ENDOGENOUS SOCIAL NETWORKS AND INEQUALITY IN AN INTERGENERATIONAL SETTING*

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Abstract. The paper examines the joint evolution of human capital investment and social networking where the network may be endogenous or exogenous. It extends Cabrales, Calvó-Armengol and Zenou (2011) to dynamic settings to allow for intergenerational transfers of human capital and social networking endowments as joint decisions. Social multipliers characterize the social equilibria. The associated intergenerational transfer elasticities exhibit rich dependence on social effects allowing the intergenerational transfers elasticity to be increasing with inequality in parents' human capitals. The stochastic steady states highlight the cross-sectional human capital distribution as outcome of shocks to cognitive and social skills coefficients.

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1 Introduction

In a world where individuals interact in myriads of ways, one wonders how the benefits of one's social connections with others combine with those conferred by individual characteristics to affect the acquisition of human capital. It is particularly interesting to be able to distinguish between consequences of connections that are the outcome of deliberate decisions by individuals and of connections being given exogenously and beyond individuals' control. Individuals may seek to form social links with others, as an objective in its own right, in order to enrich their social lives and avoid social isolation. Social links provide conduits through which benefits from interpersonal exchange can be realized. Social isolation excludes them.

The paper explores the consequences for inequality of the joint determination and evolution of social networking and human capital investments. It embeds inequality analysis in intergenerational models of endogenous social network formation in dynamic settings. The novelty of the model lies in its joint treatment of human capital investment and social network formation in dynamic settings, while distinguishing between the case of impact on human capital from endogenous as opposed to exogenous social networking. Exogenous social networks admit a natural interpretation as social immobility. The present paper aims at a deeper understanding of the consequences of social networking for inequality. For example, endogenous social networking may even decrease inequality, in general, in human capitals relative to exogenous one.

It is straightforward to recognize the difficulty of modeling social networking. For a given number of individuals I, there are $2^{\frac{I(I-1)}{2}}$ different possible networks connecting them. Thus, to a typical social group of $I = 100$ there correspond $2^{50\times 99} \approx 10^{4950}$ network configurations, some of which are not topologically distinct. As Blume, Brock, Durlauf and Jayaraman (2015) argue, there is really no general theoretical model of network formation. Relatedly, as Jackson (2014), p. 14, points out, studying endogenous network formation continues to be an important priority. One needs to be specific in order to be able to go in depth in linking differences in individual characteristics to differences in decisions and outcomes, when individuals interact socially and may have been influenced by the characteristics and decisions of those they end up being in social contact with. Jackson (2019, Ch. 6), provides numerous insights on how social network considerations are essential in understanding the contributions to inequality of the coupling of decisions and opportunities accorded by social networks.

This paper extends the static framework of Cabrales, Calvó-Armengol, and Zenou (2011),[2] CC-AZ from now on, to allow for intergenerational links in the form of intergenerational transfers of wealth jointly with those of social connections. This extension highlights the importance of endogenous setting of social connections for the cross-sectional distribution of human capital and allows us to explore how social connections may magnify the impact of the dispersion in cognitive as well as social skills coefficients on inequality. When social networking efforts are endogenous multiple equilibria become possible. At the steady state solutions associated with either high or, alternatively, low networking efforts, the distribution of human capital mirrors that of the cognitive skills coefficients. It may be arbitrary otherwise, if social networking is exogenous. There is a long-standing empirical literature on interdependence of skills with their intergenerational transmission. They are key to modeling assumptions employed by the paper. I review them briefly further below after I establish first the relation of this paper to the literature. Canen, Jackson and Trebbi (2020), who emphasize legislative activity and formation of social networks in the United States Congress, also extends CC-AZ dynamically, but is unrelated to the present paper's focus on intergenerational transfers and inequality.

1.1 Relation to the Literature

Starting from Becker and Tomes (1979) and on with Loury (1981), a number of articles have linked intergenerational transfers and the cross-section distributions of income and of wealth. Most recently, Lee and Seshadri (2019) model human capital accumulation in the presence of intergenerational transfers, while allowing for multiple stages of investment over the life cycle, such as investment during childhood, college decision and on-the-job human capital accumulation. They allow for complementarity between early and later child investments, *inter alia,* by using a model of many overlapping generations (78!), with infinitely lived altruistic dynasties. They show, using numerical simulation methods, that investment in children and parents' human capital have a large impact on the equilibrium intergenerational elasticities of lifetime earnings, education, poverty and wealth, while remaining consistent with cross-sectional inequality. That is indeed one of very few papers that take seriously Heckman's forceful suggestion [Cunha and Heckman 2007; Heckman and Mosso 2014] to allow for multiple stages of investment over the life cycle. They do not allow for social interactions.

The treatment of dynamics in the paper proceeds from a dynastic, in the style of Ramsay-Cass-Koopmans, extension of CC-AZ, which facilitates the treatment of dynamics, to models with two overlapping generations, and even more than two. Specifically, the paper also examines a variation of the two-overlapping generations model with two subperiods in order to allow for individuals to invest in augmenting the cognitive skills coefficients of their children. The impact of availability of such investments on the dynamics of evolution of human capital investments and social connections has a factorial structure, and reflects additional amplification effects via social networking on outcomes for children from their parents' decisions whenever they occur.

In addition to enriching the two workhorses of modern macroeconomics, the Ramsay-Cass-Koopmans and overlapping-generations models, by means of endogenizing the interactions among agents and predicting the evolution of inequality, the following specific results are noteworthy. The distribution of human capitals when social connections are exogenous can exhibit heavy tails. One such instance emanates from a generalization of the evolution of human capitals when cognitive and social skills coefficients are stochastic within the Ramsay-Cass-Koopmans dynastic model and pertains to the *joint* distribution of human capitals. A second instance follows again when social connections are exogenous but within the context of the overlapping-generations dynastic model when cognitive skills coefficients are stochastic. Those heavy tails results obtain for different reasons. In the former case, the assumptions made ensure that although the evolution of human capitals looks like a contraction, there is no collapse at the lower end while there is a contracting effect at the higher end of the joint distribution. In the latter case, the result pertains to the anonymized distribution of human capitals, when the stochastic evolution of human capitals is construed in a cross-sectional sense, and rests on a little known property of mixtures of normal densities (indeed of the family of exponential densities), namely that they have heavier tails than mean-matched univariate ones. Another noteworthy result is that when parents are allowed to invest in improving their children's cognitive abilities, technically in their children's cognitive skills coefficients, even when social connections are exogenous, social networking generates an amplification effect, which has not been recognized in the literature. Finally, the case of exogenous social connections while individuals can still invest in their human capitals may be interpreted as social immobility: individuals cannot change their own-social circumstances, that is whom they interact with, influence them and be influenced by them. The paper also shows that inequality associated with an arbitrary social network may be reduced if individuals are allowed to choose the intensity of interactions with others. Both models, the Ramsay-Cass-Koopmans and the overlapping generations, display similar dynamics, but the latter makes it possible to distinguish between inequality over the life cycle within a generation from intergenerational inequality.

A preview of the remainder of this paper is as follows. Section 2, Proposition 1, presents the joint evolution of human capital and social connections by means of a dynastic model. The stability properties of the two non-autarkic equilibria, namely the local instability of the higher one and the local stability of the lower one, are established and so is the social multiplier property in dynamic settings. The section also discusses an assortative networking property of the equilibrium solution. With given, that is arbitrary social connections cross-sectional inequality may be greater relative to the endogenous case, because exogeneity introduces an additional source of inequality, whose impact may not be influenced by individual actions. Intuitively, this may be interpreted as an instance of the LeChatelier Principle. Section 2.3 moves to overlapping generations models. Proposition 2 summarizes the results with endogenous human capitals and social connections. Specific results obtained include a dynamic generalization of the social multiplier property, which is the same across all individuals but varies over time and across first- and second-period human capitals, which are proportional to the respective cognitive skills coefficients. Section 3 provide a number of extensions and applications of the model, all aimed at clarifying how parents' circumstances may influence their children's wealth endowments via transfers, social networking, as well as possibly via persistence in cognitive and social skills coefficients.

Section 3.1 reports a result of relevance to the literature on intergenerational income mobility, known as "The Great Gatsby Curve:" higher earnings inequality is empirically associated with lower intergenerational mobility. The elasticity of intergenerational mobility, interpreted as that of intergenerational transfers, is increasing in the inequality of parents' human capital. Proposition 3, subsection 3.2 and additional results, reported in subsection 3.4, employ a Bayesian Nash formulation of the network interaction game to allow for stochastic shocks to cognitive skills coefficients and social skills coefficients, respectively.

Results reported in section 3.3 pertain to a salient feature of empirical income and wealth distributions, namely thick upper tails.The cross-sectional distribution of human capitals is characterized, with a key role played by the intergenerational correlation in cognitive skills coefficients. It is shown, *inter alia*, that the stochastic steady state distribution might exhibit thicker tails than normal and might not be unimodal. A different thick tails result is obtained for the joint density of human capitals. Another notable result, reported in section 3.5, affirms that if parents may invest in improving their children's cognitive skills, such investments are amplified by social networking. Indeed, all of the dynamic models developed in the paper share amplification features involving both the means and the variances. These emanate from the multiplier properties of endogenous and exogenous network effects. Section 4 outlines testable predictions of the paper. All details of derivations along with supplementary material are provided in an Online Appendix. Section 5 concludes.

1.2 Review of Relevant Empirical Findings from the Literature

This brief review limits itself to closely related studies in a large literature, starting from parental investments and social interactions, where Agostinelli (2018) and Agostinelli, Doepke, Sorrenti and Zilibotti (2020) stand out. Agostinelli (2018) studies the effects of social interactions on the dynamics of children's skills using Add Health data.[3] It focuses on child development by acknowledging endogenous peer group formation and allowing for parental investments to react to the child's social interactions. Agostinelli (2018) finds that endogenous formation of peer groups is crucial. In contrast, in the present paper it is parents' decisions that shape their children's social environment. Large general equilibrium effects that Agostinelli reports are conceptually similar to multiplier effects in my setting. In Agostinelli (2018) they originate in large-scale changes in peer group composition. Agostinelli *et al.* (2020) go beyond Agostinelli (2018) by allowing for parents to influence their children's peer groups. By allowing for interactions of parenting style and peer effects while children's skill accumulation depends on both parental inputs and peers, Agostinelli *et al.* obtain estimates that can inform policy simulations. Their findings about the impact of interventions that move children to more favorable neighborhoods, namely that the powerful effects predicted by Agostinelli (2018) may be offset by parents' equilibrium responses, are particularly interesting.

Regarding the intergenerational transmission of skills, I eschew tackling in detail the massive literature and simply refer to Adermon *et al.* (2021). Their extraordinary data set includes the entire Swedish population, links four successive generations, and maps the extended family by identifying parents' siblings and cousins, their spouses, and spouses' siblings. It enables the authors to show that traditional parent-child estimates underestimate the long-run intergenerational persistence in terms of several alternative human capital measures by at least one-third. Controlling for outcomes for more distant ancestors, they show that almost all of the persistence is captured by the parental generation. The breadth of the coverage of this study confers to it an overarching, oversize role within the universe of studies based on Scandinavian data and would likely encourage similar research in different institutional settings. Although their endogenous variables contribute to intergenerational inequality, they do not depend explicitly on inequalities in the respective social contexts. Social skills are together with cognitive skills important structural features of the model of Abbott *et al.* (2019). Both types of skills also play a prominent role in the threshold regression model in Durlauf, Kourtellos and Tan (2017) that defines status traps.[4]

2 Dynamic Model of Human Capital Investment and Networking

This section presents a dynamic extension of the model in CC-AZ, whereby productive investments in their setting are interpreted here as human capital investments which are optimized jointly with socialization efforts, interpreted here as social networking. I define a dynastic intertemporal objective function for individuals in the style of the Ramsey–Cass–Koopmans formulation and extend key results in dynamic settings, unlike the static case of CC-AZ. An alternative model, based on an overlapping-generations model that allows for intergenerational transfers of wealth and social connections, confers additional demographic richness that allows for some additional contrasts, is taken up in section 2.3 below. Both dynamic extensions are, to the best of my knowledge, novel in the context of the literature.[5]

2.1 Joint Evolution of Human Capital and Social Connections in a Ramsey-Cass-Koopmans Model

Let us consider that each decision making unit i is a dynasty that values the total discounted utility of its typical member, when utility per period is given by:

$$U_{i,t}(\mathbf{s}_{t-1}; s_{it}; \mathbf{k}_{t-1}, k_{it}) := b_{it}k_{it} + a \sum_{j=1,j\neq i}^{I} g_{ij}(\mathbf{s}_{t-1})k_{it}k_{jt-1} - c\frac{1}{2}k_{it}^2 - \frac{1}{2}s_{it}^2, \tag{2.1}$$

where $\mathbf{s}_t = (s_{1t}, \ldots, s_{it}, \ldots, s_{It})$ denotes the full vector of networking efforts, whose costs are incurred at time t but yield benefits at time $t+1$, and $\mathbf{k}_t = (k_{1t}, \ldots, k_{it}, \ldots, k_{It})$, those of human capitals. The weights of social interaction g_{ij}, the elements of a social interactions (or social adjacency) matrix $\mathbf{G}$, may be defined in terms of networking efforts in a number of alternative ways. The simplest case, with the weights being obtained axiomatically by CC-AZ, are as follows:

$$g_{ij}(\mathbf{s}) = \frac{1}{\sum_{j=1}^{I} s_j} s_i s_j, \text{ if } \forall s_i \neq 0; \quad g_{ij}(\mathbf{s}) = 0, \text{otherwise.} \tag{2.2}$$

I interpret the coefficient a of the interactive summation term in definition (2.1) as the *social synergy,* or *social skills coefficient.*[6] I refer to coefficient b_{it} as the *cognitive skills* coefficient. Networking effort may be interpreted as time spent on deliberate efforts to seek out others. For brevity, I refer to CC-AZ for numerous ways to motivate this formulation as socialization. Note that the quantity $\sum_{j=1}^{I} \frac{s_i s_j}{\sum_{j=1}^{I} s_j} = s_i$ is the weighted network counterpart of network degree. In this interpretation, individuals choose their weighted network degrees. Time lags in the effects both of human capitals of one's social contacts and of social networking confer dynamic richness. Numerous activities that may be undertaken by parents, such as sports, religious, cultural and community events, and participating in parent-teacher events (common in the U.S. for parents of primary and secondary school students) may be interpreted as having social networking consequences for their children. Analytically, the social interaction weight $g_{ij}(\mathbf{s}_{t-1})$ at time t in (2.1) expresses the notion that the current human capital investment confers benefits that depend on social networking in the preceding period. However, human capital k_{it} at t still yields benefits in the form of income reflecting own cognitive skills, but both human capital investment k_{it} and networking effort s_{it} incur adjustment costs in the form of utility loss when they are actually implemented at time t.

Period t utility is concave with respect to k_{it}, and increasing provided that variables assumes appropriate values, basically that the sum total of cognitive and non-cognitive effects is large enough; it is linear with respect to $k_{j,t-1}, j \neq i$, and increasing concave with respect to $s_{i,t-1}$ and decreasing and concave with respect to s_{it}.

In a standard dynastic interpretation, infinitely lived dynasty i avails itself of a sequence of cognitive effects $\{\ldots, b_{it}, \ldots\}_{t=0}^{\infty}$, and chooses sequences of human capital investment and networking efforts $\{\mathbf{k}_{it}\}_{t=0}^{\infty}, \{\mathbf{s}_{it}\}_0^{\infty}$, so as to maximize

$$\sum_{t=0}^{\infty} \rho^t U_{i,t}(\mathbf{s}_{t-1}; s_{it}; \mathbf{k}_{t-1}, k_{it}), \tag{2.3}$$

taking as given all other dynasties' contemporaneous decisions $\{\mathbf{k}_{-it}\}_0^{\infty}, \{\mathbf{s}_{-it}\}_0^{\infty}$, where $\rho, 0 < \rho < 1$, discounts for altruism. The dynamic analysis is summarized in Proposition 1, which follows next. The proofs of Parts A, B and E are immediate; those of Parts C and D are given in the Online Appendix, section A.1.

Proposition 1. *Agents' choices of sequences of human capital investment and networking efforts* $\{\mathbf{k}_{it}\}_0^{\infty}, \{\mathbf{s}_{it}\}_0^{\infty}$, *that maximize (2.3) taking as given all other agents' contemporaneous decisions* $\{\mathbf{k}_{-it}\}_{t=0}^{\infty}, \{\mathbf{s}_{-it}\}_{t=0}^{\infty}$, $0 < \rho < 1$, *must satisfy, when expressed in vector form:*

Part A. the system of difference equations with endogenous time-varying coefficients

$$\mathbf{k}_t = \frac{1}{c}\mathbf{b}_t + \frac{a}{c}\mathbf{G}(\mathbf{s}_{t-1})\mathbf{k}_{t-1}; \tag{2.4}$$

$$\mathbf{s}_t = a\rho[diag\, \mathbf{k}_{t+1}]\frac{\partial \mathbf{G}(\mathbf{s}_t)}{\partial \mathbf{s}_t}\mathbf{k}_t, \tag{2.5}$$

where $[diag\, \mathbf{k}_{t+1}]$ *denotes an* $I \times I$ *matrix with the elements of* $\mathbf{k}_{t+1}$ *along the main diagonal,* $g_{ij}(\mathbf{s}_t)$ *is as defined by (2.2), and* $\frac{\partial \mathbf{G}(\mathbf{s}_t)}{\partial \mathbf{s}_t}$ *denotes a matrix with the terms* $\frac{\partial \mathbf{g}_{ij}(s_{it})}{\partial \mathbf{s}_t}$ *as its* ith *row.*

Part B. If the vector of cognitive skills coefficients $\mathbf{b}_t$ *is time-invariant, then the steady state values of the system (2.4–2.5)* (k_i^*, s_i^*) *obey:*

$$k_i^* = \vartheta b_i,\ s_i^* = \varpi\vartheta b_i, \tag{2.6}$$

where the scalars (ϖ, ϑ) *satisfy the system of algebraic equations:*

$$\vartheta = \tilde{a}(\mathbf{b})^{-1}\varpi; \tag{2.7}$$

$$\vartheta = \frac{1}{c - \varpi^2}, \tag{2.8}$$

and $\tilde{a}(\mathbf{b})$ is defined as $\tilde{a}(\mathbf{b}) := \rho a \frac{\sum_i b_i^2}{\sum_i b_i}$, and may be rewritten as:

$$\tilde{a}(\mathbf{b}) = \rho a \bar{b} \left[1 + (CV_{\mathbf{b}})^2\right], \tag{2.9}$$

where $\bar{b}$ denotes the average of the components of $\mathbf{b}$ and CV_b their coefficient of variation.

