-\left(\frac{\kappa-b_i\nu_i}{\nu_i\sigma_i}\right)^{-\chi}\right].$$

The corresponding cumulative distribution function of $\max_{j\neq i}\{(b_j+\psi_j)\nu_j\}$ is written in terms of

$$\Pi_{j\neq i}\left(\text{Prob}\left\{(b_j+\psi_j)\nu_j\leq\kappa\right\}\right).$$

It follows that the expectation of $\max_{j\neq i}\{(b_j+\psi_j)\nu_j\}$ is monotonically decreasing in $\nu_j,\ j\neq i$. It thus follows that the RHS of Eq. (37) is monotonically decreasing in $\nu_j,\ j\neq i$. The ν_i's follow as a unique, in general, solution to the system of equations (41). The solutions for the networking efforts, the s_i's, follow from (39).

Contrasting "One Bad Apple Spoils the Bunch" versus "The Best Individual is *the* Role Model" can explain the possible multiplicity against uniqueness of solution. They recall the multiplicity properties of steady states in economic growth models with overlapping generations preferences or production functions.[11]

4 Dynamics

A conventional[12] dynamic analysis of such a model follows from defining an intertemporal objective function for agents, and allowing for the first-order conditions to yield equations exhibiting dynamic adjustment. Let us rewrite the definition of the utility per period (1) as:

$$U_{i,t}(\mathbf{s}_{t-1};s_{it};\mathbf{k}_{t-1},k_{it})\equiv b_{\tau(i)}k_{it}+a\sum_{j=1,j\neq i}^{I}g_{ij}(\mathbf{s}_{t-1})k_{it}k_{jt-1}-c\frac{1}{2}k_{it}^2-\frac{1}{2}s_{it}^2, \tag{42}$$

According to definition (42), it is networking efforts, that is interaction weights at time $t-1$, $\mathbf{s}_{t-1}$, that affect spillovers at time t resulting from actions at time $t-1$. Accordingly, in deciding on her networking efforts and thus interaction weights, agent i anticipates the impact on her utility in the next period. Specifically, agent i seeks to maximize

$$\sum_{t=0}^{\infty} \rho^t U_{i,\tau(i),t}(\mathbf{s}_{t-1}; s_{it}; \mathbf{k}_{t-1}, k_{it}),$$

by choosing sequences of human capital investment and networking efforts $\{\mathbf{k}_{it}\}_0^{\infty}, \{\mathbf{s}_{it}\}_0^{\infty}$, taking as given all other agents' contemporaneous decisions $\{\mathbf{k}_{-it}\}_0^{\infty}, \{\mathbf{s}_{-it}\}_0^{\infty}$, where $\rho, 0 < \rho < 1$, denotes the discount rate. This optimization problem may be modified to allow for depreciation of human capital and of links.

4.1 Joint Evolution of Human Capital and Social Connections

The first-order condition for k_{it} ignores, in the sense of Nash equilibrium, the effect that agent i's setting of k_{it} has on the spillovers to all agents in period t, taking them as given. That is, given $(\mathbf{k}_0, \mathbf{s}_0)$, we have:

$$k_{it} = \frac{1}{c} b_{\tau(i)} + \frac{a}{c} \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}_{t-1}) k_{j,t-1}. \tag{43}$$

Similarly for s_{it}, we have:

$$s_{it} = a\rho \sum_{j=1, j\neq i}^{I} k_{it+1} k_{jt} \frac{\partial g_{ij}(\mathbf{s}_t)}{\partial s_{it}}, \tag{44}$$

with $g_{ij}(\mathbf{s})$ defined by (2). The steady state values of the system (43–44) (k_i^*, s_i^*) coincide with those of the static case (10–11), provided that one adjusts for the fact that to a in (11) there corresponds $a\rho$ in (44).

It is thus straightforward to study the dynamics near a steady state. By linearizing in the standard fashion and by denoting by $\Delta x_{it} = x_{it} - x_i^*$ deviations from steady-state values, we have:

$$\Delta k_{it} = \frac{a}{c} \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}^*) \Delta k_{j,t-1} + \frac{a}{c} \sum_{j=1, j\neq i}^{I} k_j^* \sum_{h=1}^{I} \frac{\partial g_{ij}}{\partial s_h}|_{\mathbf{s}^*} \Delta s_{h,t-1}. \tag{45}$$

$$\Delta s_{it} = a\rho k_i^* \sum_{j=1, j\neq i}^{I} k_j^* \sum_{h=1}^{I} \frac{\partial^2 g_{ij}}{\partial s_i \partial s_h}|_{\mathbf{s}^*} \Delta s_{ht} + a\rho k_i^* \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_i}|_{\mathbf{s}^*} \Delta k_{jt} + \left(a\rho \sum_{j=1, j\neq i}^{I} k_j^* \frac{\partial g_{ij}}{\partial s_i}|_{\mathbf{s}^*} \right) \Delta k_{it+1}, \tag{46}$$

where except for the time-subscripted variables, all others assume their steady-state values.

The asymptotic results invoked earlier allow us to simplify these conditions.[13]

System (45–46) may now be written as follows, where we advance the time subscript for t in the first equation:

$$\Delta \mathbf{k}_{t+1} = \frac{a}{c}\mathbf{G}(\mathbf{s}^*)\Delta \mathbf{k}_t, \tag{47}$$

$$\Delta \mathbf{s}_t = \tilde{a}\vartheta \Delta \mathbf{k}_{t+1}. \tag{48}$$

By using (47) in (48) we see that the changes in networking efforts, $\Delta \mathbf{s}_t$, are determined by the contemporaneous values of the changes in human capitals, the $\Delta \mathbf{k}_t$'s. That is:

$$\Delta \mathbf{s}_t = \frac{a\tilde{a}(\mathbf{b})\rho\vartheta}{c}\mathbf{G}(\mathbf{s}^*)\Delta \mathbf{k}_t. \tag{49}$$

The dynamic evolution of the human capitals is determined by (47), and therefore of the networking efforts as well through (49). The properties of the matrix $\frac{a}{c}\mathbf{G}(\mathbf{s}^*)$ fully determines the dynamics, and its properties are in turn determined by those of the steady state solutions. We know from Cabrales *et al.* (2011) that the largest eigenvalue of $\mathbf{G}(\mathbf{s}^*)$ is equal to $\frac{\overline{x^2}(\mathbf{s}^*)}{\overline{x}(\mathbf{s}^*)}$ and corresponds to $\mathbf{s}$ as an eigenvector. Therefore, the condition

$$\frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})} = \frac{1}{c}\varpi\vartheta\tilde{a} < 1$$

is sufficient for the stability of the solution of (47). In view of (15), this condition becomes:

$$\varpi^2 < c,$$

which is satisfied, in view of (16) for *both* non-zero solutions of (15–16). Further below, I show that the basic dynamic model here also underlies models which allow for individuals to make intergenerational transfers to their children. I note that stability of the human capital process implies that of the networking efforts as well.

4.2 Evolution of Human Capital with Exogenous Social Connections

It also of interest to examine the evolution of human capital, given social connections. Assuming that $\{\mathbf{s}_t\}_{t=1}^{\infty}$ is exogenous and taking $\{\mathbf{k}_{-i,t}\}_{t=1}^{\infty}$, as given, individual i chooses $\{\mathbf{k}_{i,t}\}_{t=1}^{\infty}$, so as to maximize lifetime utility according to (42). Under the assumptions of Nash equilibrium, human capitals satisfy the sequence of difference equations (43), now rewritten in matrix form as:

$$\mathbf{k}_t = \frac{1}{c}\mathbf{b}_t + \frac{a}{c}\mathbf{G}_{-diag}(\mathbf{s}_{t-1})\mathbf{k}_{t-1}. \tag{50}$$

To see the properties of this process, let us assume that both $\mathbf{s}_t$ and $\mathbf{b}_t$ are constant, $\mathbf{s}, \mathbf{b}$. Then, (50) admits a steady state, given by:

$$\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right] c\mathbf{k} + a\text{diag}\mathbf{G}(\mathbf{s})\mathbf{k} = \mathbf{b}. \tag{51}$$

As we argued above, for a large number of agents, the diagonal elements vanish, and the second term on the lhs of (50) is approximately equal to zero.

The special properties of $\mathbf{G}(\mathbf{s})$ allow deriving conditions under which $\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1}$ exists. Specifically, since $\mathbf{G}(\mathbf{s})$ is symmetric and positive, all of its eigenvalues are real. It has a maximal simple eigenvalue, r, which is positive, and larger from the absolute values of all its other eigenvalues. Then, by Theorem III, Debreu and Herstein (1953), $\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1}$ exists is positive, if and only if

$$\frac{1}{r} > \frac{a}{c}. \tag{52}$$

As Cabrales et al. (2011), show, the maximal root is given by $\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}$ and corresponds to $\mathbf{s}$ as an eigenvector. Furthermore, by Lemma 3, Cabrales et al. (2011), p. 353,

$$\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1} = \mathbf{I} + \frac{a}{c}\frac{1}{1 - \frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\mathbf{G}(\mathbf{s}).$$

Thus, condition (52) that the maximal eigenvalue must satisfy suffices for the positivity of $\overline{x}(\mathbf{s}) - \frac{a}{c}\overline{x^2}(\mathbf{s}^2)$, and thus of the second term of the expression for the inverse above. The steady state value for $\mathbf{k}^*$ becomes:

$$\mathbf{k}^* = \frac{1}{c}\mathbf{b} + \frac{a}{c}\frac{1}{1 - \frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\mathbf{G}(\mathbf{s})\frac{1}{c}\mathbf{b}. \tag{53}$$

For the linear dynamical system (50), the unique steady state is stable, provided its maximal eigenvalue is less than 1, which is equivalent with condition (52).

Human capitals at the steady state, given by (53), consist of two terms of which the second only reflects the effects of social interactions. Inspection of the second term in the rhs of (53) suggests that it consists of a vector whose term i is

$$\frac{a}{c}\frac{s_i}{\sum_i s_i}\frac{1}{c}\frac{1}{1-\frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\,\mathbf{s}\cdot\mathbf{b}.$$

Therefore, when the social connections are not optimized, it is, of course, not surprising that the arbitrary social connections do matter. Nonetheless, the richness of what follows is interesting. The better relatively connected an individuals is, the greater the amplification of her basic human capital that she experiences. Additional richness is made possible by the different specifications explored in sections 2.2 and 2.3 above.