The system of Equations (2.7)–(2.8) admits two sets of positive solutions, $(\vartheta^*, \varpi^*; \vartheta^{**}, \varpi^{**})$, with superscript ** denoting algebraically larger quantities than those with superscript *, provided that:

$$2\left(\frac{c}{3}\right)^{\frac{3}{2}} \geq \tilde{a}(\mathbf{b}). \tag{2.10}$$

To these solutions, there correspond two sets of steady state solutions for human capitals and networking efforts, $\{k_i^*, s_i^*; k_i^{**}, s_i^{**}\}_{i=1}^I$. The zero solutions of (2.7–2.8) correspond to the autarky case, that is no social links.

Part C. The dynamic evolution of $(\mathbf{k}_t, \mathbf{s}_t)$ is given separably for each of its elements by:

$$\frac{k_{i,t+1}}{b_i} = \frac{1}{c - \rho(\tilde{a}(\mathbf{b}))^2 \frac{k_{i,t}^2}{b_i^2}}; \quad \frac{s_{it}}{b_i} = \frac{1}{\tilde{a}(\mathbf{b})} \frac{\frac{\rho a^2}{c}\frac{k_{it}}{b_i}}{1 - \frac{\rho a^2}{c}\frac{a^2}{\tilde{a}(\mathbf{b})^2}\frac{k_{it}^2}{b_i^2}} \; i = 1, \ldots, I. \tag{2.11}$$

where $\tilde{a}(\mathbf{b})$ is defined in (2.9).

Part D. The low non-autarkic steady state, $(\mathbf{k}^*, \mathbf{s}^*)$, is locally dynamically stable, and the high one, $(\mathbf{k}^{**}, \mathbf{s}^{**})$, dynamically unstable. It follows that the stable (unstable) steady state values of human capital and social networking increase (decrease) with average cognitive skills and their coefficient of variation.

Part E. If human capital does not fully depreciate and instead is subject to partial depreciation at rate δ_k, human capital k_{it} evolves according to

$$k_{i,t+1} = k_{f,it} + (1 - \delta_k)k_{it},$$

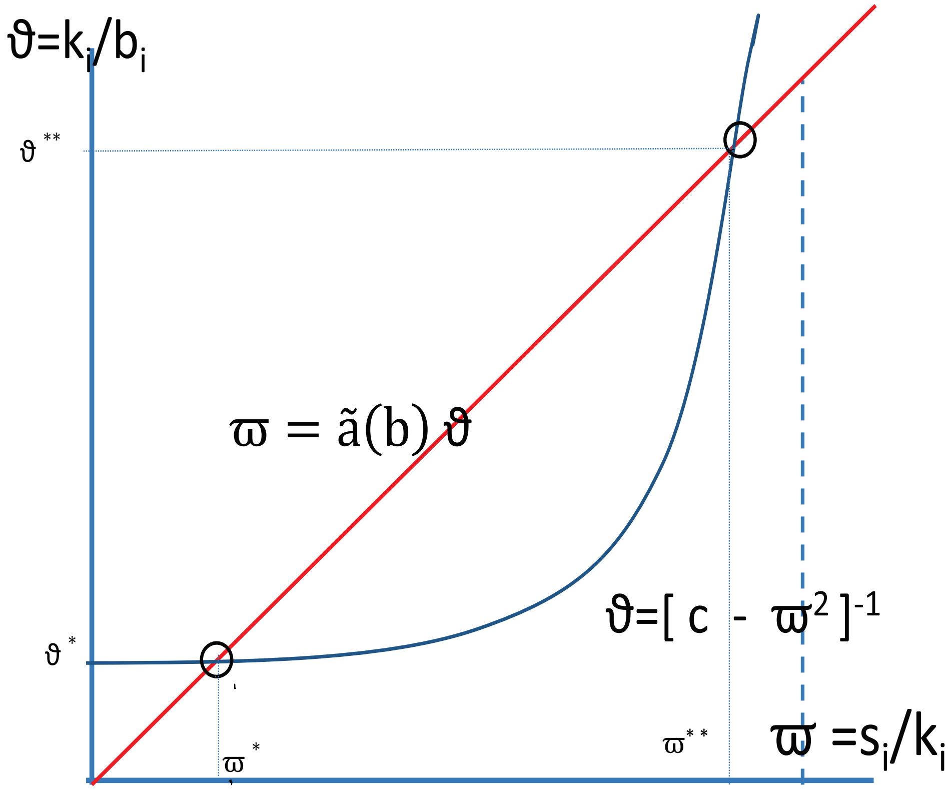

Figure 1: Solution of System (2.7–2.8) for (ϖ, ϑ)

where $k_{f,it}$ denotes the flow investment in period t, and the adjustment costs of investment in Equation (2.1) become $(c/2)k_{f,it}^2$. Optimizing with respect to the sequence of flow investments $\{k_{f,it}\}_{t=0}^{\infty}$ leaves Equation (2.5) unaffected, but Equation (2.4) hold with the modification that $(1-\delta_k)\mathbf{I}$ is added to its RHS.

Part F. If the utility per period (2.1) is modified so as the adjustment cost for networking effort becomes $(\varsigma/2)s_{it}^2$, then we have (a/ς) instead of a in Equation (2.5) and the remainder of the analysis is adjusted accordingly.

Remarks. Proposition 1, Part C shows that the dynamic evolution of k_{it} is autonomous, but that of s_{it} depends on k_{it}. Part D, corrects a claim of CC-AZ, namely that both non-autarkic equilibria of their static model are stable. See also Canen *et al.* (2020, p. 35, fn. 52), who also offer an analytical argument to demonstrate the instability property of the numerically larger steady state equilibria.[7] Figure 2.1 depicts the solutions for (ϖ, ϑ).

For the consequences of the convexity of the time map of human capital with respect to k_{it}, see Becker, Kominers, Murphy and Spenkuch (2018). Part C above predicts that human capital of the child $k_{i,t+1}$ increases, in view of (2.9), with the mean and the coefficient of variation of the cognitive skills of individuals in their social environment, and so does social networking s_{it}. Both effects work via the adjusted social skill coefficient $\tilde{a}(\mathbf{b})$ in the RHS of Equation (2.11) above. That the dispersion also matters follows directly from the complementarity associated with the synergistic effect in (2.1). This prediction is reminiscent of findings of Agostinelli (2018), most recently; *ibid.*, section 3.4. Also, Equation (2.11) implies that the child's human capital also increases with the human capital of the parent.

Part F aims at making up partially for not formally allowing for actions and socialization efforts to be subject to a constraint, which is unfortunately analytically intractable, as shown in the Online Appendix A, section A.1. By introducing instead an additional parameter, ς, in the adjustment cost of the networking effort, $(\varsigma/2)s_{it}^2$ in (2.1), we may think of competition for resources with human capital investment being handled by means of changes in ς vs. changes in c, both of which are parameters. This modification affects only Equation (2.7), which becomes

$$\vartheta = \varsigma \tilde{a}(\mathbf{b})^{-1} \varpi. \tag{2.12}$$

Comparative statics readily follow from Proposition 1, as the new parameter ς has effects in the opposite direction than the social skills coefficient a. A smaller value of ς, as for example one would associate with the proliferation of information and communication technologies (ICT), increases the optimal stable solutions for (ϑ^*, ϖ^*) and reduces the unstable ones, $(\vartheta^{**}, \varpi^{**})$. Both changes accord with intuition. If human capital is interpreted as schooling, the increased costs of educational services (the "Baumol effect") may be proxied by increasing c whereas increased ICT adoption by decreasing ς, the effects on the optimal solutions for the auxiliary variables (ϑ, ϖ) and therefore on optimal human capital and socialization efforts go in opposite directions. A lower such cost ς increases the effective value of synergy resulting in both human capitals and networking efforts to increase at the steady state.

A variation of the steady state version of this theory is consistent with an empirical finding of Agostinelli (2018, p. 35) who confirms strong homophily results. Since the social network is a weighted one, I transform the connection weights so as to interpret them as connection probabilities. By normalizing the connection weights by the sum of all weights for each individual $g_{ij}(\mathbf{s})$ yields $\bar{g}_{ij} = \frac{g_{ij}(\mathbf{s})}{\sum_{j\neq i} g_{ij}} = \frac{s_j}{\sum_{j\neq i} s_j}$. In view of Part B above, namely that the steady state values of social networking are proportional to the respective b_i's, these optimal normalized networking weights, $\bar{g}_{ij}$, are given by:

$$\bar{g}_{ij} = \frac{b_j}{\sum_{j\neq i} b_j}. \tag{2.13}$$

Thus individual i is relatively *more* likely to be connected with individual j the *greater* her relative cognitive skills coefficient, $\frac{b_j}{\sum_{j\neq i} b_j}$. This implies *assortative networking*, in effect a form of *homophily*: those with *greater* cognitive skills coefficients are more *influential*.

2.2 Dynamics of the Evolution of Human Capitals with Exogenous Social Connections

Assuming that social connections are exogenous leads to novel results. Exogenous social connections may also be interpreted as social immobility: individuals may not choose their social milieus. The evolution of human capital, given social connections, allows us to contrast with, as well as highlight, the properties with endogenous connections. Assuming that $\{\mathbf{s}_t\}_{t=1}^{\infty}$ is taken by all agents as exogenous and given, and taking $\{\mathbf{k}_{-i,t}\}_{t=1}^{\infty}$, as given, individual i chooses $\{\mathbf{k}_{i,t}\}_{t=1}^{\infty}$ so as to maximize lifetime utility according to (2.1). Under the assumptions of Nash equilibrium, human capitals satisfy the system of difference equations (2.4). To see this clearly let us assume that both $\mathbf{s}_t$ and $\mathbf{b}_t$ are time-invariant, $\mathbf{s}$ and $\mathbf{b}$. Then, for a large number of agents (2.4) admits a steady state, given by:

$$\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right] c\mathbf{k} = \mathbf{b}. \tag{2.14}$$

Since $\frac{a}{c}\mathbf{G}(\mathbf{s})$ is symmetric and positive, all of its eigenvalues are real. It has a maximal simple eigenvalue, which is positive, and larger in absolute value than all its other eigenvalues. Then, by Debreu and Herstein (1953, Theorem III), $\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1}$ exists and is positive, if and only if the maximal eigenvalue of $\frac{a}{c}\mathbf{G}(\mathbf{s})$ is less than 1. The maximal eigenvalue of $\frac{a}{c}\mathbf{G}(\mathbf{s})$ is given in *closed-form* by $\frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}$ and corresponds to $\mathbf{s}$ as an eigenvector of $\frac{a}{c}\mathbf{G}(\mathbf{s})$ [*c.f.* CC-AZ, Lemma 3, p. 353]. Thus, the condition on the maximal eigenvalue becomes:

$$\frac{\tilde{a}(\mathbf{s})}{c} < 1, \tag{2.15}$$

where $\tilde{a}(\mathbf{s})$ is defined by (2.9), with $\mathbf{s}$ taking the place of $\mathbf{b}$. That is, $\tilde{a}(\mathbf{s}) := \rho a\frac{\overline{x^2}(\mathbf{x})}{\overline{x}(\mathbf{x})}$, with the auxiliary functions $\overline{x^2}(\mathbf{s})$ and $\overline{x}(\mathbf{s})$ defined as:

$$\overline{x}(\mathbf{x}) := I^{-1}\sum_i x_i,\ \overline{x^2}(\mathbf{x}) := I^{-1}\sum_i x_i^2. \tag{2.16}$$

This feasibility condition limits the value of the social skills coefficient a, as adjusted by the coefficient of variation of the given networking efforts across agents, $\mathbf{s}$, relative to the investment cost parameter. If (2.15) is not satisfied, no positive steady state exists for the human capitals because spillovers are too strong.

Furthermore, if (2.15) holds, $\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1} = \mathbf{I} + \frac{a}{c}\frac{1}{1-\frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\mathbf{G}(\mathbf{s})$. and therefore the unique steady state solution of (2.4) becomes:

$$\mathbf{k}^* = \frac{1}{c}\mathbf{b} + \frac{a}{c^2}\,\frac{1}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\,\mathbf{s}\cdot\mathbf{b}\,\frac{\mathbf{s}}{\sum_i s_i}. \tag{2.17}$$

The mean human capital, $\bar{k}^*$ readily follows:

$$\bar{k}^* = \frac{1}{c}\bar{b} + \frac{a}{c^2}\,\frac{1}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\,\frac{\mathbf{s}\cdot\mathbf{b}}{I}. \tag{2.18}$$

Human capitals at the steady state are given as the sums of two components: the first is the autarkic term; the second reflects the effects of social interactions by being proportional to the relative networking effort $\frac{\mathbf{s}}{\sum_i s_i}$ and scaled by the term $\frac{a}{c^2}\frac{1}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\frac{\mathbf{s}\cdot\mathbf{b}}{I}$, which is common to all individuals. The larger is the social skills coefficient, or the more dispersed networking efforts are, the larger is $\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}$ (provided that the feasibility condition (2.15) is satisfied), the larger is the second term in (2.17). These effects are reflected in the expression for the variance[8] of human capitals at the steady state that readily follows from (2.17):

(2.19)

$$\mathrm{Var}_{\mathbf{k}^*} = \frac{1}{c^2}\mathrm{Var}_{\mathbf{b}} + \frac{1}{c^2}\left(\frac{\frac{a}{c}}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\,\bar{b}[\mathrm{Corr}_{\mathbf{b},\mathbf{s}}+1]\right)^2\mathrm{Var}_{\mathbf{s}} + \frac{2}{c^2}\frac{1}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\,\bar{b}[\mathrm{Corr}_{\mathbf{b},\mathbf{s}}+1]\mathrm{CoVar}_{\mathbf{b},\mathbf{s}}.$$

Equation (2.19) implies that the variance of human capitals, holding $\bar{s}, \mathrm{CoVar}_{b,s}$ and $\mathrm{Corr}_{b,s}$ constant, is increasing in the variance of s, which enters the expression via $\tilde{a}(\mathbf{s})$. Working in like manner, we obtain the following expression for the covariance of human capital and social networking $\mathrm{CoVar}_{\mathbf{k},\mathbf{s}}$:

(2.20)
$$\mathrm{CoVar}_{\mathbf{k},\mathbf{s}} = \frac{1}{c}\mathrm{CoVar}_{\mathbf{b},\mathbf{s}} + \frac{1}{c}\bar{s}\bar{b}\frac{\frac{\tilde{a}(\mathbf{s})}{c}-\frac{a}{c}}{1-\frac{\tilde{a}(\mathbf{s})}{c}}.$$

When the social networking efforts are exogenous, their given values contribute directly to the dispersion of human capitals across the population in the form of additional and indeed arbitrary terms that involve the covariance and correlation between the cognitive effects coefficients and the networking efforts, and the variance of the socialization efforts. In contrast, it is optimizing also over networking efforts, which in view of homogeneity of the weights of social networking functions, defined in (2.2), that renders both human capitals and social networking efforts proportional to the respective b's, thus limiting the dispersion of human capitals. In other words, the consequence of social immobility is proxied by the last two terms in RHS of (2.19). However, the exact comparison involves the magnitude of the endogenous quantity ϑ, according to Proposition 1, Part B: in the fully optimal case, the variance of human capitals is equal to $\vartheta^2\mathrm{Var}_{\mathbf{b}}$, which exceeds the autarkic one, the first term in the RHS of (2.19) because $\vartheta > c^{-1}$. Furthermore, from Proposition 1, ϑ is increasing in

$\mathrm{CV}_{\mathbf{b}}$. With the vector of networking efforts $\mathbf{s}$ being arbitrary (but subject to the feasibility condition (2.15)), one can find, in general, a vector of networking efforts such as $\mathrm{Var}_{\mathbf{k}^*}$ exceed the fully optimal value $\vartheta^2\mathrm{Var}_{\mathbf{b}}$.

The effect of optimizing networking efforts on the dispersion of human capitals and possibly in an unanticipated direction has an intuitive interpretation as an instance of the LeChatelier Principle [Samuelson 1960; Milgrom 2006]. When networking efforts are given, human capital investments are less responsive to the respective cognitive skills coefficients, than when they are optimized. The respective partial derivatives are ϑ and c^{-1}, where $\vartheta > c^{-1}$. As an analogy, consider cost functions when returns to scale are constant. If an input is held fixed, say capital, the marginal cost function is increasing in output. If all inputs may vary, then it is constant, which corresponds to infinite output supply elasticity.The elasticity of $k_{i,t}$ with respect to b_{it} at the steady state is equal to 1, and thus greater than its counterpart elasticity from (2.17), which is clearly less than 1.

Without optimization over social networking, Equation (2.5) are not part of the first-order conditions, Equation (2.4) does not reduce to (2.11), and no equilibrium multiplicity arises. Given social networking efforts, human capitals are uniquely defined on the transition to and at the steady state under exogenous social connections weights. Allowing for heterogeneity in parameter a, the social competence or social skills coefficient, or for its stochastic dispersion across the population, which I explore in section 3.4 further below, adds an additional exogenous source of dispersion in the evolution of human capitals.

Another perspective on exogenous social connections is to recognize that the adjacency matrix $\mathbf{G}(\mathbf{s})$ may be specified so as to study the consequences of implementing social or educational policy. E.g., the interaction weights may express education "tracking," where students with greater cognitive skills are more likely to interact with other students with high cognitive skills, or in numerous other ways by specifying the terms in Equation (2.2) as general functions of $\mathbf{b}$. This could of course be at variance with individually optimized social networking, which would be the intention of the design. Yet one can imagine designing a system that recognizes individuals' decision making and provides them with incentives to bring about a desirable outcome.

2.3 A Two-Overlapping Generations Model of Intergenerational Transfers and Social Connections

Overlapping-generations models facilitate linking generations via *intervivos* transfers while allowing via their richer demography to distinguish between the evolution of an individual's human capital over her own life cycle and that of her descendant's in each successive generation. A dynamic programming formulation using value functions that follows makes the model fully forward-looking, unlike formulations that simply include transfers as variables in the flow utility function.

Specifically, let subscripts y, o refer to individuals when they are *y*oung, *o*ld, respectively, and let time subscripts refer to when the respective quantity is operative. Member i of the generation born at t receives a transfer $k_{y,i,t}$ from her parent when young; she herself benefits at time t from social connections chosen by her parent's generation: $\mathbf{s}_{o,t}$. Her cognitive skills coefficients are given: $(b_{y,i,t}, b_{o,i,t+1})$. She chooses human capital investment and networking effort $(k_{o,i,t+1}, s_{y,i,t})$; she and her own generation benefit from $\mathbf{s}_{y,t}$ in time $t+1$. She chooses an endowment to her child in the form of human capital, $k_{y,i,t+1}$, and networking effort, $s_{o,i,t+1}$, from which her child benefits in the first period of her life at time $t+1$. The distinction between $k_{y,i,t}$ and $k_{o,i,t+1}$ is not possible within the Ramsay-Cass-Koopmans model of section 2.1, nor is it possible to distinguish between inequality over the life cycle within a generation from intergenerational inequality.[9]

I assume that the resource cost of investment $k_{o,i,t+1}$ is incurred in period t, but the adjustment costs is incurred in $t+1$ (when the benefits are also realized); consistently, the resource cost of $k_{y,i,t+1}$ is incurred in period $t+1$, but the parent anticipates that the adjustment costs are incurred by the child in $t+1$.