4.2.1 A Stochastic Extension and the Upper Tail of the Distribution of Human Capitals

We take Eq. (50) as given, that is, as an ad hoc rule for the evolution of human capital in relation to social connections. In addition, we allow for stochastic shocks to cognitive as well as non-cognitive skills. We recall the specification of cognitive shocks in section 3 above and assume that the (column) vectors $\Psi_t = (\psi_{1,t}, \ldots, \psi_{I,t})$ are defined to represent the full cognitive effect, where $\psi_{1,t} = \frac{1}{c} b_{i,t}$, with $\mathbf{b}_t$ being fully stochastic, and to be independently and identically distributed over time. The sequence of $\{\Psi_0, \ldots, \Psi_t\}$ is assumed to be a stationary stochastic process. In addition, we assume that social connections are exogenous but random. That is, the social networking efforts are denoted by $\Phi_t = (\phi_{1,t}, \ldots, \phi_{I,t})$, so that instead of (50) we now have:

$$\tilde{\mathbf{k}}_t = \Psi_t + \tilde{\mathbf{G}}(\Phi_t)\tilde{\mathbf{k}}_{t-1}, t = 1, \ldots, \tag{54}$$

with a given $\tilde{\mathbf{k}}_0$.

For the purpose of analytical convenience and without loss of generality, we assume that the social interactions matrix $\tilde{\mathbf{G}}_t = \tilde{\mathbf{G}}(\Phi_t)$ is defined to include the diagonal terms too. We assume that the pairs $\left\{\tilde{\mathbf{G}}_t, \Psi_t\right\}$ are independently and identically distributed elements of a stationary stochastic process have positive entries. We assume the additional and rather mild conditions of Theorems A and B, Kesten (1973; 1974). Adopting as matrix norm $||\cdot||$ for $I \times I$ matrices the function $||m|| = \max_{|y|=1} |ym|$, where y denotes an I row vector, and

m denotes an $I \times I$ matrix.[14] If

$$\mathcal{E} \ln^+ ||\tilde{\mathbf{G}}(\Phi_1)|| < 0,$$

then

$$\mathrm{Lim}_1 = \lim \left(\ln ||\tilde{\mathbf{G}}(\Phi_1) \cdots \tilde{\mathbf{G}}(\Phi_t)||^{\frac{1}{t}} \right) \tag{55}$$

exists, is constant and finite w.p. 1. If we assume that the $\tilde{\mathbf{G}}$'s are such that $\mathrm{Lim}_1 < 0$, then $||\tilde{\mathbf{G}}(\Phi_1) \cdots \tilde{\mathbf{G}}(\Phi_t)||$ converges to 0 exponentially fast. If $|\Psi_1|^\kappa < \infty$ for some $\kappa > 0$, with the norm $| \cdot |$ being defined as the Euclidian norm, then the series

$$\mathbf{K} \equiv \sum_{t=1}^{\infty} \tilde{\mathbf{G}}(\Phi_1) \cdots \tilde{\mathbf{G}}(\Phi_{t-1}) \Psi_t$$

converges w. p. 1, and the distribution of the solution $\tilde{\mathbf{k}}_t$ of (54) converges to that of $\mathbf{K}$, independently of $\tilde{\mathbf{k}}_0$.

In particular, from (55), if $\mathrm{Lim}_1 < 0$, then the norm of the product of t successive social interactions matrices, raised to the power of t^{-1}, is positive but less than 1. In that case, Kesten (1973) shows that the distribution of $\mathbf{K}$ can have a thick upper tail. That is, according to Kesten (1973), Theorem A, if in addition to the above conditions there exists a $\kappa_0 > 0$, for which

$$\mathcal{E} \left\{ \frac{1}{I^{\frac{1}{2}}} \min_i \left(\sum_{j=1}^{I} \tilde{\mathbf{G}}(1)_{i,j} \right) \right\}^{\kappa_0} \geq 1, \text{and } \mathcal{E} \left\{ ||\tilde{\mathbf{G}}(1)||^{\kappa_0} \ln^+ ||\tilde{\mathbf{G}}(1)|| \right\} < \infty, \tag{56}$$

then there exists a $\kappa_1 \in (0, \kappa_0]$ such that

$$\lim_{v \to \infty} \mathrm{Prob} \left\{ \max_{n \geq 0} |x \tilde{\mathbf{G}}(1) \cdots \tilde{\mathbf{G}}(n)| > v \right\} \sim X(x) v^{-\kappa_1}, \tag{57}$$

where $0 \leq X(x) < \infty$, with $X(x) > 0$, where the (row) vector x belongs to the positive orthant of the unit sphere of $I\!R^I$, exists and is strictly positive. If, in addition, the components of $\Psi(1)$ satisfy:

$$\mathrm{Prob} \left\{ \Psi(1) = 0 \right\} < 1, \ \mathrm{Prob} \left\{ \Psi(1) \geq 0 \right\} = 1, \ \mathcal{E} |\Psi(1)|^{\kappa_1} < \infty,$$

then for all elements x on the unit sphere in $I\!R^I$,

$$\lim_{v \to \infty} v^{\kappa_1} \mathrm{Prob} \left\{ x\mathbf{K} \geq v \right\} \tag{58}$$

exists, is finite and for all elements x on the positive orthant of the unit sphere in $I\!R^I$ is strictly positive.

This result establishes a Pareto (power) law for the upper tail, summarized by (57) and (58). Its intuition is straightforward.[15] Given a non-trivial initial value for the cognitive shocks, $\Psi(1)$, and an arbitrary initial value for human capitals, $\tilde{\mathbf{k}}_0$, the dynamic evolution of human capital according to (54) keeps positive the realizations of human capital, with the impact of spillovers also having an overall contracting effect that pushes the realizations and thus the distributions of human capital, too, towards 0. The distribution is prevented from collapsing at 0 by the properties of the contemporaneous cognitive shocks, Ψ_t, and from drifting to infinity by the contracting effect of the spillovers. The contracting effect results from the combination of two key requirements: First, condition (56), which requires that the minimum row sum of the social interactions matrix grows with I faster than $\sqrt{I}$; and second, the geometric mean of the norms of the social interactions matrix does not exceed 1. Thus, the upper tail is thickened by the combined effect of the contracting spillovers and tends to a power law, $\sim v^{\kappa_1}$, with a coefficient which is constant. This result is sufficient for the distribution of human capital in the entire economy to also have a Pareto upper tail. Let f_{k_i} denote the limit distribution of $k_i,\ i = 1, \ldots, I$. Then, the economy-wide distribution of human capital is given by $\sum_i \#\{i\} f_{k_i}(k)$, where $\#\{i\}$ denotes the relative proportion of types i agents.

4.2.2 A Planner's Problem

It is straightforward to develop a full model for an economy composed of I individuals, as we have been working with, where the social structure is given. However, it is also interesting to pose a planner's problem, where the planner takes as given that individuals set their human capitals according to (50), $t = 0, \ldots, \infty$, and seeks to maximize a utilitarian objective function according to the same objective, where the planner's maximization is defined in terms of the social networking efforts, $\{\mathbf{s}_t\}$:

$$\max_{\{\mathbf{s}_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \sum_{i=1}^{I} \rho^t U_{i,t}(\mathbf{s}_{-i,t}, s_{i,t}; \mathbf{k}_{-i,t}, k_{i,t}).$$

Since the planner's weights for individuals' utilities are arbitrary, they may be set equal to 1. The point of such an exercise involving the planner's setting of social connections, a kind of "social engineering," would be to identify the inefficiencies associated with individuals' setting of social interactions weights. The advantage of this specification is that it lends itself to comparisons with the results of the model with individualized decision making.

There is no consensus in the literature on the specification of the social interactions component. Therefore, a more general aim would be to consider how one could bring about a desired redistributional objective in a world where whom you know matters, defined in terms of a given planner's objective that is directly stated as a function of individuals' human capitals. The planner's problem could be stated in terms of determining for the functional specification of the social interactions component $H(\cdot)$,

$$H\left(\mathbf{s}_{t-1}; k_{i,t}, \mathbf{k}_{-i,t-1}\right),$$

that would be bringing about the planner's objective.

In view of the examples explored earlier, the function $H(\cdot)$ could be assumed to be homogeneous, but of different degrees, depending upon which specific intuition we would like to be expressed by the model. In such a case, the dynamical system (50) would be adapted as follows:

$$\mathbf{k}_{i,t} = \frac{1}{c}\mathbf{b}_{i,t} + \frac{a}{c}\frac{\partial H}{\partial k_{i,t}}\left(\mathbf{s}_{t-1}; k_{i,t}, \mathbf{k}_{-i,t-1}\right). \tag{59}$$

Steady states must satisfy the above equation, when $\mathbf{b}_t$ and the social network are time invariant. Stability properties of this dynamical system depend on the steady states associated with (59). We note that not only the full vector $\mathbf{b}$, but also the parameters a, c and the values of the social network $\mathbf{s}$ enter the steady state. Alternatively, the stability properties of this system may be sought, given the time evolution of $\mathbf{b}_t, \mathbf{s}_t$. This specification is sufficiently general to allow derivation of conditions under which the different possibilities explored by Bramoullé, Kranton and D'Amours (2014) may be obtained as special cases.

Ioannides and Soetevent (2007) study individually optimized continuous outcomes in a dynamic environment in the presence of social interactions, and where the interaction topology is exogenous and time varying. The model accommodates more general social effects than those allowed for here as well as for Gaussian stochastic shocks to the cognitive vector, the $\mathbf{b}_t$'s. Since the social interactions structure is given, the approach may rely on the well established tools of dynamic programming when the utility per period is quadratic in the individual decisions. When a conformist global effect is present, that is, when each individual suffers disutility from the gap between her own human capital and the lagged average human capital in the economy, the system involves expectations of individuals' future actions, which complicated considerably the analytics, because it involves a system of second-order difference equations with expectations. Nonetheless, the solution to such a system of equations is well understood and its properties involve conditions on the parameters of the problem. See Ioannides and Soetevent (2007), Proposition 4.[16] That approach is very much along the linear of dynamic programming problems with linear quadratic objectives and additive shocks. Alternative specifications of the planner's problem could be aimed at exploring the consequences of non-cognitive shocks. These are left for future versions of the paper.

5 Consequences for Inequality: a first pass

Sticking to an interpretation of actions as human capital investments, the variation across individual types, as expressed in the b_i's, can then be seen as a primitive determinant of the distribution of human capital across a population, that is about "what you know." Here, we see that individual human capitals are proportional to the individuals' productivities, with the factors of proportionality being functions of the distribution of the b_i's across individuals of varying complexity, as implied by different assumptions about the interaction weights. Nonetheless, individuals' utilities do depend on the distribution of the b_i's across individuals in more complicated ways. In the simplest formulation, they depend on the first and second moments of the distribution of the b_i's across types only. Individualizing the connection weights by including functions of the b_i's lead to more complicated moments of the b_i's. Fully individualizing the interaction weights, as by (24, 20, 19), or allowing for homogeneity of degree less than, or greater than, one do not change the basic conclusion, namely that the outcomes are proportional to b's, albeit with different interaction weights.

Specifically, in view of the optimal solution above for either the static or the dynamic case, we may compute the corresponding value of the individuals' objectives. That is, by using (15) and (16), the value becomes:

$$U_{i,\tau(i)}(\mathbf{s}^*, \mathbf{k}^*) = \frac{1}{2}\vartheta b^2_{\tau(i)}. \tag{60}$$

In the case of autarky, $U_{i,\tau(i),\text{aut}} = \frac{1}{c} b^2_{\tau(i)}$. Since from (16), if ϑ exists, which is ensured by the condition that (15)–(16) have at least one solution, then

$$\vartheta > \frac{1}{c}.$$

While networking dominates autarky, it is interesting that the optimum value of the quantity ϑ summarizes the impact of social networking on an individual's welfare. The larger is ϑ, the greater the contribution of social networking to individual welfare. If (15)–(16) have two solutions, then the smaller of the two, $\vartheta_{\min}$, is stable and the larger is unstable. Then, the greater is $\tilde{a}$, the greater is $\vartheta_{\min}$. Holding a constant, this occurs if $\frac{\overline{x^2}(\mathbf{b})}{\overline{x}(\mathbf{b})}$ is greater. But the feasibility condition (18) provides an upward bound on $\tilde{a}$.

Another attractive feature of this formulation is that the solution summarizes an individual's social competence. This is helpful when we examine further below the model in the presence of intergenerational transfers. The question then is how the option to let weights of influence be determined endogenously affect outcomes about human capital at the steady state, that is, "how whom you know" affects outcomes. In the models examined above, at the steady state, all outcomes are proportional to the respective b's, when social connections are optimized. So, the variation of optimal actions and optimum utility across individuals separates naturally into the impact of networking opportunities and of skill, being proportional, to the b_i, respectively b_i^2, with ϑ, the factor of proportionality, reflecting the effect of the entire distribution of the b_i's. Thus, in a model where proxies for ability are inherited [Ioannides (1986); Durlauf (2013)], this feature may be relied on, in an overlapping- or infinite-horizon model, to express inheritability. The question then becomes to what extent "what you inherit" influences "whom you know."

5.1 Unstable Social Structures

When social connections are exogenous, a great number of possibilities arises. The development in section 4.2 shows that the stability of the dynamic evolution of human capital depends on the properties of the social network, relative to the parameters of the utility function. Thus, when the social network does not satisfy conditions for stability, that is when

$$\frac{\overline{x}(\mathbf{s})}{\overline{x^2}(\mathbf{s})} < \frac{a}{c}, \tag{61}$$

and depending on initial conditions, one may think of whether it might be possible to have sets of socially networked individuals whose human capitals converge over time, and while for others they diverge. Given any given set of social networking efforts, it is straightforward to obtain conditions under which such groupings of individuals are feasible. Specifically, it is straightforward to show that given that there is a grouping of $h-1$ individuals for whom

$$\frac{\overline{x}_{h-1}(\mathbf{s})}{\overline{x^2}_{h-1}(\mathbf{s})} < \frac{a}{c}, \tag{62}$$

then the lhs above increases, that is,

$$\frac{\overline{x}_{h-1}(\mathbf{s})}{\overline{x^2}_{h-1}(\mathbf{s})} < \frac{\overline{x}_{h}(\mathbf{s})}{\overline{x^2}_{h}(\mathbf{s})},$$

if individual h is added for whom we have that:

$$s_i > \frac{\overline{x^2}_{h-1}(\mathbf{s})}{\overline{x}_{h-1}(\mathbf{s})}.$$

That is, a prospective new member of the group must have sufficiently high networking effort in order to improve social networking for the entire group she stands to join. Thus, by successive addition of such individuals the inequality sign in the infeasibility condition (62) would be reversed and the condition for stability established. Recall that the spirit of the model is that there exist may different individuals of each type. Therefore, this ought to be understood as how different types of individuals with given social networking efforts may self-organize into different social networks.

Applying these models to dynamic settings, where one may compare between given weights, perhaps representing a given social structure, and optimized weights, one may thus distinguish between given relationships, like familial ones, versus social networking across familial relationships.

5.2 Moving in Search of Desirable Interactions

Suppose that individuals may move over time in search of desirable interactions. Such interactions differ across communities, $\ell = 1, \ldots, L$, and are denoted by the respective social interactions matrix, $\mathbf{G}_\ell$ [Neal (2013)]. We may assume either that social networking efforts are given, in which case the value from joining a community may be expressed as the optimum value of

$$U_i(k_i, \mathbf{k}_{-i}; \ell) \equiv (b_i - p_{i,\ell})k_i + a \sum_{j=1, j\neq i}^{I} g_{\ell,ij} k_i k_j - c\frac{1}{2}k_i^2.$$

Individual i incurs a cost of living in community ℓ, $p_{i,\ell}$. We assume that this is set per unit of human capital, so that the cost modifies b_i, $b_i - p_{i,\ell}$.

Think of k_i as schooling,

Assume $\mathbf{G}_\ell$, given, exogenous: $U_i(k_i, \mathbf{k}_{-i}; \ell) \equiv (b_i - p_{i,\ell})k_i + a \sum_{j=1, j\neq i}^{I} g_{\ell,ij} k_i k_j - c\frac{1}{2}k_i^2$,

Individual optimum may be written as q quadratic function of $b_i - p_{i,\ell}$ and $\mathbf{G}_{\ell,i}$ (ith row of $\mathbf{G}_\ell$. A ranking of communities follows accordingly. The consequences of this formulation can be explored systematically and would likely rely on spectral properties of $\mathbf{G}_\ell$. They will be pursed further in future work.

6 Introducing Intergenerational Transfers

I consider next dynamic versions of the model that allows for intergenerational transfers. I consider first transfers of wealth, whereby individuals start their lives with a given level of wealth in the form of human capital, denoted by $k_{i,t}$, which they receive from their parents. They give birth to a child, to whom they transfer wealth equal to $k_{i,t+1}$. We let the utility function, given in (1), $U_{i,t}(\mathbf{s}_t, \mathbf{k_t})$, denote the period t payoff for individual i, let dynastic utility be identified with the value function associated with the dynamic process for each dynasty be denoted by $\mathcal{U}_{i,t}(k_{i,t})$. That is, dynastic utility is defined in the standard fashion for dynamic programming problems:

$$\mathcal{U}_{i,t}(k_{i,t}) = \max_{s_{i,t},k_{i,t+1}} : \left\{U_{i,t}(\mathbf{s}_t, \mathbf{k_t}) + \rho \mathcal{U}_{i,t}(k_{i,t+1})\right\}, \tag{63}$$

where utility per period, $U_{i,t}(\mathbf{s}_t, \mathbf{k_{i,t}})$, is given by (1):

$$U_{i,\tau(i)}(\mathbf{s}, \mathbf{k}) \equiv b_{\tau(i)} k_{i,t} + a \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}_t) k_{i,t} k_{j,t} - c\frac{1}{2}k_{i,t}^2 - \frac{1}{2}s_{i,t}^2 - k_{i,t+1}.$$

In this formulation each parent at t decides on a transfer to her child, $k_{i,t+1}$, and on the networking effort, $s_{i,t}$, that she avails herself from, given the transfer which she herself received from her own parent, $k_{i,t}$, so as to maximize her lifetime utility. Note that whereas the parent incurs the resource cost, $k_{i,t+1}$, of the transfer to the child, the child incurs the adjustment cost, that is the quantity $\frac{1}{2}k_{i,t}^2$ for the individuals who are the parents at t. Dynastic utility is defined as the sum of her own period t utility plus the discounted sum of the maximum utilities of her descendants. It is perfectly feasible to develop this model, but I note that by making the transfer to the child and her own networking efforts as simultaneous decisions, the child does not benefit from the parent's networking. In such a model, there is no human capital accumulation, since each individual lives for one period, nor growth (although exogenous growth to the productivity of human capital could be introduced). This otherwise standard model exhibits the property of the life cycle theory, in its being isomorphic to a model of a single decision maker who maximizes an infinite sum of utilities with respect to a sequence of decisions, $\{k_{i,t+1}, s_{i,t}\}_{t=0}^{\infty}$. I do not pursue this model further, but the details may be found in Appendix A.

6.1 Overlapping Generations Models of Social Networks and Intergenerational Transfers

A richer model and analytically more tractable one may be obtained if we assume that individuals have finite lifetimes and enter the economy in overlapping generations. I start with two overlapping generations, but do note however that a minimum of three overlapping generations will be necessary to express Heckman's concern about allowing for at least two periods of investment in a child's cognitive and non-cognitive skills. That is, it is critical [see Cunha and Heckman (2007)] for the acquisition of cognitive and non-cognitive skills to interact — there is dynamic complementarity among them — and investment in certain ages are more critical and then in other ages. Moreover, these come earlier for cognitive capabilities, later for non-cognitive capabilities, and vary depending on the particular biological capability. Three-overlapping generations is the minimum number that allows for direct effects between grandparents and grandchildren. Heckman and Mosso (2014) emphasize, however, there have to be at least four periods in individuals' lifetimes, with two periods for a passive child who makes no economic decisions but who benefits from parental investment in the form of goods, and two periods as a parent. This requires, of course, going beyond the standard two-overlapping generations models used by many life cycle models.

As Durlauf (2013) stresses, "the new economics of skills [see Heckman *et al.*] has two critical features. First, it employs a broad definition of skills. In particular, it differentiates between cognitive and non-cognitive skills. In this respect, the economics of skills has followed the psychology literature, in which intelligence and personality studied as distinct aspects of the mind. Many psychologists dislike the term 'non-cognitive' skills since these skills are also part of the mind, and so in their view are cognitive." Still, here I, too, follow the language of economists. Second, the literature focuses on development across the childhood and adolescence. This is obviously hard to represent within the popular two-generations paradigm widely used by economists.

The fact that parents are assumed to coexist with their children naturally allows me to model that children may avail themselves of the social connections of their parents. Such a natural "transfer in kind" can coexist with a wealth transfer. Both types of transfers are central features of the models that follow.

6.1.1 A Two-Overlapping Generations Model of Intergenerational Transfers

Let subscripts y, o refer to individuals when they are *y*oung, *o*ld, respectively, and let time subscripts refer to when the respective magnitude is operative. A member of the generation born at t receives a transfer $k_{y,i,t}$ from her parent when young; she herself takes advantage at time t of social connections made by her parent's generation: $\mathbf{s}_{y,t-1}$. Her cognitive skills are given: $b_{y,i,t}, b_{o,i,t+1}$. She chooses human capital investment and networking effort $(k_{o,i,t+1}, s_{y,i,t})$; she benefits in period $t+1$ from $k_{o,i,t+1}$; her generation benefits from $\mathbf{s}_{y,t}$ in time $t+1$. She chooses an endowment to her child in the form of human capital, $k_{y,i,t+1}$, and networking effort, $s_{o,i,t+1}$, from which her child benefits in the first period of her own life at time $t+1$.