It is important to clarify the relevant peer groups underlying this formulation. With two overlapping generations, I may define the peer groups for young generation t at time t as the members of the preceding generation who were born at $t-1$ and are old at time t. In other words, the members of generation t benefit in period t from the human capitals $\mathbf{k}_{o,t}$ and the social networking efforts of their parents' generation, $\mathbf{s}_{o,t}$. When they themselves are old in period $t+1$ they benefit from the human capitals and social networking efforts the members of their own generation themselves decided on, $\mathbf{k}_{y,t}, \mathbf{s}_{y,t}$. In their first-period decisions about social connections, individuals are aware of the fact that they themselves would benefit from their own social networking efforts when they are old; in their second-period decisions about social connections, they are aware of the fact that their children would benefit from their own second-period social connections when their children are young. Therefore, both types of second-period decisions are in effect intergenerational transfers of capital and social connections. In the absence of uncertainty, all decisions are of course made simultaneously, but being explicit about "timing" of networking efforts would be crucial with sequential resolution of uncertainty, when such uncertainty is introduced, as in section 3.2 below.

The overlapping generations formulation of the life cycle optimization problem confers the advantage of readily accommodating the distinction between *altruism* and *paternalism* in discounting intergenerational linkages. Specifically, modifying the discount factor ρ of $\mathcal{V}_i^{[t+1]}$ in Equation (2.21) below, which individuals use to discount the lifetime utility of their offspring, allows them to distinguish it from the discount factor that they themselves apply to their own second-period utility, in a way which would be in line with the formulation of altruism by Abbott *et al.* (2019). Expressing paternalism in a manner identical to that of Abbott *et al.* is harder. Still, intuitively our modeling of transfer $k_{y,i,t}$ that an individual receives from her parent may be interpreted as a paternalistic *intervivos* transfer.

The pattern of interactions allowed by the model can distinguish between vertical and oblique transmission of a social environment. Networking effort $s_{o,i,t+1}$ by parent i when she is old at time $t+1$ confers benefits to her child at time $t+1$, which may be interpreted as *vertical transmission*; it enters the social component in her child's utility function when the child is young at $t+1$. It also functions as *oblique transmission*; it enters the social component of the utility functions of the members of her child's generation, via the terms $g_{ji}(\mathbf{s}_{o,t+1})$.

In sum, the decision problem for a member of generation t, born at time t, is to choose $\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}$, given $\{k_{y,i,t}, \mathbf{k}_{o,t}, \mathbf{s}_{o,t}\}$, namely human capital an individual receives at birth, the vector of human capitals of her parents' generation, and their networking efforts, all in time period t. I obtain first-order conditions for each generation's decision variables by first defining the value functions and using the envelope property. The results are summarized in Proposition 2; the proof is in the Online Appendix, section A.2.

Proposition 2.

Part A. The value function for individual i as of time t, $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{k}_{o,t}, \mathbf{s}_{o,t})$, which involves the value function for her child as of time $t+1$, $\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{k}_{o,t+1}, \mathbf{s}_{o,t+1})$, is defined as follows:

$$\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{k}_{o,t}, \mathbf{s}_{o,t}) \tag{2.21}$$

$$= \max_{\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t}, s_{o,i,t+1}\}} \Bigg\{ b_{y,i,t} k_{y,i,t} + a \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t}) k_{y,i,t} k_{o,j,t} - \frac{1}{2} c k_{y,i,t}^2 - \frac{1}{2} s_{y,i,t}^2 - k_{o,i,t+1}$$

$$+\rho \left[b_{o,i,t+1} k_{o,i,t+1} + a \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t}) k_{o,i,t+1} k_{y,j,t} - \frac{1}{2} c k_{o,i,t+1}^2 - \frac{1}{2} s_{o,i,t+1}^2 - k_{y,i,t+1} \right]$$

$$+ \rho \mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{k}_{o,t+1}, \mathbf{s}_{o,t+1}) \Bigg\};$$

with the respective one for her child as of time $t+1$, $\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{k}_{o,t+1}, \mathbf{s}_{o,t+1})$ following by analogy (see Online Appendix for details), allow us to obtain first-order conditions with respect to $(k_{y,i,t+1}, k_{o,i,t+1})$, which once rewritten in vector form are:

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2} \mathbf{G}(\mathbf{s}_{y,t}) \mathbf{G}(\mathbf{s}_{o,t+1}) \mathbf{k}_{y,t} + \mathbf{b}^*_{y,t+1} + \frac{a}{c} \mathbf{G}(\mathbf{s}_{o,t+1}) \mathbf{b}^*_{o,t+1}; \tag{2.22}$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2} \mathbf{G}(\mathbf{s}_{o,t}) \mathbf{G}(\mathbf{s}_{y,t}) \mathbf{k}_{o,t} + \mathbf{b}^*_{o,t+1} + \frac{a}{c} \mathbf{G}(\mathbf{s}_{y,t}) \mathbf{b}^*_{y,t}, \tag{2.23}$$

where the adjusted cognitive skills coefficients vectors, defined as

$$\mathbf{b}^*_{y,t} \equiv \frac{1}{c} \left(\mathbf{b}_{y,t} - \frac{1}{\rho} \mathbf{1} \right), \quad \mathbf{b}^*_{o,t} \equiv \frac{1}{c} \left(\mathbf{b}_{o,t} - \frac{1}{\rho} \mathbf{1} \right),$$

are positive provided that $\mathbf{b}_{o,t+1} - \frac{1}{\rho}\mathbf{1} > 0, \mathbf{b}_{y,t+1} - \frac{1}{\rho}\mathbf{1} > 0$.

The first order conditions with respect to $(s_{y,i,t,}, s_{o,i,t+1})$ *are:*

$$s_{y,i,t} = \rho a k_{o,i,t+1} \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{y,t})}{\partial s_{y,i,t}} k_{y,j,t}; \tag{2.24}$$

$$s_{o,i,t+1} = \rho a k_{y,i,t+1} \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{o,t+1})}{\partial s_{o,i,t+1}} k_{o,j,t+1}. \tag{2.25}$$

Part B. The optimal paths for the human capitals $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$ *are given by:*

$$\mathbf{k}_{o,t+1} = \frac{1}{1 - \frac{a^2\rho}{c}\Upsilon_{y,t}^2}\mathbf{b}^*_{o,t+1}, \quad \mathbf{k}_{y,t+1} = \frac{1}{1 - \frac{a^2\rho}{c}\Upsilon_{o,t+1}^2}\mathbf{b}^*_{y,t+1}, \tag{2.26}$$

and for the networking efforts $(\mathbf{s}_{y,t}, \mathbf{s}_{o,t})$ *by:*

$$\mathbf{s}_{y,t} = a\rho\frac{\Upsilon_{y,t}}{1 - \frac{a^2\rho}{c}\Upsilon_{y,t}^2}\mathbf{b}^*_{o,t+1}, \quad \mathbf{s}_{o,t} = a\rho\frac{\Upsilon_{o,t}}{1 - \frac{a^2\rho}{c}\Upsilon_{o,t}^2}\mathbf{b}^*_{y,t+1}, \tag{2.27}$$

where the auxiliary variables $(\Upsilon_{y,t}, \Upsilon_{o,t+1})$,

$$\Upsilon_{y,t} \equiv \frac{\mathbf{s}_{y,t}\cdot\mathbf{k}_{y,t}}{I\bar{x}(\mathbf{s}_{y,t})}; \ \Upsilon_{o,t+1} \equiv \frac{\mathbf{s}_{o,t+1}\cdot\mathbf{k}_{o,t+1}}{I\bar{x}(\mathbf{s}_{o,t+1})},$$

encapsulate the dynamic counterpart of the social multiplier.

Part C. Sufficient conditions for the invertibility of $\mathbf{I} - \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ *and thus for the existence of meaningful steady state values* $(\mathbf{k}_y, \mathbf{k}_o)$ *of (2.22–2.23) are: the product of* $\left(\frac{a}{c}\right)^2$ *and of the largest eigenvalues of each of the positive matrices* $\mathbf{G}(\mathbf{s}_o), \mathbf{G}(\mathbf{s}_y)$ *be less than 1:*

$$\frac{1}{c^2}\tilde{a}(\mathbf{s}_y)\tilde{a}(\mathbf{s}_o) < 1. \tag{2.28}$$

Part D. The steady state solutions of (2.22–2.23) may be written out in closed form because

$$\left[\mathbf{I} - \left(\frac{a}{c}\right)^2\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\right]^{-1} = \mathbf{I} + \frac{\frac{1}{c^2}\tilde{a}(\mathbf{s}_y)\tilde{a}(\mathbf{s}_o)}{1 - \frac{1}{c^2}\tilde{a}(\mathbf{s}_y)\tilde{a}(\mathbf{s}_o)}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o), \tag{2.29}$$

with the matrix $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ *being defined via its* (i,j) *element as:*

$$\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)_{ij} = \frac{s_{y,i}}{\sum_\ell s_{y,\ell}} \frac{s_{o,j}}{\sum_\ell s_{o,\ell}} \sum_\ell s_{y,\ell} s_{o,\ell}. \tag{2.30}$$

Part E. If the vectors of cognitive skills coefficients are time-invariant, $(\mathbf{b}_y^*, \mathbf{b}_o^*)$, *and not too asymmetric a system of algebraic equations in two auxiliary variables that define the steady state values of* $(k_{y,i}, k_{o,i})$ *and* $(s_{y,i}, s_{o,i})$ *as proportional to* $(b_{y,i}^*, b_{o,i}^*)$ *and* $(b_{o,i}^*, b_{y,i}^*)$, *respectively, admit up to two sets of positive solutions. These define high-level and a low-level equilibria, from which the steady state values of human capitals and social networking efforts readily follow.*

The system of linear difference equations (2.22–2.23) is uncoupled with respect to $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$, given $(\mathbf{s}_{y,t}, \mathbf{s}_{o,t}, \mathbf{s}_{o,t+1})$. Their steady state solutions are thus easily characterized, in terms of $\left[\mathbf{I} - \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)\right]^{-1}$. Since the largest eigenvalue of $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ is bounded upwards by the product of the largest eigenvalues of $\frac{a}{c}\mathbf{G}(\mathbf{s}_o)$ and $\frac{a}{c}\mathbf{G}(\mathbf{s}_y)$ [Debreu and Herrstein (1953); Merikoski and Kumar (2006), Thm. 7, 154–155], the inverse exists, provided that the product of $\frac{a^2}{c^2}$ with the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and of $\mathbf{G}(\mathbf{s}_y)$ is less than 1. These eigenvalues exist in close form, just as in section 2.2 above.

A notable feature of the solutions from (2.29)–(2.30) for the values along the steady state of human capitals that individuals are endowed with in the first period of their lives and they themselves decide for the second period of their lives, $(\mathbf{k}_y^*, \mathbf{k}_o^*)$, are made up of components that involve the autarkic value, augmented by a common social multiplier, the coefficient of $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ in the RHS of (2.29) above, adjusted by a term that is common to all, $\sum_\ell s_{y,\ell}s_{o,\ell}$, times the product of the relative networking efforts employed by young and old agents, the term $\frac{s_{y,i}}{\sum_\ell s_{y,\ell}} \frac{s_{o,j}}{\sum_\ell s_{o,\ell}}$ in the RHS of (2.30) above. That is, the relatively better connected an individual throughout the life cycle, the more the autarkic values are amplified. Finally, the approach of section 2.2 above may again be employed to show that optimizing over social connections may reduce inequality in human capitals.

<u>Remarks</u>. Proposition 2, Part A suggests that the model is amenable to stability analysis along the lines of Proposition 1, Part C, above. Intuition suggests that the results would be

similar, with the lower level equilibria being stable and the higher level ones being unstable. Still, the algebra is more involved, as the first-period and second-period decisions are likely to be coupled.

Proposition 2, Part B shows that the optimal human capital paths track their respective adjusted cognitive skill vectors, $(\mathbf{b}^*_{y,t}, \mathbf{b}^*_{o,t})$, by time-varying factors of proportionality that are functions of all of the parameters of the model. These factors retain the interpretation of the solutions as social multipliers, now dynamic and being applied to the autarkic solutions, where the latter are now defined in terms of the adjusted cognitive skills coefficients. The time paths for the networking efforts similarly track the adjusted cognitive skills coefficients vectors but with different factors of proportionality. Unfortunately it is not possible to obtain further details on the determinants and time evolution of the auxiliary variables that define the dynamic social multipliers. Substituting from the above solutions into (2.22)–(2.23) gives a system of algebraic equations in $(\Upsilon_{y,t}, \Upsilon_{o,t})$ which are much too complicated to solve.

It is straightforward to relax the assumption of full depreciation of human capital an individual receives from his parent in the first period of his life: $k_{y,i,t}$ vanishes after $k_{o,i,t+1}$ is produced, as shown in Proposition 1, Part E, above. Also, population growth may also be introduced by modifying the intergenerational utility functions (2.21) and stipulating that the total amount of the transfer be distributed among all descendants accordingly.

2.3.1 More than Two-Overlapping Generations

This section briefly considers richer demographic structures. A minimum of three overlapping generations will be necessary to express Heckman's concern about allowing for at least two periods of investment in a child's cognitive and social skills. It is critical [see Cunha and Heckman 2007, and Cunha, Heckman and Schennach 2010] for the acquisition of cognitive and social skills coefficients to interact — there is dynamic complementarity among them — and investments at certain ages are more critical then at other ages. Moreover, these come earlier for cognitive capabilities, later for social capabilities, and vary depending on the particular biological capability. Three-overlapping generations is the minimum number of overlapping generations that allows for direct effects between grandparents and grandchildren. Heckman and Mosso (2014) emphasize, however, that there have to be at least four periods in individuals' lifetimes, with two periods for a child who makes no economic decisions but who benefits from parental investment in the form of goods, and two periods for the individual as a parent. I heed these points by first noting the analytics of the model can accommodate more than two generations. Further below in section 3.5 I allow for parents to invest in their children's cognitive skills coefficients.

In the case of three-overlapping generations,[10] that is when children coexist with their parents and their grandparents, an additional set of first-order conditions for the respective magnitudes associated with $y$outh, $a$dulthood and $o$ld-age, $(k_{y,i,t}, k_{a,i,t+1}, k_{o,i,t+2}; s_{y,i,t}, s_{a,i,t+1}, s_{o,i,t+2})$ will be derived. An individual born at t, will take as given $(k_{y,i,t}, s_{y,i,t})$ and choose

$$(k_{a,i,t+1}, k_{o,i,t+2}, k_{y,i,t+3}; s_{a,i,t+1}, s_{o,i,t+2}, s_{y,i,t+3}).$$

Intuitively, one would expect that the additional first-order conditions would introduce additional multiplicative terms to the matrix defining the dynamical system and additional terms multiplying the respective cognitive skills coefficients vectors. That is, the endowment of cognitive skills coefficients in each period of the life cycle introduce life cycle effects into the model, being weighted by the respective social interactions matrix, as in $\frac{1}{c}\frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t}$ in Equation (2.23) above. Given the pattern of recurrence, one can guess what the counterpart of (2.23) should look like. Since the respective endowments are not equal across time, steady state values for human capitals differ at different stages of the life cycle. It is therefore interesting that complicating the demographic structure of the model leaves tractable the structure that determines the dynamics of the model. Working through the derivations formally in order to derive the counterpart of (2.23) confirms, in fact, this intuition. More complex demographic structures would allow in principle for direct transfers from grandparents to grandchildren, a factor that Mare (2011) deems important for deeper understanding of intergenerational inequality.

3 Applications and Extensions

The demographic features of the model of the previous sections would become considerably richer if individuals' life cycles consist of additional periods and individuals were allowed to move across neighborhoods in the beginning of every period. One may associate each period in an individual's life cycle with a different site, each of which is characterized by a different social interactions matrix $\mathbf{G}_\ell, \ell = 1, \ldots, L$. Each agent's i contribution to the social interactions matrix of each site consists of a row and of a column. Row $i, g_{ij}, j \neq i$, expresses the interactions effects from other agents; column $i, g_{ji}, j \neq i$, expresses the interactions effects on all other agents. Endogenous choice of neighborhoods could easily be incorporated in the dynamic programming formulation of Proposition 2. The multiplicative structure of (2.22), where agents moving across sites is reflected on the coefficient of $\mathbf{k}_{y,t}$, which would now be made up of the product of the respective social interactions matrices, reflecting the effect of the three overlapping generations, and accordingly for many generations.

The models of the previous sections have several applications and extensions. Proposition 2 implies a prediction about social effects in the intergenerational wealth transfer elasticity imply an effect in the style of the Great Gatsby Curve: intergenerational wealth transfers are increasing in the inequality in the distribution of parents' human capital. The model of section 2.3 may be adapted to allow for random shocks in cognitive skills and social skills coefficients. One notable result is that the anonymized cross-section distribution of first-period human capital at the steady state may be characterized fully and shown to exhibit thick tails. A different thick tails result, namely one that applies to the joint distribution of human capitals, is obtained by means of additional assumptions about the components of the model of section 2.2. Both those thick tails results are presented in section 3.3. Finally, when Proposition 2 in section 2.3 is generalized to allow parents to invest in their children's cognitive skills an additional social multiplier emerges. I discuss these next.

3.1 Social Effects in Intergenerational Wealth Transfer Elasticities and the Great Gatsby Curve

Intergenerational wealth elasticities have attracted empirical attention. Englund *et al.* (2013) report estimated intergenerational wealth elasticities ranging between 0.296 and 0.410, based on regressions of log five-year average child's wealth against the log of five-year average parents' wealth for different age groups [*ibid.,* Table A.3], and 0.497 and 0.530, based on linear regressions [*ibid.,* Table 3]. Landersø and Heckman (2016) report locally linear regression estimates (for log incomes weighted with absolute incomes) showing intergenerational income elasticity increasing in parental income [*ibid.,* Figures 1 and 2].

Social effects on the elasticity of intergenerational wealth transfers are present when social networking is either endogenous or exogenous. They are generally not examined by the literature, with the exception of Durlauf and Sheshadri (2018). These authors focus on the residential community as the source of spillovers for human capital and skill formation. They argue that social influences on children create a nonlinear relationship between parental income and offspring income, so that increases in inequality, by altering the ways in which family income determines and interacts with social influences, reduce mobility.