We assume that the resource cost of investment $k_{o,i,t+1}$ is incurred in period t, but the adjustment costs is incurred in $t+1$ (when the benefits are also realized); consistently, $k_{y,i,t+1}$ is incurred in period $t+1$, but the parent anticipates that the adjustment costs are incurred by the child in $t+1$. This model leads to a generalization of the system of dynamic equations as the one examined earlier. It coincides with that system in the special case of cognitive skills which are equal across young and old and invariant over time: $b_i = b_{y,i,t} = b_{o,i,t+1}$.

It is important to clarify the relevant peer groups. With two overlapping generations, we may define the peer groups for young generation t at time t as the members of generation who were born at $t-1$ when they are old at time t. That is, the members of generation t benefit in period t from the human capitals $\mathbf{k}_{o,t}$ and the social networking of their parents' generation, $\mathbf{s}_{o,t}$. When they are old in period $t+1$ they benefit by the human capitals and social contacts the members of their own generation themselves decided on, $\mathbf{k}_{y,t}, \mathbf{s}_{y,t}$. In other words, in their first-period decisions about social connections, they are conscious of the fact that they themselves would benefit from their own social connections when they are old; in their second-period decisions about social connections, they are conscious of the fact that their children would benefit from their own second-period social connections when their children are young. Therefore, all second-period decisions are in effect intergenerational transfers of capital and social connections. In the absence of uncertainty, all decisions are of course made simultaneously, but being explicit about "timing" of networking efforts would be crucial with sequential resolution of uncertainty, if such uncertainty were to be introduced.

That is, the decision problem for a member of generation t, born at time t, is to choose

$$\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\},$$

given $\{k_{y,i,t}, \mathbf{s}_{o,t}\}$. We express the first-order conditions by first defining the value functions $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}), \mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})$, associated with an individual's lifetime utility when he is young at t and when he is old at $t+1$, we have:

$$\begin{aligned}
&\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}) \\
&= \max_{\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \Bigg\{ b_{y,i,t}k_{y,i,t} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck_{y,i,t}^2 - \frac{1}{2}s_{y,i,t}^2 - k_{o,i,t+1} \\
&+\rho\left[b_{o,i,t+1}k_{o,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t} - \frac{1}{2}ck_{o,i,t+1}^2 - \frac{1}{2}s_{o,i,t+1}^2 - k_{y,i,t+1}\right] + \rho\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})\Bigg\}.
\end{aligned}$$

Correspondingly,

$$\begin{aligned}
&\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}) \\
&= \max_{\{k_{o,i,t+2}, k_{y,i,t+2}; s_{y,i,t+1,}, s_{o,i,t+2}\}} \Bigg\{ b_{y,i,t+1}k_{y,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t+1})k_{y,i,t+1}k_{o,j,t+1} - \frac{1}{2}ck_{y,i,t+1}^2 - \frac{1}{2}s_{y,i,t+1}^2 - k_{o,i,t+2} \\
&+\rho\left[b_{o,i,t+2}k_{o,i,t+2} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t+1})k_{o,i,t+2}k_{y,j,t+1} - \frac{1}{2}ck_{o,i,t+2}^2 - \frac{1}{2}s_{o,i,t+2}^2 - k_{y,i,t+2}\right] + \rho\mathcal{V}_i^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+2})\Bigg\}
\end{aligned}$$

The first-order conditions with respect to $(k_{o,i,t+1}, s_{y,i,t,}; k_{y,i,t+1}, s_{o,i,t+1})$ are, respectively:

$$k_{o,i,t+1} = \frac{1}{c}b_{o,i,t+1} + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{y,j,t} - \frac{1}{c\rho}; \tag{64}$$

$$s_{y,i,t} = \rho a k_{o,i,t+1}\sum_{j=1,j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{y,i,t}}(\mathbf{s}_{y,t})k_{y,j,t}; \tag{65}$$

$$-\rho + \rho\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial k_{y,i,t+1}}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}) = 0;$$

$$-\rho s_{o,i,t+1} + \rho\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial s_{o,i,t+1}}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}) = 0.$$

Using the envelope property, the partial derivatives of the value function above,

$$\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial k_{y,i,t+1}}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}), \frac{\partial\mathcal{V}^{[t+1]}}{\partial s_{o,i,t+1}}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})$$

are equal to the partial derivatives of the respective utility per period. That is, using the envelope property, the last two equations become:

$$k_{y,i,t+1} = \frac{1}{c}b_{y,i,t+1} + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+1} - \frac{1}{c\rho}; \tag{66}$$

$$s_{o,i,t+1} = \rho a k_{y,i,t+1} \sum_{j=1,j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1}) k_{o,j,t+1}; \tag{67}$$

We can summarize the first-order conditions for the $\mathbf{k}$'s in matrix form as follows.

$$\mathbf{k}_{o,t+1} = \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{y,t} - \frac{1}{c\rho}\mathbf{1}; \tag{68}$$

$$\mathbf{k}_{y,t+1} = \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{o,t+1} - \frac{1}{c\rho}\mathbf{1}, \tag{69}$$

where $\mathbf{1}$ is a vector of 1s. From these we may obtain two single first-order difference equations: first in $\mathbf{k}_{y,t}$, by substituting for $\mathbf{k}_{o,t+1}$ from (68) in the rhs of (69), and then in $\mathbf{k}_{y,t}$, by substituting for $\mathbf{k}_{y,t}$ from (69) in the rhs of (68). That is:

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{y,t} + \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\right]\mathbf{1}. \tag{70}$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{o,t})\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{o,t} + \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\right]\mathbf{1}. \tag{71}$$

These are two well-defined uncoupled first-order linear systems for $(\mathbf{k}_{o,t}, \mathbf{k}_{o,t})$, given $(\mathbf{s}_{y,t}, \mathbf{s}_{o,t}, \mathbf{s}_{o,t+1})$. Their steady state solutions are easily characterized, in terms of the inverse of $\mathbf{I} - \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$. Since the largest eigenvalue of $\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ is bounded upwards by the product of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and $\mathbf{G}(\mathbf{s}_y)$ [Debreu and Herrstein (1953); Merikoski and Kumar (2006), Thm. 7, 154–155], the inverse exists, provided that the product of $\frac{a^2}{c^2}$ with the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and of $\mathbf{G}(\mathbf{s}_y)$ is less than 1. A sufficient condition for this is that the products of $\frac{a}{c}$ and each of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o), \mathbf{G}(\mathbf{s}_y)$ are less than 1. The characterization of the steady state solution in more detail below allows us to examine these sufficient conditions further.

Again, assuming that social networking efforts are given and using tools from the theory of stochastic linear systems in the style of Ioannides and Soetevent (2007), we can characterize the joint evolution of human capitals.

In the case of three-overlapping generations, that is when children coexist with their parents and their grandparents, we will have an additional set of equations for the respective magnitudes associated with $y$outh, $a$dulthood and $o$ld-age,

$$(k_{y,i,t}, k_{a,i,t+1}, k_{o,i,t+2}; s_{y,i,t}, s_{a,i,t+1}, s_{o,i,t+2}).$$

An individual born at t, will take as given $(k_{y,i,t}, s_{y,i,t})$ and choose

$$(k_{a,i,t+1}, k_{o,i,t+2}, k_{y,i,t+3}; s_{a,i,t+1}, s_{o,i,t+2}, s_{y,i,t+3}).$$

Intuitively, one would expect that the additional first-order conditions would introduce additional multiplicative terms to the matrix defining the dynamical system and additional terms multiplying the respective cognitive skills vectors. That is, the endowment of cognitive skills in each period of the life cycle introduce life cycle effects into the model, being weighted by the respective social interactions matrix, as in $\frac{1}{c}\frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t}$ in Eq. (71) above. Given the pattern of recurrence, we can guess what the counterpart of (71) should look like. Since the respective endowments are not equal across time, steady state values for human capitals differ at different stages of the life cycle.

It is known from research on models with more than two overlapping generations [see Azariadis et al. (2004) and references there in] that more than two overlapping generations models usher in considerably more complicated properties in general equilibrium contexts.[17] It is therefore an interesting result that complicating the demographic structure of the model leaves tractable the structure that determines the dynamics of the model. Working through the derivations formally in order to derive the counterpart of (71) confirms, in fact, this intuition.

6.1.2 Social Effects in Intergenerational Wealth Transfer Elasticities

Interpreting human capital $k_{y,i,t}$ as initial wealth for a member of the generation born at t allows us to compute intergenerational wealth elasticities. We work from (70) and define the elasticity of $k_{y,i,t+1}$ with respect to $k_{y,i,t}$ and account only for direct effects,

$$\mathrm{EL}_{k_{y,i,t}}^{k_{y,i,t+1}} = \frac{dk_{y,i,t+1}}{dk_{y,i,t}} \frac{k_{y,i,t}}{k_{y,i,t+1}},$$

that is effects on i's decisions as opposed to the impact of i's decisions on decisions of other agents, which feed back to agent i's decisions. It is easiest to see the effect under the assumption that social networking is given. Then, from (68) and (69) we have a direct effect,

$$\frac{\partial k_{y,t+1}}{\partial k_{y,i,t}} = \frac{a^2}{c^2} \left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{\mathrm{col}\ i}.$$

This effect is simply the increase in the transfer to the child, $k_{y,i,t+1}$, from an increase in first period wealth received by a member of the tth generation. This is determined from trading off the resource cost of the transfer, which is incurred by the parent in period $t+1$, with the utility increase to him from the benefit to the child when the transfer is received in period $t+1$. This is why both adjacency matrices, $\mathbf{G}(\mathbf{s}_{y,t})$ and $\mathbf{G}(\mathbf{s}_{o,t+1})$, are involved in the expression for $\frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}}$.

However, because the transfer to the child, $k_{y,i,t+1}$, and the parent's social networking effort when old, $s_{o,i,t+1}$, are jointly determined, the full benefit to the child also reflects how the change in the parent's social networking effort influences the human capital spillovers, which are associated with the parents' human capitals in period $t+1$, the second period of their lives. We see from (71) that $k_{o,i,t+1}$ is determined, given $(k_{o,i,t}, s_{y,i,t}, s_{o,i,t})$. Thus, in using the interdependence of $(k_{y,i,t+1}, s_{o,i,t+1})$, as in (67), to express the effect of $k_{y,i,t}$ on $k_{y,i,t+1}$ via $s_{o,i,t+1}$, we have:

$$\frac{\partial s_{o,i,t+1}}{\partial k_{y,i,t+1}} = \rho a \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1}) k_{o,j,t+1}, \tag{72}$$

given $\mathbf{k}_{o,t+1}, \mathbf{s}_{o,-i,t+1}$. Therefore, an effect is generated on $k_{y,i,t+1}$ due to its dependence on $s_{o,i,t+1}$, which is obtained by partially differentiating the rhs of (70) with respect to $s_{o,i,t+1}$.

In sum, for the total effect of an increase in first period wealth on the transfer to the child, we have from (70) and (67):

$$\frac{d\, k_{y,i,t+1}}{d\, k_{y,i,t}} = \frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}} \left[1 + \rho a \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1}) k_{o,j,t+1} \frac{\partial s_{o,i,t+1}}{\partial k_{y,i,t+1}} \right],$$

where the partial derivative of $\mathbf{k}_{y,t+1}$ with respect to $s_{o,i,t+1}$ is given by:

$$\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{y,t}) \frac{\partial}{\partial s_{o,i,t+1}} \mathbf{G}(\mathbf{s}_{o,t+1}) \mathbf{k}_{y,t} + \frac{\partial}{\partial s_{o,i,t+1}} \mathbf{G}(\mathbf{s}_{o,t+1}) \left[\frac{a}{c^2}\mathbf{b}_{o,t+1} - \frac{a}{\rho c^2}\mathbf{1} \right],$$

with

$$\frac{\partial}{\partial s_{o,i,t+1}} \mathbf{G}(\mathbf{s}_{o,t+1}) = \begin{bmatrix} 0 & 0 & \dots & \frac{s_{o,1,t+1}}{\sum_{j\neq 1} s_{o,1,t+1}} & \dots & 0 \\ \frac{s_{o,1,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} & \frac{s_{o,2,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} & \dots & 0 & \dots & \frac{s_{o,I,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} \\ 0 & 0 & \dots & \frac{s_{o,I,t+1}}{\sum_{j\neq I} s_{o,j,t+1}} & \dots & 0 \end{bmatrix},$$

where we have utilized the large number of agents approximation for the derivatives of the entries of $\mathbf{G}(\mathbf{s}_{o,t+1})$.

This analysis comes in handy when we examine the impact of differences in the parent's or in the child's own cognitive skills on the transfer to the child. From (69) applied for time t we have that an individual with higher first-period cognitive skills $b_{y,i,t}$ receives a larger transfers from his parent, $\frac{k_{y,i,t}}{b_{y,i,t}} = \frac{1}{c}$. This in turns induces a change in his own transfer to his child, along the lines of the effects we just computed. Working in like manner we have that an increase in the parent's own second period cognitive skills $b_{o,i,t+1}$ leads from (70) to $\frac{\mathbf{k}_{y,t+1}}{b_{o,i,t+1}} = \frac{a}{c^2} G(\mathbf{s}_{o,t+1})_{coli}$, which leads in turn from (**??**)) to a change in $s_{o,1,t+1}$, exactly as we analyzed earlier.

Therefore, we have clarified how social effects affect the elasticities of intergenerational wealth transfers. Naturally, they are present when social networking is endogenous, but also when they are exogenous. In the latter case, the intergenerational wealth elasticity becomes:

$$\mathrm{EL}_{k_{y,i,t}}^{k_{y,i,t+1}} = \frac{a^2}{c^2} \left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{ii}$$

$$\times \frac{k_{y,i,t}}{\frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{\text{row } i}\mathbf{k}_{y,t} + \frac{1}{c} b_{y,i,t+1} + \frac{a}{c^2}\left[\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{\text{row } i}\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[1 + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\text{row }_i \mathbf{1}\right]},$$

6.1.3 Steady States

Although the above single difference equation could be a starting point for the stability analysis, it is analytically more convenient to work with the full system of equations. Let us assume that the $b_{y,i,t}, b_{o,i,t}$ are time-invariant, and let

$$b_{y,i}^* \equiv b_{y,i} - \frac{1}{\rho}, \quad b_{o,i}^* \equiv b_{o,i} - \frac{1}{\rho}.$$

By applying equations (68), (65), (69), and (67) we have:

$$ck_{o,i} = b_{o,i}^* + as_{y,i} \sum_{j \neq i} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}}; \tag{73}$$

$$s_{y,i} = \rho a k_{o,i} \sum_{j=1, j\neq i}^{I} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}}; \tag{74}$$

$$ck_{y,i} = b^*_{y,i} + as_{o,i} \sum_{j \neq i} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}}; \tag{75}$$

$$s_{o,i} = \rho a k_{y,i} \sum_{j=1, j \neq i}^{I} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}}. \tag{76}$$

We define the auxiliary variables, $\psi_y = \sum_{j \neq i} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}}, \psi_o = \sum_{j \neq i} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}}$. Note that they do not depend on i. From (73) and (74), and (76) and (76), we have:

$$\rho k_{o,i}(ck_{o,i} - b^*_{o,i}) = s^2_{y,i} = \rho^2 a^2 \psi^2_y k^2_{o,i};$$

$$\rho k_{y,i}(ck_{y,i} - b^*_{y,i}) = s^2_{o,i} = \rho^2 a^2 \psi^2_o k^2_{y,i}.$$

We may thus solve for $k_{y,i}, k_{o,i}$, and then by using the definitions of ψ_y, ψ_o, for $s_{y,i}, s_{o,i}$, as follows:

$$k_{y,i} = \frac{b^*_{y,i}}{c - \rho a^2 \psi^2_o}, k_{o,i} = \frac{b^*_{o,i}}{c - \rho a^2 \psi^2_y}; \tag{77}$$

$$s_{y,i} = \rho a \psi_y \frac{b^*_{o,i}}{c - \rho a^2 \psi^2_y}, s_{o,i} = \rho a \psi_o \frac{b^*_{y,i}}{c - \rho a^2 \psi^2_o}. \tag{78}$$

Thus, human capitals and networking efforts by young and old, $(k_{y,i}, k_{o,i}; s_{y,i}, s_{o,i})$, are uniquely defined in terms of the auxiliary variables (ψ_y, ψ_o) and parameters. Human capitals $(k_{y,i}, k_{o,i})$ are proportional to their respective cognitive skills, $(b_{y,i}, b_{o,i})$, though with different factors of proportionality. In contrast, networking efforts, $(s_{y,i}, s_{o,i})$, are proportional to the cognitive skills corresponding to the life cycle period when individuals avail of them. That is, when individuals are old, and when their children are young, $(b_{o,i}, b_{y,i})$, again with different factors of proportionality.

Finally, by substituting back into the definitions of ψ_y, ψ_o, we obtain obtain third-degree equations in ψ_y, ψ_o :

$$\psi_y = \frac{1}{c - \rho a^2 \psi^2_o} \frac{\mathbf{b}^*_y \cdot \mathbf{b}^*_o}{I \overline{x}(\mathbf{b}^*_o)}; \tag{79}$$

$$\psi_o = \frac{1}{c - \rho a^2 \psi^2_y} \frac{\mathbf{b}^*_y \cdot \mathbf{b}^*_o}{I \overline{x}(\mathbf{b}^*_y)}, \tag{80}$$

where $\mathbf{b}^*_y \cdot \mathbf{b}^*_o = \sum b^*_{y,i} b^*_{o,i}$.

Equations (79–80) have at most two solutions in (ψ_y, ψ_o), which can be characterized easily but not solved for explicitly. The steady state values of all endogenous variables then follow from (77–78). Note that the life cycle model is crucial for the result. ψ_y and ψ_o would be equal to one another, were it not for the fact that, $b_{y,i} \neq b_{o,i}$, first-period and second-period cognitive skills are in general different from one another. Similarly, interesting complications follow if cognitive skills may be influenced by means of investment, which the section that follows explores.

If we were to assume, as in section 4.2, that the social networks are given exogenously, in this case those of young and of old agents, with values not necessarily coinciding with the steady state ones, then a number of additional results are possible. First, under the assumption that the social networking efforts are constant over time, $(\mathbf{s}_y, \mathbf{s}_o)$, the system of equations (70–71) implies that a single equation for aggregate capital $\mathbf{k}_t = \mathbf{k}_{y,t} + \mathbf{k}_{o,t}$, may be obtained. The dynamics are exactly the same as in each of the two systems and no further discussion is necessary. Second, we may reformulate the evolution of human capitals in stochastic terms, as in the analysis of section 4.2.1 but now in terms of $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$. Similar results regarding stochastic limits in the form of a power law are likely to be obtained.

Such results may be strengthened in the following way. Intuitively, as the number of overlapping generations increases, the matrix for human capitals in the laws of motion (70), (71), becomes the product of increasing number of factors. In the limit, as the number of overlapping generations tends to infinity, the product of stochastic matrices may be handled by techniques similar to those of section 4.2.1, leading to power laws in every period.

6.2 Stochastic Shocks to Cognitive Skills

A natural extension of the model is to allow for the vector of cognitive skills to be stochastic. For simplicity in demonstrating the basic issues that this extension entails, we assume that the distribution of $\mathcal{B}_{i,t} = (b_{y,i,t}, b_{o,i,t+1}; b_{y,i,t+1})$ is multivariate normal, whose means $\mathbf{b}_m = (\mathbf{b}_{m,y}, \mathbf{b}_{m,o}; \mathbf{b}_{m,y+})$, with $(m_{y,i}, m_{o,i}, m_{y+,i})$ as the components of the respective vectors, and variance-covariance matrix $\mathbf{\Sigma}$ depend on i. That is, put concisely, the variance-covariance matrix for $\mathcal{B}_i = (b_{y,i,t}, b_{o,i,t+1}; b_{y,i,t+1})$ looks as follows:

$$\begin{bmatrix} \sigma_b^2 & \rho_o\sigma_b\sigma_o & \rho_b\sigma_b\sigma_{b+} \\ \rho_o\sigma_b\sigma_o & \sigma_o^2 & 0 \\ \rho_o\sigma_b\sigma_{b+} & 0 & \sigma_{b+}^2 \end{bmatrix}.$$

The realizations $\mathcal{B}_i$ are independent across individuals. The covariance $\mathrm{COV}(b_{y,i,t}, b_{y,i,t+1})$ expresses the inheritability of first-period cognitive skills from parents to children; the covariance $\mathrm{COV}(b_{y,i,t}, b_{o,i,t+1})$ expresses the dependence between first- and second-period cognitive skills for the same individual.

The economy evolves as follows: at time t, individual i is born and her cognitive skills, $b_{y,i,t}$, and wealth transfer from her parent $k_{y,i,t}$ are realized. Individual i avails herself of social interactions in exactly the same way as in the deterministic model above. I simplify the model by assuming that socialization efforts remain constant over time, $(s)_o$ by the old, and $(s)_y$ by the young.