I investigate next the predictions of the model for social effects in intergenerational wealth transfer elasticities. I obtain a Great Gatsby Curve-type result: the elasticity of intergenerational transfers is increasing in the inequality of human capitals of parents.

Working from Proposition 2, Equations (2.22), the elasticity of $k_{y,i,t+1}$, the transfer that an individual born at t makes to his child when the child is born at $t+1$, with respect to $k_{y,i,t}$, the transfer that the individual himself received from his own parent, is defined as:

$$\mathrm{EL}_{k_{y,i,t}}^{k_{y,i,t+1}} = \frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}} \frac{k_{y,i,t}}{k_{y,i,t+1}}.$$

However, it is also interesting to consider the cross elasticity of $k_{y,i,t+1}$ with respect to the human capitals of others:

$$\mathrm{EL}_{k_{y,j,t}}^{k_{y,i,t+1}}|_{j\neq i} = \frac{\partial k_{y,i,t+1}}{\partial k_{y,j,t}} \frac{k_{y,j,t}}{k_{y,i,t+1}}.$$

It is easiest to see the effect under the assumption that social networking is given. From (2.22) and (2.23), $\frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}} = \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{i,i}$. This expresses a trade off between the resource cost of the transfer, which is incurred by the parent in period $t+1$, the increase in utility the parent enjoys from the benefit to the child when the transfer is received in period $t+1$. This is why both adjacency matrices, $\mathbf{G}(\mathbf{s}_{y,t})$ and $\mathbf{G}(\mathbf{s}_{o,t+1})$, are involved in the expression for $\frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}}$.

From (2.22), applied for time t, I have that an individual with higher first-period cognitive skills coefficients $b_{y,i,t}$ receives a larger transfers from his parent, $\frac{k_{y,i,t}}{b_{y,i,t}} = \frac{1}{c}$. This in turns induces a change in his own transfer to his child, along the lines of the effects I just derived. Working in like manner I have that an increase in the parent's own second period cognitive skills coefficients $b_{o,i,t+1}$ leads from (2.22) to $\frac{\partial \mathbf{k}_{y,t+1}}{\partial b_{o,i,t+1}} = \frac{a}{c^2} G(\mathbf{s}_{o,t+1})_{col\ i}$, which leads in turn to a change in $s_{o,1,t+1}$, exactly as I analyzed earlier.

The properties of the intergenerational wealth elasticity are summarized by Proposition .3, whose complete statement and proof are given in the Online Appendix, section A.3. The expression for the intergenerational elasticity when networking efforts are endogenous, $\mathrm{EL}_{ii,en}$, from Proposition A.3, Part D, reproduced here for convenience,

$$\mathrm{EL}_{ii,en} = \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}^*_y)\mathbf{G}(\mathbf{s}^*_o)\right]_{ii} = \frac{a^2}{c^2}\frac{s^*_{y,i}s^*_{o,i}}{\sum_h s^*_{y,h}\sum_h s^*_{o,h}}\sum_h s^*_{y,h}s^*_{o,h},$$

readily leads to a Great Gatsby curve type of result, if we were to assume that the invariant cognitive effects coefficients are also equal to one another: $\mathbf{b}^* := \mathbf{b}^*_y = \mathbf{b}^*_o$. Then, if ϱ^* denotes the common coefficient of proportionality according to Proposition 2, Part E, that is $s^*_{,i} = s^*_{o,i} = \varrho^* b^*_i$, then the own intergenerational elasticity from (A.25) and in view of (2.9) becomes

$$\mathrm{EL}_{ii,en} = \frac{a^2}{c^2}\varrho^2\frac{(b^*_i)^2}{I\bar{b}}\tilde{a}(\mathbf{b}) = \frac{a^2}{c^2}\varrho^2(b^*_i)^2 I^{-1}a[1 + CV^2_{\mathbf{k}_y}],$$

where I used the fact that because of proportionality, $CV_{\mathbf{k}} = CV_{\mathbf{b}}$. Therefore, the elasticity of intergenerational mobility, interpreted as that of intergenerational wealth transfers (via human capital) from parents to their children when they are young, is increasing is the inequality of parents' human capitals, as measured by its coefficient of variation.[11] This finding is in the spirit of the literature on the Great Gatsby curve[12] [Krueger (2012); Corak (2013)]. However, it pertains to the prediction for the relationship between own intergenerational wealth transfer elasticity and inequality. A similar result may be obtained about the cross-elasticity.

The Great Gatsby curve shows that across countries the intergenerational earnings elasticity increases with inequality. In particular, Corak (2013, Figure 1), plots the intergenerational elasticity of earnings, against the Gini coefficient of earnings (after taxes and transfers), for a number of OECD countries. It shows that the greater the inequality of earnings the greater the intergenerational elasticity and therefore the less the mobility in terms of earnings.

Because fits reported by the literature are not particularly tight, despite the curve's popularity, it is quite possible that a host of other effects are present.[13] The evidence on the Great Gatsby Curve is normally reported from direct estimations of elasticities from regressions in terms of log of incomes. In that case, working with the logs of transfers in terms of the logs of cognitive effects would require a reformulation of the basic model but is likely to strengthen the results in favor of a stronger effect along the lines of the Great Gatsby curve. This finding is also akin to the spirit of other findings that have been motivated by the Great Gatsby curve literature. E.g., Narayan *et al.* (2018) explore several cases of intergenerational mobility, like pertaining to income, education, etc.

3.2 Human Capital and Intergenerational Transfers with Shocks to Cognitive Skills Coefficients and Exogenous Social Connections

The evolution of human capitals when the vectors of cognitive skills, $(\mathbf{b}_{y,t}, \mathbf{b}_{o,t})$, are assumed to be stochastic and social connections are exogenous may be studied as a linear-quadratic dynamic programming problem under uncertainty as the solution to an underlying Bayesian Nash game. This is a special case of the static model in Blume *et al.* (2015).[14] The introduction of stochastic shocks to skills coefficients fits nicely a point made by James

Heckman on the importance of distinguishing between heterogeneity and uncertainty in the determination of inequality.[15] Parents' attempts to deal with uncertainty determines vertical transmission, and also results in oblique transmission via spillovers, as discussed above in section 3.1. This extension leads to a thick tails result for the "anonymized" cross-sectional distribution of human capitals at the stochastic steady state.

Individual i born at time t realizes cognitive skills coefficients $b_{y,i,t}$, an exogenous random state variable, and a wealth transfer from her parent $k_{y,i,t}$, an endogenous state variable whose evolution is described in detail below. Individual i avails herself of social interactions in exactly the same way as in the deterministic model above, $\mathbf{s}_o$ by the old, and $\mathbf{s}_y$ by the young.

Given the realizations of $(b_{y,i,t}, k_{y,i,t})$, individual i's own second-period cognitive skills, $b_{o,i,t+1}$, a random variable to be realized in period $t+1$, is distributed, conditional on $b_{y,i,t}$, according to $N(b_{m,o,i} + \frac{\sigma_o}{\sigma_y}\rho_o(b_{y,i,t} - b_{m,y,i}), \sigma_o^2(1-\rho_o^2))$. The cognitive skills coefficients of individual i's child, denoted by $b_{y,i,t+1}$ and realized in period $t+1$, is assumed to follow an $AR(1)$ process,

$$b_{y,i,t+1} = \bar{b}_{y,i} + \rho_b b_{y,i,t} + \epsilon_{y,i,t+1}, \tag{3.1}$$

where $\bar{b}_{y,i}$ is constant, and the stochastic shock $\epsilon_{y,i,t+1}$ is IID across i,t with distribution $N(0,\sigma_\epsilon^2)$. The unconditional distribution of $b_{y,i,t}$ is $N(b_{m,y,i}, \sigma_b^2)$, where $\sigma_b^2 = \frac{1}{1-\rho_b^2}\sigma_\epsilon^2$. Thus, conditional on $b_{y,i,t}$, $b_{y,i,t+1}$ is distributed according to $N((1-\rho_b)b_{m,y,i} + \rho_b b_{y,i,t}, \sigma_\epsilon^2)$, where $b_{m,y,i} = \frac{1}{1-\rho_b}\bar{b}_{y,i}$. Let $\mathbf{b}_m = (\mathbf{b}_{m,y}, \mathbf{b}_{m,o})$, with $(b_{m,y,i}, b_{m,o,i})$, as the components of the respective vectors. The unconditional variance-covariance matrix of $\mathbf{b}_{y,t}$ is $\sigma_b^2\mathbf{I}$. Therefore, the conditional expectations $\mathcal{E}[b_{o,i,t+1}|b_{y,i,t}]$ and $\mathcal{E}[b_{y,i,t+1}|b_{y,i,t}]$ are known once $b_{y,i,t}$ is realized, and are sufficient to characterize the individual's decision problem, which admits a standard dynamic programming with a linear-quadratic utility per period under uncertainty. The utility per period is a special case of the basic model of a Bayesian Nash social interaction game in Blume *et al.* (2015, Theorem 1).

<u>Proposition 3.</u> *Individual i chooses second period human capital and transfer to her child,*

$(k_{o,i,t+1}, k_{y,i,t+1})$, *given the realization of* $b_{y,i,t}$, *and subject to uncertainty with respect to her own second period cognitive skills coefficients and her child's first period skills,* $(b_{o,i,t+1}, b_{y,i,t+1})$, *so as to maximize her expected utility.*

Part A. Reformulating the individual's decision problem of Proposition 2 under shocks to cognitive skills coefficients yields first-order conditions for $\mathbf{k}_{y,t}$,

$$\text{(3.2)} \quad \mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \frac{1}{c}\mathcal{E}[\mathbf{b}_{y,t+1}|t] + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathcal{E}[\mathbf{b}_{o,t+1}|t] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1},$$

and expected steady state means,

$$\text{(3.3)} \quad \mathbf{k}_y^* = \frac{1}{c}\left[\mathbf{I} + \frac{c^{-2}\tilde{a}(\mathbf{s}_o)\tilde{a}(\mathbf{s}_y)}{1 - c^{-2}\tilde{a}(\mathbf{s}_o)\tilde{a}(\mathbf{s}_y)}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\left[\frac{1}{c}\mathbf{b}_{m,y} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,o} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}\right].$$

The deviations $\Delta\mathbf{k}_{y,t} = \mathbf{k}_{y,t} - \mathbf{k}_y^*$, *obey multivariate normal stationary limit distributions with means* $\mathbf{0}$ *and variance covariance matrix* $\mathbf{\Sigma}_{y,\infty}$:

$$\text{(3.4)} \quad \mathbf{\Sigma}_{y,\infty} = \left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^2 \cdot \tilde{a}(\mathbf{s}_y)^2\tilde{a}(\mathbf{s}_o)^2}{1 - \left(\frac{a}{c}\right)^2 \tilde{a}(\mathbf{s}_y)^2\tilde{a}(\mathbf{s}_o)^2}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\frac{1}{c^2}\left[\rho_b\mathbf{I} + \rho_o\frac{\sigma_o}{\sigma_b}\frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right],$$

where the matrix $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ *is defined in (2.30) via its* (i,j) *element. The evolution of* $\mathbf{k}_{o,t}$, *and that of* $\Delta\mathbf{k}_{o,t} = \mathbf{k}_{o,t} - \mathbf{k}_o^*$, *are obtained in like manner.*

Part B. The "anonymized" cross-sectional distribution of first-period human capitals at the stochastic steady-state, the $k_{y,i}$*'s, that is when individuals' identities are not distinguished, is given by a mixture of univariate normals, with weights equal to* I^{-1}, *the relative proportion of agent types in the population, with mean and variance given, respectively, by:*

$$\text{(3.5)} \quad Mean_{k_y} = \frac{1}{I}\sum_i k_{y,i}^*,\ \ Var_{k_y} = \frac{1}{I}\left[\left(\sum_i k_{y,i}^*\right)^2 - \sum_i (k_{y,i}^*)^2\right] + \frac{1}{I}\mathrm{trace}\left(\mathbf{\Sigma}_{y,\infty}\right).$$

The respective anonymized cross-sectional distribution of second-period human capitals as well as that of the joint distribution of the first-period human capital individuals receive and the transfer they make to their children are obtained in like manner.[16]

Remarks. The conditional expectations on the RHS of Equation (A.27) are expressed in terms of $\mathbf{b}_{y,t}$, and are thus known once the $b_{y,i,t}$'s are realized. This allows us to solve out for these expectations and rewrite (A.27) in the form of (A.29). By using the envelope theorem in the derivations of Part A, the cognitive skill of an individual's child, $b_{y,i,t+1}$, enters via its expectation only, while the inheritability parameter does enter the derivations. Online Appendix A, section A.4, provides additional details and a full statement and proof of Proposition 3, stated as Proposition A.4.

It is straightforward to generalize the above results if different individuals' first- and second-period cognitive skills coefficients are not independent and identically distributed draws from the same distribution. Even if the components of $(\mathbf{b}_y, \mathbf{b}_o)$ are independent and identically distributed random variables, the variance covariance matrices of human capitals display a lot of richness, because of the exogenous social interactions structure. It is also interesting to examine the dependence of the cross-section distribution of human capitals on the inheritability of cognitive skills coefficients.

3.3 Two Thick Tails Results

A salient feature of observed income and wealth distributions is that they exhibit thick upper tails. It is interesting to note that the "anonymized" cross-sectional distribution of first-period human capitals at the stochastic steady state, the $k_{y,i}$'s, may be seen as a convex mixture of normal densities. It thus exhibits heavier tails than normal[17], when compared with a mean-matched unicomponent distribution, and would not be unimodal if the means of the constituent distributions differ. Note, again, that the term in brackets above reflects the relative networking efforts of agents when they are young and old. Therefore, both the mean and variance increase with heterogeneity in social connection efforts, as measured by $\frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}$, other things being equal. But the expression for the variance includes an additional effect, the correlation of first- and second-period social interactions efforts in the numerator of the above expression. Thus, the greater this correlation, *cet. par.*, the greater the variance of the cross-sectional distribution at the steady state. Also, the greater is the social effects coefficient a, the larger are the elements of $\mathbf{\Sigma}_{y,\infty}$, according to (A.35) above.

A different thick tails result may be obtained for the joint distribution of human capitals. Under suitable assumptions, it obeys a multivariate Pareto law. The details of this result are presented in Online Appendix B, and rest on appropriate limiting stochastic assumptions about each of the two terms in the RHS of (2.4) above. That is, rewriting that system of difference equations as

$$\tilde{\mathbf{k}}_t = \mathbf{\Psi}_t + \tilde{\mathbf{G}}(\mathbf{\Phi}_t)\tilde{\mathbf{k}}_{t-1}, t = 1, \dots, \tag{3.6}$$

admits the interpretation that the first term denotes the direct effect of stochastic cognitive skills coefficients, while the matrix coefficient the second term, $\tilde{\mathbf{G}}(\mathbf{\Phi}_t)$, represents the stochastic effect of social networking. Online Appendix B establishes that under appropriate assumptions, the upper tails of the *joint* limit distribution of human capitals obey a Pareto law. The significance of the result lies in that a power law is obtained for a random *vector*, a novel result (to the best of my knowledge), and not for just a random scalar, as in the voluminous Pareto law literature.

The rough intuition of the result is that given a non-trivial initial value for the cognitive shocks $\mathbf{\Psi}_1$ in (3.6) and an arbitrary initial value for human capitals, $\tilde{\mathbf{k}}_0$, the dynamic evolution of human capital according to (3.6) keeps positive the realizations of human capital, while the impact of spillovers is having an overall contracting effect that pushes the realizations and thus the joint distribution of human capitals, too, towards 0. The distribution is prevented from collapsing at 0 by the properties of the contemporaneous cognitive shocks, Ψ_t, and from drifting to infinity by the contracting effect of the spillovers.[18]

The persistence of nontrivial dispersion results from the joint effect of two key requirements. First, it is assumed that there exists a positive constant κ_0, such that the expectation of the minimum row sum of the social interactions matrix raised to the power of κ_0 grows with the number of agents I faster than $\sqrt{I}$, roughly speaking; and second, the geometric mean of the limit of the sequence of norms of the social interactions matrix is positive but less than 1. The former condition ensures that there are sufficiently strong spillovers on each agent from all others and therefore no collapse occurs at the lower end of the distribution. The latter condition ensures the presence of a contracting effect at the upper end of the distribution.

3.4 Human Capital and Intergenerational Transfers under Shocks to Social Skills Coefficients and Exogenous Social Connections

The social competence, or social skills coefficient a, weights the value an individual attaches to social interactions, whose value reflects personality traits. I review next the results obtained by individualizing coefficient a and allowing it to differ over the life cycle, $\mathbf{a}_{it} = (a_{y,i,t}, a_{o,i,t+1})$, and treating them as random variables. The full analytical details are in the Online Appendix A, section A.5, in the form of Proposition A.5.

Accordingly, I redefine the value functions of Proposition 2, Part A, when choosing $(k_{o,i,t+1}, k_{y,i,t+1})$, and consequently the individual treats as random the value that she would derive from $k_{o,i,t+1}$ in her second period of her own life and the value accruing to her child from the transfer $k_{y,i,t+1}$. The former depends on the human capitals of others, $k_{y,j,t}, j \neq i$, which are known when she makes the decision at time t, but the effect depends on the realization of $a_{y,i,t+1}$. The latter depends on the cognitive skills coefficients of the child at time $t + 1$ and the realization of the social interactions effect $a_{o,i,t+2}$ at time $t + 2$. The proof is immediate.

Allowing for randomness in $\mathbf{a}_{it} := (a_{y,i,t}, a_{o,i,t+1|})$ does not change substantially the first-order conditions. In the special case when only the conditional means enter, the difference from the deterministic case is noteworthy only if the random variables $a_{i,t}$ were not IID over individuals and time. E.g., if $a_{i,t}$ is serially correlated over time, the system of first-order conditions becomes stochastic. It is also conceptually straightforward to allow for correlation between between $a_{i,t}$ and first-period cognitive skills, $b_{y,i,t}$, and therefore with $(b_{y,i,t+1}, b_{o,i,t+1})$, as well. Such a generalization may be accommodated by the tools employed by Proposition A.5, Online Appendix, section A.5. Although the derivations would not be trivial extensions of Proposition 5, they would still be tractable. The steady state means and variance covariance matrix would reflect the stochastic dependence parameters. Although solving the system of first-order conditions for the steady states is no longer so straightforward as before, the three sources of variation are clear. For $(k_{y,i}, k_{o,i})$, the respective period autarkic solution $\frac{1}{c}b_{o,i}$ is augmented by means of a component that reflects social interactions in both periods multiplicatively, adjusted by the mean social skills, and a component that reflects $\frac{1}{c}b_{o,i}$, adjusted by the social interactions weights associated with the second period in individuals' lifetimes, and by the mean social skills coefficients. Thus, individuals' social skills coefficients have spillovers on other individuals' behavior.