Modifying the individual's decision problem in the obvious way allows us to derive first order conditions, the stochastic counterpart of (66)–64). They are as follows:

$$k_{y,i,t+1} = \frac{1}{c}\mathcal{E}[b_{y,i,t+1}|b_{y,i,t};t] + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_o)\mathcal{E}[k_{o,j,t+1}|i,t] - \frac{1}{c\rho}; \tag{81}$$

$$k_{o,i,t+1} = \frac{1}{c}\mathcal{E}[b_{o,i,t+1}|b_{y,i,t};t] + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_y)\mathcal{E}[k_{y,j,t}|i,t] - \frac{1}{c\rho}. \tag{82}$$

We rewrite (82) in terms of j. Under the assumption that all agents observe the realization of $\mathbf{b}_{y,t}$ and that each individual i's conditional expectations of $(b_{o,j,t+1}; b_{y,j,t+1})$, $j \neq i$, are independent of i,

$$\mathcal{E}[b_{o,j,t+1}|i,t] = \mathcal{E}[b_{o,j,t+1}|j,t], \forall i,j,$$

and by using (82) to substitute into the rhs of (81) we have:

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \frac{1}{c}\mathcal{E}[\mathbf{b}_{y,t+1}|t] + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathcal{E}[\mathbf{b}_{o,t+1}|t] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}, \tag{83}$$

where each component of the vectors of conditional expectations is given by:

$$\mathcal{E}[\mathbf{b}_{y,i,t+1}|t] = \mathcal{E}[\mathbf{b}_{y,i,t+1}|b_{y,i,t}], \mathcal{E}[\mathbf{b}_{o,i,t+1}|t] = \mathcal{E}[\mathbf{b}_{o,i,t+1}|b_{y,i,t}].$$

These conditional expectations may be written in terms of the parameters $(\mathbf{b}_m, \Sigma)$ of the $\mathcal{B}_t$, the vector form of $\mathcal{B}_{i,t}$. This is a stochastic linear system that may be solved in closed form in a standard fashion, as we see further below. The conditional expectations are written, in the standard fashion, as:

$$\mathcal{E}[\mathbf{b}_{y,i,t+1}|b_{y,i,t}] = m_{y+,i} + \rho_b \frac{\sigma_{b+}}{\sigma_b}(b_{y,i,t} - m_{y,i}), \mathcal{E}[\mathbf{b}_{o,i,t+1}|b_{y,i,t}] = m_{o,i} + \rho_o \frac{\sigma_o}{\sigma_b}(b_{y,i,t} - m_{y,i}). \quad (84)$$

We can therefore use (84) in (83) to rewrite the system equations (83) as follows:

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \mathbf{G}_{\text{adj}}(\mathbf{s}_o)\mathbf{b}_{y,t} + \mathbf{C}, \quad (85)$$

where

$$\mathbf{G}_{\text{adj}}(\mathbf{s}_o, \Sigma) = \rho_b \frac{\sigma_{b+}}{c\sigma_b}\mathbf{I} + +\rho_o \frac{\sigma_o}{\sigma_b}\frac{a}{c^2}\mathbf{G}(\mathbf{s}_o);$$

$$\mathbf{C}(\mathbf{s}_o, \mathbf{b}_m, \Sigma) = \frac{1}{c}\mathbf{b}_{m,y+} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,o} - \rho_b \frac{\sigma_{b+}}{c\sigma_b}\mathbf{b}_{m,y} - \frac{a}{c^2}\rho_o \frac{\sigma_o}{\sigma_b}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,y} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}$$

The version of the system equations (85) is in the standard form for stochastic dynamical systems. The state has two components, $(\mathbf{k}_{y,t}, \mathbf{b}_{y,t})$, the first of which is predetermined and the second is random and realized at time t. By relying on the tools of linear stochastic systems we can express the steady state distribution of human capitals (and of all other endogenous variables of interest) in terms of the parameters of the stochastic process of shocks $\mathcal{B}_{y,t}$. These results are microfounded within a model of intergenerational transfers.

Specifically, we may transform system (85) in terms of deviations of human capitals from their deterministic steady state values, given by:

$$\mathbf{k}_y^* = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_y^* + \mathbf{C}(\mathbf{s}_o, \mathbf{b}_m, \Sigma). \quad (86)$$

By Proposition 4.1 of Bertsekas (1995), $\Delta\mathbf{k}_{y,t} = \mathbf{k}_{y,t} - \mathbf{k}_y^*$ has a multivariate normal limit distribution with mean $\mathbf{0}$ and variance covariance matrix Σ_∞ that satisfies:

$$\Sigma_\infty = \frac{a^4}{c^4}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\Sigma_\infty\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y) + \mathbf{G}_{\text{adj}}(\mathbf{s}_o)\Sigma\mathbf{G}_{\text{adj}}^{\mathbf{T}}(\mathbf{s}_o). \quad (87)$$

The properties of $\mathbf{G}(\mathbf{s}_y), \mathbf{G}(\mathbf{s}_o)$ are crucial determinants of the properties of the mean, $\mathbf{k}_y^*$, and of the variance covariance of the limit distribution, Σ_∞. It is thus clear that social networking has a profound effect on the steady state distribution of human capitals. Both their limit mean and variance display complex dependence on the properties of social networking and of the parameters $(\mathbf{b}_m, \mathbf{\Sigma})$ of the process of cognitive shocks $\mathcal{B}_{y,t}$.

6.3 Investment in Cognitive Skills in a Model of Two Overlapping Generations with Two Subperiods Each

We allow for the possibility that individuals may use resources to influence the cognitive skills of their children, while we retain the feature that their social networking decisions also influence their children's social networks, via the social structure which influences the child but results from parents' decision, which we interpret as influence via non-cognitive skills. We retain the overlapping generations structure and assume that youth and adulthood lasts for two subperiods each, early youth and youth, and adulthood and old age. So, each adult at time t, who was born at time $t-2$, gives birth to a child at time t, in her third subperiod of her life who in turn lives for four subperiods, $t, t+1, t+2, t+3$, during two of which she overlaps with the parent who is still alive, for two more subperiods. She then in turn gives birth to her own children at time $t+2$, when she herself is an adult. Individuals make decisions affecting the household only in adulthood and old age. For a child born at time t, her cognitive skills when she become an adult at time $t+2$ are determined by a given input at birth, $b_{y,i,t}$, which may be constant, and investments $(\iota_{c1,t}, \iota_{c2,t+1})$, and given by:

$$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \beta_1 \iota_{c1,t} + \beta_2 \iota_{c2,t+1},$$

where $\beta_0, \beta_1, \beta_2$ are parameters, and $(\iota_{c1,t}, \iota_{c2,t+1})$ are resource costs, which are incurred, contemporaneously with the respective adjustment costs, in time periods t, and $t+1$, the first and second subperiods in a child's life time, $\frac{1}{2}\gamma_1 \iota_{c1,t}^2$, $\frac{1}{2}\gamma_1 \iota_{c2,t+1}^2$, respectively. Thus, in addition to the first order conditions as above, we need to obtain conditions for the optimization of $(\iota_{c1,t}, \iota_{c2,t+1})$.[18]

An individual born at t takes cognitive skills and human capital as given, $(b_{y,i,t}, k_{y,i,t})$, and benefits from the networking efforts of the parents' generation, $\mathbf{s}_{o,t-1}$, who are in the third subperiod of their lives when she is born. She chooses at time t the second subperiod human capital and the first subperiod transfer received by the child at time $t+2$, respectively $\{k_{o,i,t+1}, k_{y,i,t+2}\}$; and the first and second subperiod networking efforts, $\{s_{y,i,t,}, s_{o,i,t+1}\}$, respectively. These benefit herself in the second subperiod of her life, and benefit her child too, when the child is in her first subperiod of her life and she herself in her third subperiod of her life. For analytical convenience, I assume that the adjustment costs for decisions $\{s_{y,i,t,}, k_{o,i,t+1}\}$, are both incurred in period t. The optimization problem implies that the cognitive skills, $b_{y,i,t+2}$, of the individual's child and the transfer she receives when she becomes an adult, $k_{y,i,t+2}$, are determined simultaneously. The definition of the problem now changes to:

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t-1}) = \max_{\{k_{o,i,t+1}, k_{y,i,t+2}; \iota_{c1,t}, \iota_{c2,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \Big\{ \rho^2 \mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})$$

$$+ b_{y,i,t} k_{y,i,t} + a \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t-1}) k_{y,i,t} k_{o,j,t} - \frac{1}{2} c k_{y,i,t}^2 - \frac{1}{2} s_{y,i,t}^2 - k_{o,i,t+1} - \iota_{c1,t} - \frac{1}{2} \gamma_1 \iota_{c1,t}^2 +$$

$$\rho \left[b_{o,i,t+1} k_{o,i,t+1} + a \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t}) k_{o,i,t+1} k_{y,j,t} - \frac{1}{2} c k_{o,i,t+1}^2 - \frac{1}{2} s_{o,i,t+1}^2 - k_{y,i,t+2} - \iota_{c1,t+1} - \frac{1}{2} \gamma_1 \iota_{c1,t+1}^2 \right] \Big\}.$$

The first order conditions for $\iota_{1,t}, \iota_{2,t+1}$ are:

$$-1 - \gamma_1 \iota_{c1,t} + \rho^2 \frac{\partial \mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}} \left[\frac{\partial b_{y,i,t+2}}{\partial \iota_{c1,t}} + \frac{\partial b_{o,i,t+3}}{\partial \iota_{c1,t}} \right] = 0.$$

$$-\rho[1 - \gamma_2 \iota_{c2,t+1}] + \rho^2 \frac{\partial \mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}} \left[\frac{\partial b_{y,i,t+2}}{\partial \iota_{c2,t+1}} + \frac{\partial b_{o,i,t+3}}{\partial \iota_{c2,t+1}} \right] = 0.$$

Using the envelope property we rewrite the partial derivation of the value function above and get:

$$-1 - \gamma_1 \iota_{1,t} + \rho^2 \beta_1 \left[k_{y,i,t+2} + \rho k_{o,i,t+3} \right] = 0.$$

$$-1 - \gamma_2 \iota_{2,t+1} + \rho \beta_2 \left[k_{y,i,t+2} + \rho k_{o,i,t+3} \right] = 0.$$

Solving for $\iota_{1,t}, \iota_{2,t+1}$ yields:

$$\iota_{1,t} = \frac{1}{\gamma_1}(\rho^2 \beta_1 [k_{y,i,t+2} + \rho k_{o,i,t+3}] - 1); \iota_{2,t+1} = \frac{1}{\gamma_2}(\rho \beta_2 [k_{y,i,t+2} + \rho k_{o,i,t+3}] - 1).$$

This in turn yields:

$$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \rho \rho_\beta [k_{y,i,t+2} + \rho k_{o,i,t+3}] - \rho_\beta, \tag{88}$$

where the auxiliary parameter ρ_β is defined as $\rho_\beta \equiv \left(\rho \frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2} \right)$. For some of the analysis below we assume that $b_{y,i,t}$ is constant, so that cognitive skills do not necessarily steadily increase. Of course, such a figure could be incorporated.