The stochastic structure for the $(\mathbf{a}_{y,t}, \mathbf{a}_{o,t+1})$ may be generalized to allow for persistent heterogeneity in addition to period-specific randomness. This would allow one to compare the empirical performance of such extensions of the model with alternative formulations that allow for amplification of social interactions effects either intergenerationally, as suggested by the results of Lindahl *et al.* (2015), or within and across social groups, as elaborated by Calvó-Armengol and Jackson (2004) and Ioannides and Loury (2004). A glimpse of the likely results from such extensions may be obtained by using the first-order conditions to predict how changes in the parameter values at time t would affect the future paths of the human capitals. Finally, endogenizing social connections in the presence of stochastic shocks to skills, either cognitive, social or both, would lead to a refinement of the social multiplier results. The endogenous social multipliers would likely reflect portfolio considerations in social networking. This deserves attention in future research.

3.5 Parents Investing in their Children's Cognitive Skills Coefficients

The models examined so far take cognitive and social skills coefficients as given. However, research by James Heckman and others has established that early age interventions may improve both types of skills. I examine the possible impact of such interventions by reformulating the model to allow individuals to invest in the cognitive skills coefficients of their children. This leads to a novel social multiplier effect associated with parents' investment in their own children's cognitive skills coefficients.

I modify the overlapping generations structure to assume that youth and adulthood last for two subperiods each, early youth and youth, and adulthood and old age, respectively. With t indexing subperiods, an adult at time t, who was born at time $t-2$ and is in her third subperiod of her life, gives birth to a child. The child lives for four subperiods, $t, t+1, t+2, t+3$, during the first two of which she overlaps with her parent who is still alive, and then lives on for two more subperiods. She in turn gives birth to her own child at time $t+2$, when she herself is an adult. Individuals make decisions affecting the household only in adulthood and old age. For a child born at time t, her cognitive skills coefficients when she become an adult at time $t+2$ are determined as a linear function, a special case of Heckman and Mosso (2014), of the given input at birth, $b_{y,i,t}$, and of investments $(\iota_{c1,t}, \iota_{c2,t+1})$:

$$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \beta_1 \iota_{c1,t} + \beta_2 \iota_{c2,t+1}, \tag{3.7}$$

where $\beta_0, \beta_1, \beta_2$ are positive parameters. The resource costs of these investments, which are decision variables, are incurred contemporaneously with the respective adjustment costs, in time periods t, and $t+1$, $\frac{1}{2}\gamma_1 \iota_{c1,t}^2$, and $\frac{1}{2}\gamma_1 \iota_{c2,t+1}^2$, respectively. The technical details of this formulation are stated as Proposition A.6, Online Appendix A, section A.6. See Agostinelli (2018) for empirical results along similar lines.

Optimal parental investments $(\iota_{c1,t}, \iota_{c2,t+1})$ turn out to be linear functions of $k_{y,i,t+2} + \rho k_{o,i,t+3}$. This in turn implies that the first order condition for $\mathbf{k}_{o,t+3}$ involves the matrix

$$\frac{a}{\rho^* c}\mathbf{G}(\mathbf{s}_{y,t+2}) \left[\mathbf{I} + \frac{\hat{a}}{c} \frac{1}{1 - \frac{\hat{a}}{ac}\tilde{a}(\mathbf{s}_{y,t+2})} \mathbf{G}(\mathbf{s}_{y,t+2}) \right] \frac{a}{\tilde{\rho} c_{cs}} \mathbf{G}(\mathbf{s}_{o,t+2}),$$

where $\rho^* \equiv 1 - \frac{\rho^2 \rho_\beta}{c}$, $\tilde{\rho} \equiv 1 - \frac{\rho \rho_\beta}{c_{cs}}$, $c_{cs} \equiv c - \rho\rho_\beta$, and $\hat{a} \equiv \frac{a\rho^2\rho_\beta}{\rho^* \tilde{\rho} c_{cs}} \left(1 - \frac{\rho^3 \rho_\beta^2}{\rho^* \tilde{\rho} c c_{cs}}\right)^{-1}$ are constants. It follows that the first-order condition for $k_{y,i,t+2}$ must reflect the influence that decision has, as implied by the optimization problem, on $b_{y,i,t+2}$. Since $b_{y,i,t+2} = b_{o,i,t+3}$ the utility per period from the last two subperiods of the child's lifetime contribute to the first-order conditions.

I note a key role for social networking, that has so far not been recognized, benefiting individuals when young. The product $\mathbf{G}(\mathbf{s}_{y,t+2})\mathbf{G}(\mathbf{s}_{o,t+2})$ is augmented by $\left[\mathbf{I} + \frac{\hat{a}}{c}\frac{1}{1-\frac{\hat{a}}{ac}\tilde{a}(\mathbf{s}_{y,t+2})}\mathbf{G}(\mathbf{s}_{y,t+2})\right]$. Intuitively, the option parents have to optimize their children's cognitive skills coefficients produces an amplification of the effects of social networking when the child is young and in the first subperiod of the child's life, which are expressed via $\mathbf{G}(\mathbf{s}_{o,t+2})$, magnified by

$$\frac{\hat{a}}{c}\frac{1}{1-\frac{\hat{a}}{ac}\tilde{a}(\mathbf{s}_{y,t+2})}\mathbf{G}(\mathbf{s}_{y,t+2}).$$

In contrast to Agostinelli (2018), it does not depend on social network endogeneity. A similar but equally hitherto unexplored effect would likely obtain if a parent may invest in improving her child's social skills coefficient. It is tempting to interpret the multiplier effects identified by this section as conceptually related to multiplier effects by Agostinelli *et al.* (2020). The latter effects are best seen as general equilibrium effects emanating from counterfactual experiments involving non-marginal changes in policy parameters. I see theirs and my results being in principle compatible.

4 Testable Implications

Social networking plays an important role in the paper as a determinant of inequality of human capital outcomes, but is hard to measure. It does vary across cultures, ethnic groups and nations, and to the extent that it could be analyzed by means of proxies for social structure, its impact on human capital accumulation may be assessed. The paper does suggest several testable propositions.

Human capital accumulation is affected by the properties of the social structure at several stages over the life cycle. The paper predicts that reduced frictions in social networking may reduce inequality in human capital and make it more sensitive to the endowment of cognitive skills. In other words, greater influence over "whom you know" may leverage the consequences of "what you know." The simple model of networking may be augmented in numerous ways that may account for richer dependence of inequality on social interactions.

A particularly promising route is to model neighborhood choice, with different concepts of neighborhood being allowed, as a way to control for a component of social networking. The importance of variations in the quality of the residential neighborhood of upbringing and of the intensity of exposure to neighborhood effects appear to differ across countries. They are not as important in Sweden [Adermon *et al.* (2021)], but very important in the U.S. [e.g., Chetty and Hendren (2018)]. The total effect of the extended family, horizontally as well as vertically construed, may also be distinguished from that of the residential neighborhood, as patterns of familial living arrangements differ markedly across countries.

It is straightforward to test the predictions of Proposition 1, Part C, that children's human capital child $k_{i,t+1}$ increases not only with the mean but also the coefficient of variation of the cognitive skills of individuals in their social environment. The potential impact on social networking s_{it} could also be explored.

Another prediction of the paper is about the determinants of the elasticity of intergenerational transfers with respect to inequality in underlying parameters. Data exist, such as from the Panel Study of Income Dynamics, that would allow linking intergenerational transfers to a wide range of individual and contextual social effects. Earlier research, e.g. Altonji *et al.* (1997), has examined the determinants of inter vivos transfers but not the particular effects suggested by the present paper.

The paper predicts that social multipliers are relevant not only for human capital accumulation but also for interventions aimed at improving skills, cognitive as well as social, when they take place. Existing research on such interventions could be seen through such a lens as well as extended accordingly. Such a role of the social context deserves special attention.

Thick tails in the distribution of income and wealth have received attention, but their presence has not been linked to social interactions. While most such studies aim at univariate distributions, the particular finding of the paper that pertains to joint distributions may be examined with data for entire social groups made up of identifiable individuals.

Lastly, in the context of empirical investigations, it is worth mentioning that exogeneity of social connections is not a serious drawback when social networking is not set in anticipation of magnitudes, like labor market outcomes and the like, when associated magnitudes are the objects of empirical study. This point is made by Blume *et al.* (2015). This also applies when such exogeneity proxies for social immobility.

5 Conclusions

The dynamic models analyzed by this paper offer a novel view of the joint evolution of human capital investment, intergenerational transfers and social networking. In the dynamic models with overlapping generations, each individual receives a transfer from their parents in the first period of their lives and avail themselves of the social connections that their parents have chosen for the second period of their own lifetimes. They in turn choose their own second-period human capital, own second-period social connections, and transfer to their own children. The endogeneity of the social structure makes that analysis much richer. Still, the tools of the paper do allow us to study the underlying steady states for individuals' life cycle accumulation, intergenerational transfers, and social connections for themselves and for their children in great detail. The elasticity of the intergenerational transfer received by an individual is increasing in the intergenerational transfer received by the parent, exhibits rich dependence on social effects, and is positive and less than 1. It is also increasing in the inequality of parents' human capitals, a Great Gatsby curve type of result.

The paper thus offers a novel view of the consequences for inequality of the joint evolution, endogenous or exogenous, of social connections and human capital investments. It allows for intergenerational transfers of both human capital and social networking endowments in dynamic and steady-state settings of dynastic overlapping-generations models of increasing demographic complexity. The paper highlights the separable effects on human capital of dispersion of social networking efforts alone, when they are exogenous, as distinct from when they are jointly optimized with human capital. To the best of my knowledge, clarification of the significance of this simple decomposition is a new finding.

Indeed, the consequences for inequality of the endogeneity of social connections are underscored by examining outcomes when social connections are assumed to be exogenous. In that case, individuals' human capital reflect an arbitrarily more general dependence on social connections across individuals. The dependence highlights both the relative importance of "whom you know" as opposed to "what you know" in the determination of individual human capitals and their steady-state distribution. When individuals optimize over their social connections, their actions make up for the arbitrariness of outcomes and thus reduce dependence to a smaller set of fundamentals. Endogeneity of social connections ushers in multiplicity of equilibria, with low-level equilibria being stable and high-level ones unstable.

Several aspects of the present paper deserve further attention in future research. To name a few, it would be interesting to develop more general stochastic formulations, and, in addition, to fully explore the interfaces between social networking and neighborhood choice, where one must also account for clustering into different types of neighborhoods; another would be to allow individuals to learn from others' social competence; yet another would be to examine how the network formation process might be influenced by public policy. In this context, it would be particularly fruitful to explore introducing homophily, perhaps in the style of Canen *et al.* (2020). Modeling explicitly the acquisition of social jointly with cognitive skills is also interesting. Although no general theory of network formation is available, endogenous networks may be defined for those different classes of problems, all of which bear upon the emergence and persistence of inequality via the social structure.

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Notes

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[2]Albornoz, Cabrales, and Hauk (2014) develop a conceptually similar use of the Cabrales *et al.* model, but in a static context.

[3]https://addhealth.cpc.unc.edu/data/

[4]Abbott, Gallipoli, Meghir and Violante (2019) specify the intergenerational transmission of noncognitive skills by means of probabilities of transitioning into social terciles, conditional on mother's education and on the child's cognitive skills [*ibid.*, section 3.3]. However, the data reported by Abbott *et al.* (2019, Table G.4), which originate in the NLSY97 data set, make a powerful case for the joint effect of cognitive and noncognitive skills on college graduation rates.

[5] An important adaptation of CC-AZ by Canen *et al.* (2020) to the study of legislative activity does propose a dynamic model in the style of best-response.

[6]The term noncognitive skills would be more inclusive; see Deming (2017, p. 1601), who considers social and leadership skills in noncognitive skills. This modeling is also consistent with the formulation of wage regression equations by Heckman, Stixrud and Urzua (2006, T.1, p. 418).

[7]I was unaware of the Canen *et al.*'s argument, which is indeed set in a dynamic model of legislative activity and social interactions. I thank Antonio Cabrales who directed my attention to it after I had shared with him my own finding. Canen *et al.* work with an extension of the Cabrales *et al.* that allows for

homophily. Relatedly, I thank the editor, Iourii Manovskii, for bringing to my attention Merlino (2019), who also establishes the instability property of the numerically larger steady state and the stability of the numerically smaller, the Pareto-dominated one.

[8]This derivation relies on elementary and intuitive steps here, but it is an instance of anonymizing the cross-sectional distribution of a vector of random variables; see Proposition A.5, Part B, Online Appendix A, and section 2.6.3, Ioannides (2013). Relatedly, I note that for the purpose of the equations immediately below, I have abused notation to avoid proliferation of symbols and use Var, CoVar, CV, Corr as pertaining to components of the respective vectors as individual data samples.

[9] The assumption that individuals benefit when they are young from the social interactions environment that has been chosen by their parents is consistent with Agostinelli *et al.* (2020), who allow for parents to restrict whom their children can interact with through their parenting style decisions.

[10]In fact, Samuelson (1958) itself is cast in terms of three-overlapping generations.

[11]The term I^{-1} should be interpreted broadly as denoting as the share of a large number of identical types.

[12]The Great Gatsby curve plots the intergenerational income elasticity, that is, the elasticity of the child's income with respect to that of the parent, against a country's cross sectional income inequality. As Krueger puts it, "the points cluster around an upward sloping line, indicating that countries that had more inequality across households also had more persistence in income from one generation to the next." The naming is ironic, if not euphemistic, as Gatsby, the character in J. Scott Fitzgerald's novel, rises from low to high social status.

[13]Corak (2013, Figures 2 and 3) shows that in the United States, sons raised by top and bottom decile fathers are more likely to occupy the same position as their fathers. For sons of top (bottom) earning decile fathers, the probability that their sons' income fall in different deciles increases (decreases) with the income decile.

[14]There is an obvious interpretation of this formulation as individuals' seeking to form optimal risk portfolios, when the risks come from exposure to social networking. In this context, optimizing with respect to social connections, too, would yield an unconstrained portfolio problem, at the cost of giving up the attractive linear-quadratic formulation of the problem. This angle is not pursued further here. The model could be generalized in the style of Blume *et al.* (2015).

[15] See Heckman's presentation in the Lindau-Nobel 2017 Meetings: https://tinyurl.com/nmpprkzk

[16]I thank Vassilis Hajivassiliou for his help with the proof of Proposition A.4, Part E, Online Appendix.

[17]This claim rests on Shaked (1980) and applies to two-parameter exponential families of distributions, of which the normal is a member.

[18]The scalar counterpart of the conditions of Kesten's theorems have been extensively invoked in the economics literature. See Gabaix (1999, pp. 761–762), whose approach can be the starting point for linking the magnitude of the Pareto exponent approximating the upper tail to the parameters of the underlying distribution of interest.

Online Appendix to

ENDOGENOUS SOCIAL NETWORKS AND INEQUALITY IN AN INTER-GENERATIONAL SETTING

Yannis M. Ioannides

A Appendix A: Proofs of Propositions 1, 2 and 3, and Additional Supplementary Material

A.1 Proposition 1. Part C and D. Proof

Part C.

Condition (2.5) of Proposition 1 in the main text, when when taken at the limit of $I \to \infty$, may be written as:

$$s_{it} = a\rho k_{i,t+1} \frac{\mathbf{s}_t \cdot \mathbf{k}_t}{I\bar{x}(\mathbf{s}_t)}. \tag{A.1}$$

By using (2.4) to substitute for $k_{i,t+1}$, and defining the auxiliary variable

$$\Upsilon_t := \frac{\mathbf{s}_t \cdot \mathbf{k}_t}{I\bar{x}(\mathbf{s}_t)}, \tag{A.2}$$

(A.1) yields:

$$s_{it} = \frac{a\rho}{c} b_i \Upsilon_t + \frac{a^2\rho}{c} s_{it} \Upsilon_t^2.$$

Solving for s_{it} yields:

$$s_{it} = b_i \frac{\frac{a\rho}{c}\Upsilon_t}{1 - \frac{a^2\rho}{c}\Upsilon_t^2}. \tag{A.3}$$

Similarly, for $k_{i,t+1}$ from (A.1) and (A.3) we have:

$$k_{i,t+1} = \frac{a\rho}{c} b_i + \frac{a^2 \rho}{c} \Upsilon_t^2 k_{i,t+1}.$$

Solving for $\frac{k_{i,t+1}}{b_i}$ yields:

$$\text{(A.4)} \qquad \frac{k_{i,t+1}}{b_i} = \frac{\frac{1}{c}}{1 - \frac{a^2\rho}{c}\Upsilon_t^2}.$$

Thus, using the definition of Υ_t, (A.2), we obtain the law of motion for Υ_t:

$$\text{(A.5)} \qquad \Upsilon_t = \frac{1}{ac} \tilde{a}(\mathbf{b}) \frac{1}{1 - \frac{a^2\rho}{c}\Upsilon_{t-1}^2}.$$

Since the slope of the time map rises from being equal to 0, at $\Upsilon_{t-1} = 0$, to tending to infinity as $\Upsilon_{t-1} \to \frac{c}{a^2\rho}$, it follows that if there exist two positive steady states, the slope of the time map is less than 1 at the lower steady state, and greater than 1 at the higher one. Clearly, if two distinct positive steady states for Υ_t exist, $(\Upsilon^*, \Upsilon^{**})$, they satisfy

$$\Upsilon \leq \left(\frac{c}{a^2 \rho} \right)^{\frac{1}{2}}.$$

They correspond to the two positive roots of the cubic equation resulting by taking the law of motion at a steady state. A sufficient condition for existence may be obtained in the same way as in the static case. That is, the slope of the tangent from $(0,0)$ to the time map must have slope less than 1. The condition (2.10) is modified trivially, where instead of just $\tilde{a}$ we have $\rho\tilde{a}$.

The analysis of stability of the steady state for human capitals and socialization efforts, normalized by the respective cognitive effect, $\left(\frac{k_{it}}{b_i}, \frac{s_{it}}{b_i}\right)$, rests on the respective property of Υ_t. By linearizing Equations (A.3) and (A.4) around a steady state, we may write the deviations of these variables from their steady states in terms of the deviation of Υ_t from its steady state. That is:

$$\Delta\left(\frac{s_{i,t+1}}{b_i}\right)=\frac{\partial\left(\frac{s_{i,t+1}}{b_i}\right)}{\partial\Upsilon_{t+1}}|_{\{\Upsilon_{t+1}=\Upsilon^*\}}\Delta\Upsilon_{t+1};$$

$$\Delta\left(\frac{k_{i,t+1}}{b_i}\right)=\frac{\partial\left(\frac{k_{i,t+1}}{b_i}\right)}{\partial\Upsilon_{t}}|_{\{\Upsilon_{t}=\Upsilon^*\}}\Delta\Upsilon_{t}.$$

Since the RHS of (A.3) and (A.4) are increasing functions of Υ_t, the derivatives in the RHS of the linearized laws of motion for $\frac{s_{i,t}}{b_i}$ and $\frac{k_{i,t}}{b_i}$ above are positive and finite at the fixed points. It is thus clear that the dynamics of the evolution of $\frac{s_{i,t}}{b_i}$ and $\frac{k_{i,t}}{b_i}$ depend entirely on that of $\Delta\Upsilon_t$. From the discussion following Equation (A.5) above, the low steady states are stable and the high ones unstable. Q.E.D.