It follows that the first-order condition for $k_{y,i,t+2}$ must reflect the influence that decision has, as implied by the optimization problem, on $b_{y,i,t+2}$. Since $b_{y,i,t+2} = b_{o,i,t+3}$ the utility per period from the last two subperiods of the child's lifetime contribute to the first-order conditions. The first order conditions are:

$$-\rho + \rho^2 \frac{\partial \mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial k_{y,i,t+2}} + \rho^2 \frac{\partial \mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}} \frac{\partial b_{y,i,t+2}}{\partial k_{y,i,t+2}} = 0.$$

After using the envelope property and (88), this yields the following:

$$-1 + \rho \left[b_{y,i,t+2} + a \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t+1}) k_{o,j,t+2} - c k_{y,i,t+2} \right] + \rho^2 \rho_\beta k_{y,i,t+2} + \rho^3 \rho_\beta k_{o,i,t+3} = 0.$$

This condition is rewritten as:

$$k_{y,i,t+2} = \frac{1}{c_{cs}} b_{y,i,t+2} + \frac{a}{c_{cs}} \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t+1}) k_{o,j,t+2} + \frac{\rho^2}{c_{cs}} \rho_\beta k_{o,i,t+3} - \frac{1}{\rho c_{cs}}, \tag{89}$$

where the auxiliary variable c_{cs} is defined as: $c_{cs} \equiv c - \rho\rho_\beta$. This condition may be rewritten by using (88) to eliminate $b_{y,i,t+2}$ by expressing it in terms of $(k_{y,i,t+2}, k_{o,i,t+3})$.

In addition, the first-order conditions for $k_{o,i,t+1}, s_{y,i,t,}, s_{o,i,t+1}$, Eq. (**??**) are as follows:

$$k_{o,i,t+1} = \frac{1}{c} b_{o,i,t+1} + \frac{a}{c} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t}) k_{y,j,t} - \frac{1}{c\rho}. \tag{90}$$

$$s_{y,i,t} = \rho a k_{o,i,t+1} \sum_{j=1, j \neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{y,t})}{\partial s_{y,i,t}} k_{y,j,t}; \tag{91}$$

$$s_{o,i,t+1} = \rho a k_{y,i,t+1} \sum_{j=1, j \neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{o,t+1})}{\partial s_{o,i,t+1}} k_{o,j,t+1}. \tag{92}$$

Conditions (91) and (92) are similar, respectively, to (65) and 67) and thus may be manipulated at the steady state in like manner to the steady state analysis in section 6.1.3 above. It is more convenient to write Eq. (90) by advancing the time subscript as follows:

$$k_{o,i,t+3} = \frac{1}{c} b_{o,i,t+3} + \frac{a}{c} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t+2}) k_{y,j,t+2} - \frac{1}{c\rho}. \tag{93}$$

By using (88) to write for $b_{o,i,t+3}$ in terms of its solution in terms of $(k_{y,i,t+2}, k_{o,i,t+3})$ and rewriting the conditions for $(k_{y,i,t+2}, k_{o,i,t+3})$ in matrix form, we have:

$$\mathbf{k}_{o,t+3} = \frac{\beta_0}{\rho^* c} \mathbf{b} - \frac{\rho_\beta}{\rho^*} \mathbf{i} + \left[\frac{\rho\rho_\beta}{\rho^* c} \mathbf{I} + \frac{a}{\rho^* c} \mathbf{G}(\mathbf{s}_{y,t+2}) \right] \mathbf{k}_{y,t+2}, \tag{94}$$

where $\rho^* \equiv 1 - \frac{\rho^2\rho_\beta}{c}$.

$$\mathbf{k}_{y,t+2} = \frac{\beta_0}{\tilde{\rho}c_{cs}}\mathbf{b} - \frac{1}{\tilde{\rho}\rho c_{cs}}\mathbf{i} + \frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2} + \frac{\rho^2\rho_\beta}{c_{cs}}\mathbf{k}_{o,t+3}, \tag{95}$$

where $\tilde{\rho} \equiv 1 - \frac{\rho\rho_\beta}{c_{cs}}$. However, by substituting from (94) for $\mathbf{k}_{o,t+3}$ in the rhs of (95), we have:

$$\left[\left(1 - \frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)\mathbf{I} - \frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}cc_{cs}}\mathbf{G}(\mathbf{s}_{y,t+2})\right]\mathbf{k}_{y,t+2}$$

$$= \beta_0\left[\frac{\rho^2\rho_\beta}{\tilde{\rho}\rho^* cc_{cs}} + \frac{1}{\tilde{\rho}c_{cs}}\right]\mathbf{b} - \left[\frac{1}{\tilde{\rho}\rho c_{cs}} + \frac{\rho^2\rho_\beta^2}{\tilde{\rho}\rho^* c_{cs}}\right]\mathbf{i} + \frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}.$$

By dividing through by $1 - \frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}$ and denoting

$$\hat{a} \equiv \frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}c_{cs}}\left(1 - \frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)^{-1},$$

we may solve the previous equation with respect to $\mathbf{k}_{y,t+2}$ as follows:

$$\mathbf{k}_{y,t+2} = \left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\left[\mathbf{b}'_{\text{eff}} + \frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}\right],$$

where $\mathbf{b}'_{\text{eff}}$ is the resulting new constant. By substituting into the rhs of (94), we obtain a single first-order linear difference system in $\mathbf{k}_{o,t+2}$:

$$\mathbf{k}_{o,t+3} = \mathbf{b}_{\text{eff}} + \frac{a}{\rho^* c}\mathbf{G}(\mathbf{s}_{y,t+2})\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}, \tag{96}$$

where $\mathbf{b}_{\text{eff}}$ denotes the resulting constant. Thus, this equation depends on both networking efforts by the young and the old in two successive periods, $\mathbf{G}(\mathbf{s}_{y,t+2}), \mathbf{G}(\mathbf{s}_{o,t+2})$.

In a notable difference from the previous model, we now see a key new role for the social networking that individuals avail of when young. The product $\mathbf{G}(\mathbf{s}_{y,t+2})\mathbf{G}(\mathbf{s}_{o,t+2})$ is adjusted by $\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}$. Intuitively, this effect acts to reinforce the effects of social networking when young. This readily follows from (94) and (94) above. Feedbacks are generated due to the investment in cognitive skills. Mathematical results invoked upon earlier can still be used to determine the stability of (96). That is, $\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})\right]^{-1}$ admits a simple expression, following steps similar to (8) above, provided that

$$\frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})} < 1.$$

Thus:

$$\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})\right]^{-1} = \mathbf{I} + \frac{\hat{a}}{c}\frac{\overline{x}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2}) - \frac{\hat{a}}{c}\overline{x^2}(\mathbf{s}_{y,2})}\mathbf{G}(\mathbf{s}_{y,2}).$$

Thus, the stability of (96) rests on the spectral properties of

$$\frac{a}{\rho^* c}\,\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{y,2})\mathbf{G}(\mathbf{s}_{o,2}) + \frac{a}{\rho^* c}\,\frac{\hat{a}}{c}\,\frac{a}{\tilde{\rho}c_{cs}}\frac{\overline{x}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2}) - \frac{\hat{a}}{c}\overline{x^2}(\mathbf{s}_{y,2})}\mathbf{G}(\mathbf{s}_{y,2})^2\mathbf{G}(\mathbf{s}_{o,2}).$$

By Theorem 1, Merikoski and Kumar (2004), 151–152, the maximal eigenvalue of the sum of two real symmetric (Hermitian) matrices is bounded upwards by the sum of the maximal eigenvalues of the respective matrices. Thus, a condition for the stability of (96) readily follows and involves $(\mathbf{s}_{y,2}, \mathbf{s}_{o,2})$ along with the other parameters of the model.

7 Conclusions

The dynamic models analyzed by this paper offer a novel view of the joint evolution of human capital investment and social networking. Those of our models that are embedded in overlapping generations frameworks display all strengths and weaknesses of that workhorse of modern growth theory and macroeconomics. Our analysis first takes advantage of formal similarities between infinite horizon dynastic life cycle modeling and overlapping generations models with intergenerational transfers. The dynamic models of the paper share the important feature namely that individuals' lifetime capital accumulation plans are distinguished from intergenerational transfers, while allowing for an endogenous social structure. The model where endogenous investment influence the cognitive skills of one's child is analytically considerably more complicated than when cognitive skills are given, however, because of additional dynamic complexity. In our basic model with overlapping generations, individuals receive a transfer from their parents in the first period of their lives and avail themselves of the social connections that their parents chose at that same period. They in turn choose their own second-period human capital, own second-period social connections, and transfer to their children. Even so, the dynamical system involving the vectors of life cycle accumulation and transfers, given the social network, is still linear in those magnitudes and tractable. The endogeneity of the social structure makes that analysis quite complicated but considerably richer, but the tools of the paper allows us to study the underlying steady states for individuals' life cycle accumulation, intergenerational transfers, and social connections for themselves and for their children in great detail.

All these models share the property that human capital accumulation, transfers and social connections, when all are optimized, are proportional to cognitive skills. However, the model with intergenerational transfers yields that human capital accumulation is proportional to ones's cognitive skills in the second period and intergenerational transfers, in effect human capital endowment of their children, is proportional to their children's cognitive skills in the first period of the children's lives, consistent with the logic of the model. In contrast, social connections for one's second period are proportional to the respective own cognitive skills, and social connections for the first period of one's children are proportional to their children's first period cognitive skills. Thus, intergenerational transfers of both human capital endowments and social networking endowments are jointly determined. Interestingly, the consequences for inequality of the endogeneity of social connections are underscored by examining the models when it is assumed that they are exogenous. When social connections are not optimized, individuals' human capital reflect a much more general dependence on social connections across individuals. Social effects are also shown to be present in intergenerational wealth transfer elasticities. The dependence does not reduce to aggregate statistics and highlights both "whom you know" and "what you know" in the determination of individual incomes.

Notes

[1]A partial list of papers that emphasize empirical aspects of network formation is as follows. Goldsmith-Pinkham and Imbens (2013), which also contains a dynamic network formation model, where the utility for i of forming a link with j depends on the distance between the two of them in the covariate space, $|\mathbf{X}_i - \mathbf{X}_j|$, on whether the two were friends in th previous period, and on whether they had friends in common in the previous period; the discussion that follows the paper is particular interesting, including contributions by Bramoullé (2013), Graham (2013), Jackson (2014), Kline and Tamer (2013), and Sacerdote (2013). Additional contributions by Baetz (2013), Blume, Brock, Durlauf and Jayaraman (2014) (section 6), Boucher and Fortin (2014), Christakis, Fowler, Imbens and Kalyanaram (2010), Hiller (2014), and Tarbush and Teytelboym (2015). The last of these papers emphasizes online social network formation, where a fixed number of agents interact in overlapping social groups.