Part D.

We express the RHS of (A.5) in terms of $k_{i,t+1}$ by using (A.4). So, (A.5) becomes:

$$\Upsilon_t=\frac{\tilde{a}(\mathbf{b})}{ab_i}k_{i,t+1}.$$

Applying this for $t-1$ and using it back in (A.4) yields a time-invariant difference equation in $k_{i,t}$ alone:

$$\text{(A.6)}\qquad \frac{k_{i,t+1}}{b_i}=\frac{1}{c-\rho\tilde{a}^2\frac{k_{i,t}^2}{b_i^2}}.$$

Its dynamics are fully determined; see Proposition 1, Part C.

Finally, from (A.5) and (A.3) the dynamic evolution of s_{it} is entirely determined by that of Υ_t. The low non-autarkic steady state $\mathbf{s}^*$ is dynamically stable and the high one $\mathbf{s}^{**}$ dynamically unstable. However, the algebra of deriving the time map is more complicated. Whereas from (A.3), s_{it} is a well-defined function of Υ_t, inverting it in order to express Υ_t in terms of s_{it} leads to identifying it as the positive root of the associated quadratic equation in Υ_t. So, substituting back into (A.5) leads to a fourth-degree equation. Its properties may be examined by means of the implicit function theorem.

Imposing a constraint for actions and networking efforts is intractable unfortunately. Suppose, for example, that individual i is endowed with one unit of labor, of which s_i which may be allocated to networking effort and $p_k k_i$ to producing human capital. The individual's problem is otherwise the same except that it is subject to a constraint: $p_k k_i + s_i = 1$. Adjoining the constraint by means of a Lagrange multiplier λ_i leads to first-order conditions which now include the λ_i's. Working in the usual way to solve for the λ_i's leads to a linear equation for $\mathbf{\Lambda}$, the vector of λ_i's, which unfortunately does not imply a simple solution like in our treatment above and therefore does not simplify to allow us to derive useful conclusions. A simplification fails to occur because the formulation does not yield homogeneous equations which lead to solutions for human capitals and efforts as proportional to the cognitive skills coefficients. Specifically, the equation for $\mathbf{\Lambda}$ becomes:

$$\left[2\mathbf{I} + \frac{a}{c}\frac{1}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\mathbf{G}(\mathbf{s})\right]\mathbf{\Lambda} = \left[\mathbf{I} + \frac{a}{c}\frac{1}{1-\frac{\tilde{a}(\mathbf{s})}{c}}\mathbf{G}(\mathbf{s})\right]\cdot\frac{1}{c}\mathbf{b} + a\frac{\mathbf{s}\cdot\mathbf{k}}{I\bar{x}(\mathbf{s})}\mathbf{k} - \mathbf{1},$$

which is impossible to solve in closed form.

A.2 Proposition 2. Proof

The decision problem for a member of generation t, born at time t, is to choose

$$\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\},$$

given $\{k_{y,i,t}, \mathbf{s}_{o,t}\}$. We express the first-order conditions by first defining the value functions $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}), \mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})$, associated with an individual's lifetime utility when he is young at t and when he is old at $t+1$. That is:

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t})$$

$$= \max_{\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \left\{ b_{y,i,t}k_{y,i,t} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck_{y,i,t}^2 - \frac{1}{2}s_{y,i,t}^2 - k_{o,i,t+1}\right.$$

$$+\rho\left[b_{o,i,t+1}k_{o,i,t+1}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t}-\frac{1}{2}ck_{o,i,t+1}^2-\frac{1}{2}s_{o,i,t+1}^2-k_{y,i,t+1}\right]+\rho\mathcal{V}_i^{[t+1]}(k_{y,i,t+1},\mathbf{s}_{o,t+1})\right\}.$$

Correspondingly,

$$\mathcal{V}_i^{[t+1]}(k_{y,i,t+1},\mathbf{s}_{o,t+1})$$

$$=\max_{\{k_{o,i,t+2},k_{y,i,t+2};s_{y,i,t+1},,s_{o,i,t+2}\}}\left\{b_{y,i,t+1}k_{y,i,t+1}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{o,t+1})k_{y,i,t+1}k_{o,j,t+1}-\frac{1}{2}ck_{y,i,t+1}^2-\frac{1}{2}s_{y,i,t+1}^2-k_{o,i,t+2}\right.$$

$$+\rho\left[b_{o,i,t+2}k_{o,i,t+2}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{y,t+1})k_{o,i,t+2}k_{y,j,t+1}-\frac{1}{2}ck_{o,i,t+2}^2-\frac{1}{2}s_{o,i,t+2}^2-k_{y,i,t+2}\right]+\rho\mathcal{V}_i^{[t+2]}(k_{y,i,t+2},\mathbf{s}_{o,t+2})\right\}$$

Part A readily follows. The first-order conditions with respect to $(k_{o,i,t+1}, s_{y,i,t,}; k_{y,i,t+1}, s_{o,i,t+1})$ are, respectively:

(A.7)
$$k_{o,i,t+1}=\frac{1}{c}b_{o,i,t+1}+\frac{a}{c}\sum_{j\neq i}g_{ij}(\mathbf{s}_{y,t})k_{y,j,t}-\frac{1}{c\rho};$$

(A.8)
$$s_{y,i,t}=\rho a k_{o,i,t+1}\sum_{j=1,j\neq i}^{I}\frac{\partial g_{ij}}{\partial s_{y,i,t}}(\mathbf{s}_{y,t})k_{y,j,t};$$

$$-\rho+\rho\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial k_{y,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1})=0;$$

$$-\rho s_{o,i,t+1}+\rho\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial s_{o,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1})=0.$$

Using the envelope property, the partial derivatives of the value function above,

$$\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial k_{y,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1}),\ \frac{\partial\mathcal{V}^{[t+1]}}{\partial s_{o,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1})$$

are equal to the partial derivatives of the respective utility per period. That is, using the envelope property, the last two equations become:

(A.9)
$$k_{y,i,t+1}=\frac{1}{c}b_{y,i,t+1}+\frac{a}{c}\sum_{j\neq i}g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+1}-\frac{1}{c\rho};$$

(A.10)
$$s_{o,i,t+1} = \rho a k_{y,i,t+1} \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{o,i,t+1}} (\mathbf{s}_{o,t+1}) k_{o,j,t+1}.$$

We can summarize the first-order conditions for the $\mathbf{k}$'s in matrix form as follows.

(A.11)
$$\mathbf{k}_{o,t+1} = \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{y,t} - \frac{1}{c\rho}\mathbf{1};$$

(A.12)
$$\mathbf{k}_{y,t+1} = \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{o,t+1} - \frac{1}{c\rho}\mathbf{1},$$

where $\mathbf{1}$ is a $I-$ vector of 1's. From these we may obtain two single first-order difference equations: first in $\mathbf{k}_{y,t}$, by substituting for $\mathbf{k}_{o,t+1}$ from (A.11) in the RHS of (A.12), and then in $\mathbf{k}_{y,t}$, by substituting for $\mathbf{k}_{y,t}$ from (A.12) in the RHS of (A.11). That is, (2.22 – 2.23) in the main text follow, reproduced here as well for clarity:

(A.13)
$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{y,t} + \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\right]\mathbf{1}.$$

(A.14)
$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{o,t})\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{o,t} + \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\right]\mathbf{1}.$$

Part B. The proof of Proposition 1 motivates the definitions of the following auxiliary variables:
$$\Upsilon_{y,t} \equiv \frac{\mathbf{s}_{y,t}\cdot\mathbf{k}_{y,t}}{I\bar{x}(\mathbf{s}_{y,t})};\ \Upsilon_{o,t+1} \equiv \frac{\mathbf{s}_{o,t+1}\cdot\mathbf{k}_{o,t+1}}{I\bar{x}(\mathbf{s}_{o,t+1})}.$$

Using these definitions, (2.24–2.25) are simplified as follows:

$$s_{y,i,t} = a\rho k_{o,i,t+1}\Upsilon_{y,t},\ s_{o,i,t+1} = a\rho k_{y,i,t+1}\Upsilon_{o,t+1}.$$

Substituting back into (A.11–A.12) yields:

(A.15)
$$\mathbf{k}_{o,t+1} = \frac{1}{1-\frac{a^2\rho}{c}\Upsilon_{y,t}}\mathbf{b}^*_{o,t+1},\ \ \mathbf{k}_{y,t+1} = \frac{1}{1-\frac{a^2\rho}{c}\Upsilon_{o,t+1}}\mathbf{b}^*_{y,t+1}.$$

Part C. Since the largest eigenvalue of $\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ is bounded upwards by the product of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and $\mathbf{G}(\mathbf{s}_y)$ [Debreu and Herrstein (1953); Merikoski and Kumar (2006, Theorem. 7, 154–155)], the inverse exists, provided that the product of $\frac{a^2}{c^2}$ with the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and of $\mathbf{G}(\mathbf{s}_y)$ is less than 1. A sufficient condition for this is that the products of $\frac{a}{c}$ and each of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o), \mathbf{G}(\mathbf{s}_y)$ are less than 1.

Part D. We follow the line of proof in Cabrales *et al.* (2011, Lemma 3, p. 353), and explore whether $\left[\mathbf{I} - \left(\frac{a}{c}\right)^2 \mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\right]^{-1}$ may be written in close form. Writing out the generic element of the matrix product $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ yields

$$\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)_{i,j} = \frac{\sum_\ell s_{y,\ell} s_{o,\ell}}{I\overline{x}(\mathbf{s}_o)} \frac{s_{y,i} s_{o,j}}{I\overline{x}(\mathbf{s}_y)}.$$

For the higher powers of $\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ we use the symmetry of each of the matrices $\mathbf{G}(\mathbf{s}_y), \mathbf{G}(\mathbf{s}_o)$ and the result in *ibid.* to write for the generic element of $\mathbf{G}(\mathbf{s}_y)$ (and similarly for $\mathbf{G}(\mathbf{s}_o)$) as follows:

$$[\mathbf{G}(\mathbf{s}_y)]^2_{i,j} = \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)} [\mathbf{G}(\mathbf{s}_y)]_{i,j}.$$

Thus by trivial induction and provided that condition (2.28) in the main text holds, the power expansion for the above matrix converges and is given by (2.29) in the main text.

Part E. By applying equations (A.11), (A.42), (A.12), and (A.43) we have:

(A.16) $$ck_{o,i} = b^*_{o,i} + a s_{y,i} \sum_{j \neq i} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}};$$

(A.17) $$s_{y,i} = \rho a k_{o,i} \sum_{j=1, j\neq i}^{I} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}};$$

(A.18) $$ck_{y,i} = b^*_{y,i} + a s_{o,i} \sum_{j \neq i} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}};$$

$$s_{o,i} = \rho a k_{y,i} \sum_{j=1, j\neq i}^{I} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}}. \tag{A.19}$$

We define a pair of auxiliary variables, ψ_y, ψ_o,

$$\psi_y = \frac{1}{c - \rho a^2 \psi_o^2} \frac{\mathbf{b}_y^* \cdot \mathbf{b}_o^*}{I \overline{x}(\mathbf{b}_o^*)}; \tag{A.20}$$

$$\psi_o = \frac{1}{c - \rho a^2 \psi_y^2} \frac{\mathbf{b}_y^* \cdot \mathbf{b}_o^*}{I \overline{x}(\mathbf{b}_y^*)}, \tag{A.21}$$

where $\mathbf{b}_y^* \cdot \mathbf{b}_o^* = \sum b_{y,i}^* b_{o,i}^*$ does not depend on i. From (A.16) and (A.17), and (A.19) and (A.19), we have:

$$\rho k_{o,i}(c k_{o,i} - b_{o,i}^*) = s_{y,i}^2 = \rho^2 a^2 \psi_y^2 k_{o,i}^2;$$

$$\rho k_{y,i}(c k_{y,i} - b_{y,i}^*) = s_{o,i}^2 = \rho^2 a^2 \psi_o^2 k_{y,i}^2.$$

We may thus solve for $k_{y,i}, k_{o,i}$, and then by using the definitions of ψ_y, ψ_o, for $s_{y,i}, s_{o,i}$, we obtain solutions for $k_{y,i}, k_{o,i}$ and $s_{y,i}, s_{o,i}$ in terms of (ψ_y, ψ_o) and $(b_{y,i}, b_{o,i})$. Finally, substituting back into the definitions of ψ_y, ψ_o, yields third-degree equations in ψ_y, ψ_o, (A.20–A.21).

Equations (A.20–A.21) have at most two solutions in (ψ_y, ψ_o), provided that

$$\frac{\mathbf{b}_y^* \cdot \mathbf{b}_o^*}{I \overline{x}(\mathbf{b}_y^*)} < \frac{c}{a} \left(\frac{c}{\rho}\right)^{\frac{1}{2}}; \frac{\mathbf{b}_y^* \cdot \mathbf{b}_o^*}{I \overline{x}(\mathbf{b}_o^*)} < \frac{c}{a} \left(\frac{c}{\rho}\right)^{\frac{1}{2}}.$$

Q.E.D.

Equations (A.20–A.21) have at most two solutions in (ψ_y, ψ_o), which may be characterized easily but not solved for explicitly. The steady state values of all endogenous variables then follow. Note that the life cycle model is crucial for the result: ψ_y and ψ_o would be equal to one another, were it not for the fact that, $b_{y,i} \neq b_{o,i}$; first-period and second-period cognitive skills coefficients are in general not equal to one another. Similarly, interesting complexity and accordant richness follow if cognitive skills coefficients may be influenced by means of investment, which I explore in section 3.5 of the main text.

If we were to assume, as in section 2.2, that the social networking efforts are given exogenously, whose counterparts in the present case are those of young and of old agents, with values not necessarily coinciding with the steady state ones, then a number of additional results are possible. E.g, under the assumption that the social networking efforts are constant over time, $(\mathbf{s}_y, \mathbf{s}_o)$, the system of equations (2.22–2.23) implies that a single equation for aggregate capital $\mathbf{k}_t = \mathbf{k}_{y,t} + \mathbf{k}_{o,t}$, may be obtained. The dynamics are exactly the same as in each of the two systems and no further discussion is necessary.

Such results may be strengthened in the following way. Intuitively, as the number of overlapping generations increases, the matrix for human capitals in the laws of motion (2.22), (2.23), becomes the product of increasing number of factors. In the limit, as the number of overlapping generations tends to infinity, the product of stochastic matrices may be handled by means of standard techniques for products of matrices.

A.3 Proposition A.3

Proposition A.3 supplements section 3.1.

Proposition A.3. *The own elasticity of the transfer to a child, $k_{y,i,t+1}$, with respect to the transfer the parent herself received from her own parent, $k_{y,i,t}$, is given by*

Part A.

$$EL^{k_{y,i,t+1}}_{k_{y,i,t}} = \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{ii} \frac{k_{y,i,t}}{ELK_{i,t}(b_{y,i,t+1})}, \tag{A.22}$$

where: $ELK_{i,t}(b_{y,i,t+1})$

$$\equiv \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{row\, i}\mathbf{k}_{y,t} + \frac{1}{c}b_{y,i,t+1} + \frac{a}{c^2}\left[\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{row\, i}\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[1 + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})_{row\, i}\mathbf{1}\right].$$

Part B. The cross elasticity of the transfer to a child, $k_{y,i,t+1}$, with respect to the transfer *someone else's parent herself received from her own parent, $k_{y,j,t},\ j \neq i$, is given by*

$$\text{(A.23)} \qquad EL_{k_{y,j,t}}^{k_{y,i,t+1}}(b_{y,i,t+1}) = \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{ij}\frac{k_{y,j,t}}{ELK_{i,t}(b_{y,i,t+1})}.$$

Part C.

(A.24)

$$a:\ 0 < EL_{k_{y,i,t}}^{k_{y,i,t+1}} < 1,\ \frac{\partial}{\partial k_{y,i,t}}EL_{k_{y,i,t}}^{k_{y,i,t+1}} > 0; b:\ 0 < EL_{k_{y,j,t}}^{k_{y,i,t+1}} < 1,\ \frac{\partial}{\partial k_{y,j,t}}EL_{k_{y,j,t}}^{k_{y,i,t+1}} > 0.$$

The ratio of the own to the cross elasticity is given by: $\frac{k_{y,i,t}}{k_{y,j,t}}\frac{[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})]_{ii}}{[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})]_{ij}}$.

Part D. If the elasticity $\frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}}\frac{k_{y,i,t}}{k_{y,i,t+1}}$ *is evaluated in terms of deviations from the steady state of* $k_{y,i,t}$*, then (A.22) leads to simple expressions for both the cases of endogenous and exogenous social connections when the cognitive effects coefficients are time invariant:*

$$\text{(A.25)} \qquad EL_{ii,en} = \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_y^*)\mathbf{G}(\mathbf{s}_o^*)\right]_{ii} = \frac{a^2}{c^2}\frac{s_{y,i}^* s_{o,i}^*}{\sum_h s_{y,h}^* \sum_h s_{o,h}^*}\sum_h s_{y,h}^* s_{o,h}^*;$$

$$\text{(A.26)} \qquad EL_{ii,ex} = \frac{a^2}{c^2}\left[\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\right]_{ii} = \frac{a^2}{c^2}\frac{s_{y,i} s_{o,i}}{\sum_h s_{y,h} \sum_h s_{o,h}}\sum_h s_{y,h} s_{o,h}.$$

Part A readily follows from the derivations in the main text and the following derivation, for the total effect of an increase in first period wealth on the transfer to the child. That is, from (2.22) and (2.25) we have:

$$\frac{d\ k_{y,i,t+1}}{d\ k_{y,i,t}} = \frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}}\left[1 + \rho a \sum_{j=1,j\neq i}^{I}\frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1})k_{o,j,t+1}\frac{\partial s_{o,i,t+1}}{\partial k_{y,i,t+1}}\right],$$

where the partial derivative of $\mathbf{k}_{y,t+1}$ with respect to $s_{o,i,t+1}$ is given by:

$$\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\frac{\partial}{\partial s_{o,i,t+1}}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{y,t} + \frac{\partial}{\partial s_{o,i,t+1}}\mathbf{G}(\mathbf{s}_{o,t+1})\left[\frac{a}{c^2}\mathbf{b}_{o,t+1} - \frac{a}{\rho c^2}\mathbf{1}\right],$$

with

$$\frac{\partial}{\partial s_{o,i,t+1}}\mathbf{G}(\mathbf{s}_{o,t+1}) = \begin{bmatrix} 0 & 0 & \dots & \frac{s_{o,1,t+1}}{\sum_{j\neq 1} s_{o,1,t+1}} & \dots & 0 \\ \frac{s_{o,1,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} & \frac{s_{o,2,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} & \dots & 0 & \dots & \frac{s_{o,I,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} \\ 0 & 0 & \dots & \frac{s_{o,I,t+1}}{\sum_{j\neq I} s_{o,j,t+1}} & \dots & 0 \end{bmatrix}.$$

Part B follows by inspection of (A.22), and provided that the sufficient conditions for the positivity of $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$ in Part B, of Proposition 3, hold. Q.E.D.

Parts C and D readily follow from the definitions.

A.4 Proposition A.4

This section provides, in the from of Proposition A.4, the full statement and proof of Proposition 3 in the main text. The assumptions stated in the beginning of section 3.2 continue to hold.

Proposition A.4. *Individual i chooses second period human capital and transfer to her child, $(k_{o,i,t+1}, k_{y,i,t+1})$, given the realization of $b_{y,i,t}$, and subject to uncertainty with respect to her own second period skills coefficients and her child's first period skills, $(b_{o,i,t+1}, b_{y,i,t+1})$, so as to maximize her expected utility.*

Part A. Defining the individual's decision problem of Proposition 3 under uncertainty yields the first-order conditions in vector form, the stochastic counterpart of (2.22–2.23):

$$\text{(A.27)}\quad \mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \frac{1}{c}\mathcal{E}[\mathbf{b}_{y,t+1}|t] + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathcal{E}[\mathbf{b}_{o,t+1}|t] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1};$$

$$\text{(A.28)}\quad \mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)\mathbf{k}_{y,t} + \frac{1}{c}\mathcal{E}[\mathbf{b}_{o,t+1}|t] + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_y)\mathcal{E}[\mathbf{b}_{y,t}|t] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}.$$

Part B. Given social connections $(\mathbf{G}(\mathbf{s}_y), \mathbf{G}(\mathbf{s}_o))$, the state of the economy is described by *the stochastic system for* $(\mathbf{k}_{y,t}, \mathbf{b}_{y,t})$, *where* $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$ *evolve according to*

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \mathbf{G}_{adj,y}(\mathbf{s}_o)\mathbf{b}_{y,t} + \mathbf{C}_y, \tag{A.29}$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \mathbf{G}_{adj,o}(\mathbf{s}_y)\mathbf{b}_{y,t} + \mathbf{C}_o, \tag{A.30}$$

where:

$$\mathbf{G}_{adj,y}(\mathbf{s}_o) = \frac{1}{c}\left[\rho_b\mathbf{I} + \rho_o\frac{\sigma_o}{\sigma_b}\frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right];$$

$$\mathbf{C}_y = \frac{1}{c}\left[(1-\rho_b)\mathbf{I} - \frac{a}{c}\rho_o\frac{\sigma_o}{\sigma_b}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{b}_{m,y} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,o} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}; \tag{A.31}$$

$$\mathbf{G}_{adj,o}(\mathbf{s}_y) = \frac{1}{c}\left[\frac{\sigma_o}{\sigma_y}\rho_o\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_y)\right];$$

$$\mathbf{C}_o = \frac{1}{c}\left[\mathbf{b}_{m,o} - \frac{a}{c}\rho_o\frac{\sigma_o}{\sigma_b}\mathbf{b}_{m,y}\right] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}; \tag{A.32}$$

and $\mathbf{b}_{y,t}$ *is an exogenous vector stochastic process, denoting first-period cognitive skills, introduced above.*

Part C. Under the additional assumption that the vector of means and the variance-covariance matrix are time invariant, the stationary steady state is given by:

(A.33)

$$\mathbf{k}_y^* = \frac{1}{c}\left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^2 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1 - \left(\frac{a}{c}\right)^2 \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\left[\frac{1}{c}\mathbf{b}_{m,y} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,o} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}\right].$$

The deviation $\Delta\mathbf{k}_{y,t} = \mathbf{k}_{y,t} - \mathbf{k}_y^*$ *has a multivariate normal stationary limit distribution with mean* $\mathbf{0}$ *and variance covariance matrix* $\mathbf{\Sigma}_{y,\infty}$ *that satisfies:*

$$\mathbf{\Sigma}_{y,\infty} = \frac{a^4}{c^4}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{\Sigma}_{y,\infty}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y) + \mathbf{G}_{adj,y}(\mathbf{s}_o)\sigma_b^2\mathbf{I}\mathbf{G}^T_{adj,y}(\mathbf{s}_o). \tag{A.34}$$

A necessary and sufficient condition for the existence of a positive definite matrix $\mathbf{\Sigma}_\infty$ *is that the matrix* $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *be stable, for which by Proposition 2, Part C, a sufficient condition is that* $\frac{a^2}{c^2}$ *times the product of the largest eigenvalue of* $\mathbf{G}(\mathbf{s}_y)$ *and of* $\mathbf{G}(\mathbf{s}_o)$ *be less than 1. For the special case of (2.2) this condition is identical to (2.28).*

If $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *is stable, then:*

$$\mathbf{\Sigma}_{y,\infty} = \left[\mathbf{I} - \left(\frac{a}{c}\right)^4 \mathbf{G}(\mathbf{s}_y)^2\mathbf{G}(\mathbf{s}_o)^2\right]^{-1} \mathbf{G}_{adj,y}(\mathbf{s}_o)\mathbf{G}^T_{adj,y}(\mathbf{s}_o)\sigma_b^2$$

$$= \left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^4 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1-\left(\frac{a}{c}\right)^4\left[\frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}\right]^2}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\mathbf{G}_{adj,y}(\mathbf{s}_o)\mathbf{G}^T_{adj,y}(\mathbf{s}_o)\sigma_b^2, \tag{A.35}$$

where the matrix $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ *is defined via its* (i,j) *element in (2.30).*

Part D. Under the above assumptions the vector the stationary steady state for $\mathbf{k}_{o,t}$ *satisfies*

(A.36)

$$\mathbf{k}_o^* = \left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^2 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1-\left(\frac{a}{c}\right)^2 \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\left[\mathbf{k}_y^* + \frac{1}{c}\left[\mathbf{b}_{m,o} + \frac{a}{c}\mathbf{G}(\mathbf{s}_y)\mathbf{b}_{m,y} - \frac{1}{\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}\right]\right].$$

The deviation $\Delta\mathbf{k}_{o,t} = \mathbf{k}_{o,t} - \mathbf{k}_o^*$ *has a multivariate normal stationary limit distribution with mean* $\mathbf{0}$ *and variance covariance matrix* $\mathbf{\Sigma}_{o,\infty}$ *that is given by an expression as in (A.35), with* $\mathbf{G}_{adj,o}$, *defined in (A.32), in the place of* $\mathbf{G}_{adj,k}$. *where* $\sigma_o^2\mathbf{I}$ *denotes the variance covariance matrix of* $\mathbf{b}_o$. *A necessary and sufficient condition for the existence of a positive definite matrix* Σ_∞ *is that the matrix* $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *be stable, for which by Proposition 3, Part C, a sufficient condition is that* $\frac{a^2}{c^2}$ *times the product of the largest eigenvalue of* $\mathbf{G}(\mathbf{s}_y)$ *and of* $\mathbf{G}(\mathbf{s}_o)$ *be less than 1. For the special case of (2.2) this condition is (2.28). If* $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *is stable,* $\mathbf{\Sigma}_{o,\infty}$ *is given by the counterpart of (A.35) with* σ_o^2 *in the place of* σ_y^2.

Part E. The cross-sectional distribution of first-period human capitals at the stochastic steady-state, the $k_{y,i}$*'s, when we do not distinguish individuals' identities, is given by a mixture of univariate normals, with weights equal to* I^{-1}, *the relative proportion of agent* *types in the population, with mean and variance given by:*

$$\text{(A.37)} \qquad Mean_{k_y} = \frac{1}{I}\sum_i k^*_{y,i}, \;\; \text{Var}_{k_y} = \frac{1}{I}\sum_i (k^*_{y,i})^2 + \frac{1}{I} trace\,(\mathbf{\Sigma}_{y,\infty}) - \left(\frac{1}{I}\sum_i k^*_{y,i}\right)^2.$$

The respective cross-sectional distribution of second-period human capitals as well as that of the joint distribution of the first-period human capital individuals receive and the transfer they make to their children is obtained in like manner.[19]

Proposition A.4. Proof

Part A. Transforming the individual's decision problem in the obvious way allows us to derive first order conditions, the stochastic counterpart of (2.22)–2.23). They are as follows:

$$\text{(A.38)} \qquad k_{y,i,t+1} = \frac{1}{c}\mathcal{E}[b_{y,i,t+1}|b_{y,i,t};t] + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_o)\mathcal{E}[k_{o,j,t+1}|i,t] - \frac{1}{c\rho};$$

$$\text{(A.39)} \qquad k_{o,i,t+1} = \frac{1}{c}\mathcal{E}[b_{o,i,t+1}|b_{y,i,t};t] + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_y)\mathcal{E}[k_{y,j,t}|i,t] - \frac{1}{c\rho}.$$

These conditions may be rewritten readily as in the main text.

For Part B, from the stochastic assumptions we have that:

$$\mathcal{E}[b_{o,i,t+1}|b_{y,i,t}] = m_{o,i} + \frac{\sigma_o}{\sigma_y}\rho_o(b_{y,i,t} - b_{m,y,i}); \mathcal{E}[b_{y,i,t+1}|b_{y,i,t}] = (1-\rho_b)b_{m,y,i} + \rho_b b_{y,i,t}.$$

These expressions are used to write (A.27)–(A.28) by defining $\mathbf{G}_{\text{adj,y}}(\mathbf{s}_o), \mathbf{G}_{\text{adj,o}}(\mathbf{s}_y)$, in the form of (A.29)–(A.30).

Parts C and D readily by Proposition 4.1 of Bertsekas (1995): $\Delta\mathbf{k}_{y,t} = \mathbf{k}_{y,t} - \mathbf{k}^*_y$ has a multivariate normal limit distribution with mean $\mathbf{0}$ and variance covariance matrix Σ_∞ that satisfies (A.33) in the main text. The explicit solutions for $\mathbf{\Sigma}_{y,\infty}, \mathbf{\Sigma}_{o,\infty}$ follow by iterating (A.33), if the matrix $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ is stable.

For Part E, consider the discrete random variable J taking values in $\{1, 2, \ldots, I-1, I\}$, with equal probabilities I^{-1}, and define the random vector $D = (d_1, d_2, \ldots, d_i, \ldots, d_I)$, with

$d_i = 1,$ iff $i = J; d_i = 0,$ iff $i \neq J$. We assume that the shocks introduced in the main part of Proposition 3 are statistically independent of the random index J and of the corresponding dummy random vector D. Finally, consider the univariate random variable $\mathcal{Z}_t$ that consists of randomly selecting elements the human capital vector, that is "anonymizing" this vector:

$$\mathcal{Z}_t = D^{\mathrm{T}} \Delta \mathbf{k}_{y,t} = \sum_i D_i \Delta k_{y,i,t}.$$

In this representation, one and only one of the D_i binary random variables will take the value 1 and all the others will be 0, so $\mathcal{Z}_t = \Delta k_{y,i,t}$ with equal probability I^{-1}. Since the D_i's are fully independent of the $\Delta k_{y,i,t}$'s, and each D_i takes the value 1 with equal probability I^{-1}, the expressions in (A.37) in the main text follow. The full probability density and distribution functions of $\mathcal{Z}_t$ follow directly from its definition. They are a mixture of univariate normal distributions. It is important to note that since only one of the $\Delta k_{y,i,t}$ is realized at any one time, the covariance/correlation structure between the $\Delta k_{y,i,t}$ is irrelevant. Only the individual variances matter. Still, the results reported in (A.37) reflect the social structure and, in addition, ensure a much richer outcome, as the cross-sectional distribution might no longer be unimodal and will in general exhibit thick tails. For the general case, see Shaked (1980); for conditions on the mixing distribution, see Antonov and Koksharov (2017).

Similar derivations readily follow for $k_{o,i,t}$ and the joint distribution of $(k_{y,i,t}, k_{y,i,t+1})$, the latter being the stochastic counterpart of Social Effects in Intergenerational Wealth Transfer Elasticities, discussed in section 3.1 of the main text. This involves deriving an expression for the covariance of $(\Delta k_{y,i,t+1}, \Delta k_{y,i,t})$ as:

$$\mathrm{Covar}(\Delta k_{y,i,t+1}, \Delta k_{y,i,t}) = \mathcal{E}[k_{y,i,t+1} k_{y,i,t}] - (k^*_{y,i})^2,$$

which involves elementary but tedious derivations. Q.E.D.

A.5 Proposition A.5

I redefine the individual's decision problem to individualize a as $(a_{y,i,t}, a_{o,i,t+1})$ and assume them to be random variables. Accordingly, I redefine the value functions of Proposition 3, Part A, when choosing $(k_{o,i,t+1}, k_{y,i,t+1})$, and consequently the individual treats as uncertain the value that she would derive from $k_{o,i,t+1}$ in her second period of her own life and the value accruing to her child from the transfer $k_{y,i,t+1}$. The former depends on the human capitals of others, $k_{y,j,t}, j \neq i$, which are known when she makes the decision at time t, but the effect depends on the realization of $a_{y,i,t+1}$. The latter depends on the cognitive skills coefficients of the child at time $t+1$ and the realization of the social interactions effect $a_{o,i,t+2}$ at time $t+2$.

The value function for individual i as of time t and for her child as of time $t+1$ readily follow as in Proposition A.4, Part A.

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}; a_{y,i,t})$$

$$= \max_{\mathcal{E}\{k_{y,i,t+1}, k_{o,i,t+1}; s_{y,i,t}, , s_{o,i,t+1}\}} \mathcal{E} \left\{ b_{y,i,t} k_{y,i,t} + a_{y,i,t} \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t}) k_{y,i,t} k_{o,j,t} - \frac{1}{2} c k_{y,i,t}^2 - \frac{1}{2} s_{y,i,t}^2 - k_{o,i,t+1} \right.$$

$$+\rho \left[b_{o,i,t+1} k_{o,i,t+1} + a_{o,i,t+1} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t}) k_{o,i,t+1} k_{y,j,t} - \frac{1}{2} c k_{o,i,t+1}^2 - \frac{1}{2} s_{o,i,t+1}^2 - k_{y,i,t+1} \right]$$

$$\left. +\rho \mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}; a_{y,i,t+1}) | a_{i,t} \right\};$$

$$\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}; a_{y,i,t+1})$$

$$= \max_{\{k_{y,i,t+2}, k_{o,i,t+2}; s_{y,i,t+1}, , s_{o,i,t+2}\}} \mathcal{E} \left\{ b_{y,i,t+1} k_{y,i,t+1} + a_{y,i,t+1} \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t+1}) k_{y,i,t+1} k_{o,j,t+1} - \frac{1}{2} c k_{y,i,t+1}^2 - \frac{1}{2} s_{y,i,t+1}^2 \right.$$

$$-k_{o,i,t+2} + \rho \left[b_{o,i,t+2} k_{o,i,t+2} + a_{o,i,t+2} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t+1}) k_{o,i,t+2} k_{y,j,t+1} - \frac{1}{2} c k_{o,i,t+2}^2 - \frac{1}{2} s_{o,i,t+2}^2 - k_{y,i,t+2} \right]$$

$$\left. +\rho \mathcal{V}_i^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+2}; a_{y,i,t+2}) | a_{y,i,t+1} \right\}.$$

The results are summarized by Proposition A.5, which follows next.

Proposition A.5. Under the assumption that the social interactions coefficients in the problem defined by Proposition A.4, Part A, are random variables, $(a_{y,i,t}, a_{o,i,t+1})$, the definitions of the value functions, the counterparts of those in Proposition 3, for individual i as of time t and for her child as of time $t+1$, $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}; a_{i,t})$, $\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}; a_{i,t+1})$, that are associated with an individual's lifetime utility when he is young at t and when he is old at $t+1$, are modified accordingly.

Part A. The first-order conditions with respect to the human capital decisions $(k_{y,i,t+1}, k_{o,i,t+1})$ are given by:

$$\rho\mathcal{E}\left[a_{y,i,t+1}\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+1} \mid a_{y,i,t}\right] = \rho + \rho c k_{y,i,t+1}; \tag{A.40}$$

$$\rho b_{o,i,t+1} + \rho\mathcal{E}\left[a_{o,i,t+1}\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{y,i,t} \mid a_{y,i,t}\right] = 1 + c k_{o,i,t+1}. \tag{A.41}$$

The first-order conditions with respect to the socialization efforts $(s_{y,i,t}, s_{o,i,t+1})$ are:

$$s_{y,i,t} = \rho\mathcal{E}\left[a_{o,i,t+1}k_{o,i,t+1}\sum_{j=1,j\neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{y,t})}{\partial s_{y,i,t}} k_{y,j,t} \mid a_{y,i,t}\right]; \tag{A.42}$$

$$s_{o,i,t+1} = \rho\mathcal{E}\left[a_{y,i,t+1}k_{y,i,t+1}\sum_{j=1,j\neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{o,t+1})}{\partial s_{o,i,t+1}} k_{o,j,t+1} \mid a_{y,i,t}\right]. \tag{A.43}$$

Part B. The first-order conditions for human capitals $(k_{y,i,t+1}, k_{o,i,t+1})$, under the assumptions that the networking efforts are constant, $\mathbf{s}_{y,t} = \mathbf{s}_y, \mathbf{s}_{o,t} = \mathbf{s}_o$, and the parameters $(a_{y,i,t}, a_{o,i,t+1})$ are approximated by their means, then conditions (A.40)–(A.41) may be written in vector form as:

(A.44)
$$\mathbf{k}_{y,t+1} = \frac{\bar{\mathbf{A}}_{y,t+1}\bar{\mathbf{A}}_{o,t+1}}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{\bar{\mathbf{A}}_{y,t+1}}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{\bar{\mathbf{A}}_{y,t+1}}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1};$$

$$\text{(A.45)}\quad \mathbf{k}_{o,t+1} = \frac{\bar{\mathbf{A}}_{y,t+1}\bar{\mathbf{A}}_{o,t+1}}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)\mathbf{k}_{o,t} + \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{\bar{\mathbf{A}}_{y,t}}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{b}_{y,t} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_y)\right]\mathbf{1},$$

where $\bar{\mathbf{A}}_{y,t+1},\ \bar{\mathbf{A}}_{o,t+1}$ *denote the diagonal matrices composed of the conditional means*

$$\mathcal{E}[a_{y,i,t+1}|t],\ \mathcal{E}[a_{o,i,t+1}|t].$$

Part C. If the means of parameters $(a_{y,i,t}, a_{o,i,t+1})$ *are time invariant, so are the vectors denoting their means,* $\bar{\mathbf{A}}_{y,t}, \bar{\mathbf{A}}_{o,t}$*, and sufficient conditions for the existence of meaningful steady state values of* $(\mathbf{k}_y, \mathbf{k}_o)$ *amount to sufficient conditions for the invertibility of*

$$\text{(A.46)}\qquad \mathbf{I} - c^{-2}\bar{\mathbf{A}}_y\bar{\mathbf{A}}_o\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y),$$

namely that the product of the largest $\frac{\bar{a}_{y,i}\bar{a}_{o,i}}{c^2}$ *times the largest eigenvalues of each of the matrices* $\mathbf{G}(\mathbf{s}_o), \mathbf{G}(\mathbf{s}_y)$ *be less than 1.*

Proposition A.5. Proof

Part A. Proof. The first-order conditions (A.40)–(A.41) and (A.42)–(A.43) readily follow by differentiation and application of the envelope property along with the principle of optimality.