[2]Albornoz, Cabrales, and Hauk (2014) develop a conceptually similar use of the Cabrales *et al.* model, but in a static context.

[3] This basic model may be augmented to account for a variety of motivations, such as altruism, conformism and habit formation. See Ioannides (2013), Ch. 2.

[4]Cabrales *et al.* follow standard practice in this literature and define a finite number of types of players and work with an $m-$replica game, for which the total number of individuals is a large multiple of the number of types. In this fashion, as we see further below, it is possible to increase the number of individuals in order to reduce the influence of any single one of them and be able to characterize outcomes in a large economy. *Ibid.*, p. 341.

[5]In $(\varpi, \vartheta)-$ space, the tangent from the origin to the graph of (16) must have slope less than $\tilde{a}^{-1}$.

[6]Formulations of determinants of interactions with rich demographics may be helpful in accommodating the range of empirical issues broached by Ioannides and Loury (2004).

[7]This recalls the controversy in the literature regarding the returns to scale properties of the matching function. See Blanchard and Diamond (1990) and Petrongolo and Pissarides (2001).

[8]The so-called CES structure is in turn a special case of a mean value with an arbitrary function [Hardy *et al.* (1952), p. 65]. That is, let $y(k)$ be a function, which is assumed to be continuous and strictly monotonic, in which case so is its inverse, $y^{-1}(k)$. The CES structure defined here is simply $y^{-1}\left(\sum gy(k)\right)$, for $y(k) = k^{1-\frac{1}{\xi}}$.

[9]In fact, a feature such as the last one is relied upon by Lucas and Moll (2014), where individuals divide their time between producing goods using their existing knowledge and interacting with others in search of new productivity-enhancing ideas. Such interactions take the form of pairwise meetings, which is simply

an opportunity for each individual to observe the productivity of someone else. If that is higher than his own, he adopts it in place of the one he came in with. To ensure that the growth generated by the process is sustained, Lucas and Moll assume that the stock of good ideas to be discovered is inexhaustible. It is possible to introduce this set of possibilities once we have allowed for shocks that in effect renew the set of productive ideas. More on this, later.

[10]This is similar to the feature in Lucas and Moll (2014), where agents retain the best lessons from their contacts.

[11]See Tao and Lee (2014) for the econometrics of a social interactions model based on extreme-order statistics.

[12]Cabrales *et al.* apply a dynamic analysis due to Corchón and Mas-Colell (1996), according to which agents adjust their strategies along directions which they know will cause their utilities will grow. This amounts to equating the time derivatives of agents' decisions to the partial derivatives of the payoff with respect to the same variable. That is:

$$\frac{\partial s_i(t)}{\partial t} = \frac{\partial u_i(\mathbf{s}(t), \mathbf{k}(t))}{\partial s_i(t)}, \quad \frac{\partial k_i(t)}{\partial t} = \frac{\partial u_i(\mathbf{s}(t), \mathbf{k}(t))}{\partial k_i(t)}. \tag{97}$$

Linearizing around the individual optimum exploits the fact that the first partial derivatives are equal to zero at the optimum. The resulting system of equations are comprised of terms that are linear in $(s_j(t) - s_j^*)$ and $(k_j(t) - k_j^*)$ with coefficients the respective second partial derivatives $\frac{\partial^2 u_i}{\partial s_i \partial s_j}$. In the limit for large I, the own cross-partial derivatives vanish, except for the one with respect to s_i, k_i. So, the only non-zero derivatives are

$$\frac{\partial^2 u_i}{\partial s_i^2} = -1, \frac{\partial^2 u_i}{\partial s_i \partial k_i} = a(\mathbf{b})k, \frac{\partial^2 u_i}{\partial k_i^2} = -c.$$

Cabrales *et al.* show that with this kind of stability, these equations "imply that there is no feedback from the changes in one individuals' strategies to any of the others' when close to either equilibria. Then, the local dynamics are entirely driven by one individual's strategy and thus local stability is the same as local maxima, which is why the condition for stability is the same as the condition that guarantees that the second-order conditions are satisfied" [*ibid.*, p. 346]. According to this analysis, both non-autarkic solutions are stable equilibria with respect to the feasible perturbations postulated.

[13]First, we note that:

$$\frac{\partial g_{ij}}{\partial s_h} = -\frac{s_i s_j}{\left(\sum_{h=1}^I s_h\right)^2}, \quad h \neq i, j;$$

$$\frac{\partial g_{ij}}{\partial s_i} = \frac{s_j \sum_{h \neq i} s_h}{\left(\sum_{h=1}^I s_h\right)^2};$$

$$\frac{\partial g_{ij}}{\partial s_j} = \frac{s_i \sum_{h \neq i} s_h}{\left(\sum_{h=1}^I s_h\right)^2};$$

The terms $s_j \sum_{h\neq i} s_h$ tend to $I\varpi^2\vartheta^2\bar{\mathbf{b}}$, and the terms $\left(\sum_{h=1}^{I} s_h\right)^2$ tend to $\varpi\vartheta I^2(\bar{\mathbf{b}})^2$. Therefore, as $I \to \infty$, the derivatives above tend to zero and the second term in the rhs of (45) and of (46) vanish. Similarly, the first term in the rhs of (46) also vanishes.

[14] The notation $\ln^+$, defined as follows: $\ln^+ x = \min\{\ln x, 0.\}$

[15]This argument is reminiscent of arguments explaining the emergence of power laws elsewhere in the economics literature. See for the city size distribution case Ioannides (2013), Ch. 8.

[16]That paper also examines endogenous social connections, but does not specify them in sufficient detail in order to obtain specific results.

[17]In fact, Samuelson (1958) itself is cast in three-overlapping generations. Azariadis, Bullard and Ohanian (2004) find additional properties in economies with many overlapping generations, in particular with respect to monotonicity (or non-monotonicity) of the equilibrium price when consumptions in different periods are weak gross substitutes.

[18]This very special case of infinite substitutability of investments in cognitive skills is in contrast to Heckman and Mosso (2014), but is made for analytical convenience.

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APPENDIX

A Model with Intergenerational Transfers

Using intuition gained earlier, it is straightforward to see that one can obtain a simple rule for $\mathbf{s}_{i,t}$ the networking efforts, as functions of the respective $k_{i,t}$'s in the same period, which is the same as the one obtained earlier, (10):

$$\varpi_t = \frac{s_{i,t}}{k_{i,t}} = a\frac{\sum_j s_{jt}k_{jt}}{\sum_j s_{jt}}. \tag{98}$$

Then, using this result we may express utility in period t as follows:

$$b_{\tau(i)}k_{i,t} + a\varpi_t k_{i,t}\sum_{j=1,j\neq i}^{I}\frac{k_{j,t}}{K_t} - \left(\frac{1}{2}c + \frac{1}{2}\varpi_t^2\right)k_{i,t}^2 - k_{i,t+1}$$

$$= (b_{\tau(i)} + a\varpi_t)k_{i,t} - \left(\frac{1}{2}c + \frac{1}{2}\varpi_t^2 + a\varpi_t\frac{1}{K_t}\right)k_{i,t}^2 - k_{i,t+1},$$

where $K_t = \sum_{j=1}^{I} k_{i,t}$. Note that for simplicity, the resource cost of the transfer $k_{i,t+1}$ is accounted for when the transfer is made, but the adjustment cost $\frac{1}{2}ck_{i,t+1}^2$ is incurred when the investment takes place in period $t+1$. In this interpretation, utility per period at time t is output from investing $k_{i,t}$ net of the cost networking effort and adjustment cost of investment, whose resource cost was incurred in the previous period.

By utilizing the envelope property, we can write the first-order conditions for the $k_{i,t+1}$'s as follows:

$$(b_{i,t+1} + a\varpi_{t+1}) - \left(c + \varpi_{t+1}^2 + 2a\varpi_{t+1}\frac{1}{K_{t+1}}\right)k_{i,t+1} = \frac{1}{\rho}.$$

These equations may be solved in terms of each of the $k_{i,t}$'s as follows. By summing up the above equation over all i's, we can solve the resulting linear equation for the sum of all human capitals, $K_{t+1} = \sum_{j=1}^{I} k_{i,t+1}$, in terms of ϖ_{t+1} and $B_{t+1} = \sum_i b_{i,t+1}$. Then returning to the above equation allows us solve for each of the $k_{i,t+1}$'s. We also need to impose a positivity constraint on the utility per period. We then return to the the first-order conditions for the $s_{i,t+1}$'s in order to determine ϖ_{t+1}.

That is, we have:

$$K_{t+1} = \frac{B_{t+1} + I\left(a\varpi_{t+1} - \frac{1}{\rho} - 2a\varpi_{t+1}\right)}{c + \varpi_{t+1}^2}, \tag{99}$$

$$k_{i,t+1} = \frac{b_{i,t+1} + a\varpi_{t+1} - \frac{1}{\rho}}{c + \varpi_{t+1}^2} \frac{B_{t+1} + I\left(a\varpi_{t+1} - \frac{1}{\rho}\right) - 2a\varpi_{t+1}}{B_{t+1} + I\left(a\varpi_{t+1} - \frac{1}{\rho}\right)}. \tag{100}$$

Finally, by substituting back into (98), the necessary condition for $s_{i,t+1}$ we have:

$$\varpi_{t+1} = a\frac{B_{t+1} + I\left(a\varpi_{t+1} - \frac{1}{\rho}\right) - 2a\varpi_{t+1}}{c + \varpi_{t+1}^2} \frac{\sum_i b_{i,t+1}^2 + 2a\left(\varpi_{t+1} - \frac{1}{\rho}\right)B_{t+1} + I\left(\varpi_{t+1} - \frac{1}{\rho}\right)^2}{B_{t+1} + I\left(a\varpi_{t+1} - \frac{1}{\rho}\right)}.$$

This is a quartic (fourth-degree) equation in ϖ_{t+1} that involves parameters a, c, ρ, and B_{t+1} the aggregate of the cognitive skills. It is known since the time of Evariste Galois that such equations may be solved in closed form. Unfortunately, it is also known that the solutions are too complicated algebraically.

However simplistic, this model does account for intergenerational transfers, which becomes the human capital with which the next generation works. Networking efforts and thus connection intensities are determined given human capitals in the respective period. Note that the model is solved completely in the deterministic case, however unwieldy the solution is. Individual human capitals depend on individual cognitive skills $(b_{i,t}, b_{i,t+1})$, as well as aggregate cognitive skills, B_t, both directly and via the factor of proportionality between networking efforts and human capitals. As I show next, the culprit for the algebraic complication here is the assumptions about the timing of the individuals' decisions. These are relaxed in the exposition that follows and yield an economically more interesting model.