Part B. Proof. This follows readily, as in Proposition A.4, Part B.

Part C. Proof. This follows readily, as in Proposition A.4, Part D.

A.6 Proposition A.6

Proposition A.6 supplements Section 3.5, which allows for parents to invest in their children's cognitive skills coefficients. This is handled by extending the demographic structure as per Section 3.5.

Proposition A.6. For an individual born at t*, her cognitive skills coefficients and human capital in period* t *are given,* $(b_{y,i,t}, k_{y,i,t})$*; she benefits from the networking efforts of the parents' generation,* $\mathbf{s}_{o,t-1}$*, who are in the third subperiod of their lives. She chooses at time* t *her own second subperiod human capital and the first subperiod transfer to her own child* at time $t+2$, respectively $\{k_{o,i,t+1}, k_{y,i,t+2}\}$; and the first and second subperiod networking efforts, $\{s_{y,i,t,}, s_{o,i,t+1}\}$, respectively. She benefits herself in her own second subperiod and her child benefits when the child is in her own first subperiod of her life and the parent herself in her third subperiod of her life. The adjustment costs for decisions $\{s_{y,i,t,}, k_{o,i,t+1}\}$ are incurred in period t. The optimization problem treats the cognitive skills, $b_{y,i,t+2}$, of the individual's child and the transfer she receives when she becomes an adult, $k_{y,i,t+2}$, as being determined simultaneously.

Part A. The first order conditions for $(\iota_{c1,t}, \iota_{c2,t+1})$ yield:

(A.47) $$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \rho\rho_\beta[k_{y,i,t+2} + \rho k_{o,i,t+3}] - \rho_\beta,$$

where parameter ρ_β is defined as $\rho_\beta \equiv \left(\rho\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right)$.

Part B. The first-order conditions with respect to $(\mathbf{k}_{y,t+2}, \mathbf{k}_{o,t+2})$ yield a first-order linear difference system in $\mathbf{k}_{o,t+2}$:

(A.48) $$\mathbf{k}_{o,t+3} = \mathbf{b}_{\text{eff}} + \frac{a}{\rho^* c}\mathbf{G}(\mathbf{s}_{y,t+2})\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2},$$

where $\rho^* \equiv 1 - \frac{\rho^2\rho_\beta}{c}$, $\tilde{\rho} \equiv 1 - \frac{\rho\rho_\beta}{c_{cs}}$, $c_{cs} \equiv c - \rho\rho_\beta$, and $\hat{a} \equiv \frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}c_{cs}}\left(1 - \frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)^{-1}$, and $\mathbf{b}_{\text{eff}}$ are constant. The optimal $\mathbf{k}_{y,t+2}$ follows from $\mathbf{k}_{o,t+2}$ according to:

(A.49) $$\mathbf{k}_{y,t+2} = \left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\left[\mathbf{b}'_{\text{eff}} + \frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}\right],$$

where $\mathbf{b}'_{\text{eff}}$ is a constant.

Part C. The stability of (A.48) rests on the spectral properties of

(A.50) $$\frac{a}{\rho^* c}\,\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{y,2})\mathbf{G}(\mathbf{s}_{o,2})\left[\mathbf{I} + \frac{\hat{a}}{c}\frac{1}{1 - \frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})}}\mathbf{G}(\mathbf{s}_{y,2})\right],$$

provided that $\frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})} < 1$. A sufficient condition for the stability of (A.48) is that $\frac{a}{\rho^* c}\,\frac{a}{\tilde{\rho}c_{cs}}$ times the product of the maximal eigenvalue of $\mathbf{G}(\mathbf{s}_{y,2})$ and of $\mathbf{G}(\mathbf{s}_{o,2})$ times 1 plus the *maximal eigenvalue of* $\frac{\hat{a}}{c}\frac{1}{1-\frac{\hat{a}}{c}\frac{\overline{x^2(\mathbf{s}_{y,2})}}{\overline{x}(\mathbf{s}_{y,2})}}\mathbf{G}(\mathbf{s}_{y,2})$ *be less than 1.*

Proposition A.6. Proof

An individual born at t takes cognitive skills coefficients and human capital as given, $(b_{y,i,t}, k_{y,i,t})$, and benefits from the networking efforts of the parents' generation, $\mathbf{s}_{o,t-1}$, who are in the third subperiod of their lives when she is born. She chooses at time t the second subperiod human capital and the first subperiod transfer received by the child at time $t+2$, respectively $\{k_{o,i,t+1}, k_{y,i,t+2}\}$; and the first and second subperiod networking efforts, $\{s_{y,i,t,}, s_{o,i,t+1}\}$, respectively. These benefit herself in the second subperiod of her life, and benefit her child too, when the child is in her first subperiod of her life and she herself in her third subperiod of her life. For analytical convenience, I assume that the adjustment costs for decisions $\{s_{y,i,t,}, k_{o,i,t+1}\}$ are both incurred in period t. The optimization problem implies that the cognitive skills, $b_{y,i,t+2}$, of the individual's child and the transfer she receives when she becomes an adult, $k_{y,i,t+2}$, are determined simultaneously. The definition of the value function for the problem now changes to:

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t-1}) = \max_{\{k_{o,i,t+1},k_{y,i,t+2};\iota_{c1,t},\iota_{c2,t+1};s_{y,i,t,},s_{o,i,t+1}\}} \Big\{\rho^2\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})$$

$$+b_{y,i,t}k_{y,i,t} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t-1})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck^2_{y,i,t} - \frac{1}{2}s^2_{y,i,t} - k_{o,i,t+1} - \iota_{c1,t} - \frac{1}{2}\gamma_1\iota^2_{c1,t}+$$

$$\rho\left[b_{o,i,t+1}k_{o,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t} - \frac{1}{2}ck^2_{o,i,t+1} - \frac{1}{2}s^2_{o,i,t+1} - k_{y,i,t+2} - \iota_{c1,t+1} - \frac{1}{2}\gamma_1\iota^2_{c1,t+1}\right]\Bigg\}.$$

The first order conditions for $\iota_{1,t}, \iota_{2,t+1}$ are:

$$-1 - \gamma_1\iota_{c1,t} + \rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}}\left[\frac{\partial b_{y,i,t+2}}{\partial\iota_{c1,t}} + \frac{\partial b_{o,i,t+3}}{\partial\iota_{c1,t}}\right] = 0;$$

$$-\rho[1 - \gamma_2\iota_{c2,t+1}] + \rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}}\left[\frac{\partial b_{y,i,t+2}}{\partial\iota_{c2,t+1}} + \frac{\partial b_{o,i,t+3}}{\partial\iota_{c2,t+1}}\right] = 0.$$

Using the envelope property we rewrite the partial derivation of the value function above and get:

$$-1-\gamma_1\iota_{1,t}+\rho^2\beta_1\left[k_{y,i,t+2}+\rho k_{o,i,t+3}\right]=0;$$

$$-1-\gamma_2\iota_{2,t+1}+\rho\beta_2\left[k_{y,i,t+2}+\rho k_{o,i,t+3}\right]=0.$$

Solving for $\iota_{1,t}, \iota_{2,t+1}$ yields:

$$\iota_{1,t}=\frac{1}{\gamma_1}(\rho^2\beta_1[k_{y,i,t+2}+\rho k_{o,i,t+3}]-1);\iota_{2,t+1}=\frac{1}{\gamma_2}(\rho\beta_2[k_{y,i,t+2}+\rho k_{o,i,t+3}]-1).$$

This in turn yields condition (A.47) in the main text:

$$b_{y,i,t+2}=b_{o,i,t+3}=\beta_0 b_{y,i,t}+\rho\rho_\beta[k_{y,i,t+2}+\rho k_{o,i,t+3}]-\rho_\beta, \tag{A.51}$$

where the auxiliary parameter ρ_β is defined as $\rho_\beta\equiv\left(\rho\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)$. For some of the analysis below we assume that $b_{y,i,t}$ is constant, so that cognitive skills coefficients do not necessarily steadily increase. Of course, such a feature could be incorporated.

It follows that the first-order condition for $k_{y,i,t+2}$ must reflect the influence that decision has, as implied by the optimization problem, on $b_{y,i,t+2}$. Since $b_{y,i,t+2}=b_{o,i,t+3}$ the utility per period from the last two subperiods of the child's lifetime contribute to the first-order conditions. The first order conditions are:

$$-\rho+\rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2},\mathbf{s}_{o,t+1})}{\partial k_{y,i,t+2}}+\rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2},\mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}}\frac{\partial b_{y,i,t+2}}{\partial k_{y,i,t+2}}=0.$$

After using the envelope property and (A.51), this yields the following:

$$-1+\rho\left[b_{y,i,t+2}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+2}-ck_{y,i,t+2}\right]+\rho^2\rho_\beta k_{y,i,t+2}+\rho^3\rho_\beta k_{o,i,t+3}=0.$$

This condition is rewritten as:

$$k_{y,i,t+2}=\frac{1}{c_{cs}}b_{y,i,t+2}+\frac{a}{c_{cs}}\sum_{j\neq i}g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+2}+\frac{\rho^2}{c_{cs}}\rho_\beta k_{o,i,t+3}-\frac{1}{\rho c_{cs}}, \tag{A.52}$$

where the auxiliary variable c_{cs} is defined as: $c_{cs}\equiv c-\rho\rho_\beta$. This condition may be rewritten by using (A.51) to eliminate $b_{y,i,t+2}$ by expressing it in terms of $(k_{y,i,t+2}, k_{o,i,t+3})$.

In addition, the first-order conditions for $k_{o,i,t+1}, s_{y,i,t,}, s_{o,i,t+1}$ are as follows:

$$k_{o,i,t+1} = \frac{1}{c} b_{o,i,t+1} + \frac{a}{c} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t}) k_{y,j,t} - \frac{1}{c\rho}; \tag{A.53}$$

$$s_{y,i,t} = \rho a k_{o,i,t+1} \sum_{j=1, j \neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{y,t})}{\partial s_{y,i,t}} k_{y,j,t}; \tag{A.54}$$

$$s_{o,i,t+1} = \rho a k_{y,i,t+1} \sum_{j=1, j \neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{o,t+1})}{\partial s_{o,i,t+1}} k_{o,j,t+1}. \tag{A.55}$$

Conditions (A.54) and (A.55) are similar, respectively, to (2.24) and (2.25) and thus may be manipulated at the steady state in like manner to the steady state analysis in Proposition A.4, Part E above. It is more convenient to write Equation (A.53) by advancing the time subscript as follows:

$$k_{o,i,t+3} = \frac{1}{c} b_{o,i,t+3} + \frac{a}{c} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t+2}) k_{y,j,t+2} - \frac{1}{c\rho}. \tag{A.56}$$

By using (A.47) to write for $b_{o,i,t+3}$ in terms of its solution in terms of $(k_{y,i,t+2}, k_{o,i,t+3})$ and rewriting the conditions for $(k_{y,i,t+2}, k_{o,i,t+3})$ in matrix form, we have:

$$\mathbf{k}_{o,t+3} = \frac{\beta_0}{\rho^* c} \mathbf{b} - \frac{\rho_\beta}{\rho^*} \mathbf{i} + \left[\frac{\rho \rho_\beta}{\rho^* c} \mathbf{I} + \frac{a}{\rho^* c} \mathbf{G}(\mathbf{s}_{y,t+2}) \right] \mathbf{k}_{y,t+2}, \tag{A.57}$$

where $\rho^* \equiv 1 - \frac{\rho^2 \rho_\beta}{c}$, and

$$\mathbf{k}_{y,t+2} = \frac{\beta_0}{\tilde{\rho} c_{cs}} \mathbf{b} - \frac{1}{\tilde{\rho} \rho c_{cs}} \mathbf{i} + \frac{a}{\tilde{\rho} c_{cs}} \mathbf{G}(\mathbf{s}_{o,t+2}) \mathbf{k}_{o,t+2} + \frac{\rho^2 \rho_\beta}{c_{cs}} \mathbf{k}_{o,t+3}, \tag{A.58}$$

where $\tilde{\rho} \equiv 1 - \frac{\rho \rho_\beta}{c_{cs}}$. However, by substituting from (A.57) for $\mathbf{k}_{o,t+3}$ in the RHS of (A.58), we have:

$$\left[\left(1-\frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)\mathbf{I}-\frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}cc_{cs}}\mathbf{G}(\mathbf{s}_{y,t+2})\right]\mathbf{k}_{y,t+2}$$

$$=\beta_0\left[\frac{\rho^2\rho_\beta}{\tilde{\rho}\rho^*cc_{cs}}+\frac{1}{\tilde{\rho}c_{cs}}\right]\mathbf{b}-\left[\frac{1}{\tilde{\rho}\rho c_{cs}}+\frac{\rho^2\rho_\beta^2}{\tilde{\rho}\rho^*c_{cs}}\right]\mathbf{i}+\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}.$$

By dividing through by $1-\frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}$ and denoting

$$\hat{a}\equiv\frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}c_{cs}}\left(1-\frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)^{-1},$$

we may solve the previous equation with respect to $\mathbf{k}_{y,t+2}$ as follows:

$$\mathbf{k}_{y,t+2}=\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\left[\mathbf{b}'_{\text{eff}}+\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}\right],$$

where $\mathbf{b}'_{\text{eff}}$ is the resulting new constant. By substituting into the RHS of (A.57), we obtain a single first-order linear difference system in $\mathbf{k}_{o,t+2}$:

(A.59) $$\mathbf{k}_{o,t+3}=\mathbf{b}_{\text{eff}}+\frac{a}{\rho^*c}\mathbf{G}(\mathbf{s}_{y,t+2})\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2},$$

where $\mathbf{b}_{\text{eff}}$ denotes the resulting constant. Thus, this equation depends on both networking efforts by the young and the old in two successive periods, $\mathbf{G}(\mathbf{s}_{y,t+2}),\mathbf{G}(\mathbf{s}_{o,t+2})$.

In a notable difference from the previous models, we now see a key new role for the social networking that individuals avail of when young. The product $\mathbf{G}(\mathbf{s}_{y,t+2})\mathbf{G}(\mathbf{s}_{o,t+2})$ is adjusted by $\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}$. Intuitively, this effect acts to reinforce the effects of social networking when young. This readily follows from (A.57) and (A.57) above. Feedbacks are generated due to the investment in cognitive skills. Mathematical results invoked upon earlier can still be used to determine the stability of (A.59). That is, $\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})\right]^{-1}$ admits a simple expression, following steps similar to those employed above, provided that the maximal eigenvalue of $\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})$ is less than 1, that is:

$$\frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})}<1.$$

Thus:

$$\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})\right]^{-1} = \mathbf{I} + \frac{\hat{a}}{c}\frac{\overline{x}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2}) - \frac{\hat{a}}{c}\overline{x^2}(\mathbf{s}_{y,2})}\mathbf{G}(\mathbf{s}_{y,2}).$$

Thus, the stability of (A.59) rests on the spectral properties of

$$\frac{a}{\rho^* c}\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{y,2})\mathbf{G}(\mathbf{s}_{o,2}) + \frac{a}{\rho^* c}\frac{\hat{a}}{c}\frac{a}{\tilde{\rho}c_{cs}}\frac{\overline{x}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2}) - \frac{\hat{a}}{c}\overline{x^2}(\mathbf{s}_{y,2})}\mathbf{G}(\mathbf{s}_{y,2})^2\mathbf{G}(\mathbf{s}_{o,2}).$$

By Theorem 1, Merikoski and Kumar (2004, pp. 151–152), the maximal eigenvalue of the sum of two real symmetric matrices is bounded upwards by the sum of the maximal eigenvalues of the respective matrices. Thus, a condition for the stability of (A.59) readily follows and involves $(\mathbf{s}_{y,2}, \mathbf{s}_{o,2})$ along with the other parameters of the model. Q.E.D.

B Appendix B: A Multivariate Pareto Law for the Joint Distribution of Human Capitals

Under the assumption of stochastic cognitive and social skills coefficients I can obtain a multivariate Pareto law for the upper tail of the distribution of human capitals at a stochastic steady state. In order to do so I specify that cognitive skills coefficients are stochastic by assuming that the (column) vectors $\mathbf{\Psi}_t = (\psi_{1,t}, \ldots, \psi_{I,t})^{\mathrm{T}}$ are defined to represent the full cognitive effect, where $\psi_{i,t} = \frac{1}{c}b_{i,t}$, with $\mathbf{\Psi}_t$ being a random column vector that is independently and identically distributed over time. That is, the sequence of $\{\mathbf{\Psi}_0, \ldots, \mathbf{\Psi}_t\}^{\mathrm{T}}$ is assumed to be a stationary vector stochastic process. In addition, I assume that networking efforts are exogenous but random. That is, the social networking efforts are denoted by $\mathbf{\Phi}_t = (\phi_{1,t}, \ldots, \phi_{I,t})^{\mathrm{T}}$, so that instead of (2.4) I now have:

(B.1) $$\tilde{\mathbf{k}}_t = \mathbf{\Psi}_t + \tilde{\mathbf{G}}(\mathbf{\Phi}_t)\tilde{\mathbf{k}}_{t-1}, t = 1, \ldots,$$

with a given $\tilde{\mathbf{k}}_0$. For the purpose of analytical convenience and without loss of generality, I assume that the social interactions matrix $\tilde{\mathbf{G}}_t = \tilde{\mathbf{G}}(\mathbf{\Phi}_t)$ is defined to include the diagonal terms too. Although the source of randomness in networking efforts represented by $\mathbf{\Phi}_t$ is not specified here, it could originate in randomness in parameter a; see Proposition A.5 above. Proposition B.1 establishes a thick upper tail of the joint distribution of human capitals. The proof, which adapts Theorems A and B, Kesten (1973), and all associated conditions, follows next.

Proposition B.1. *Let the pairs* $\left\{\tilde{\mathbf{G}}_t, \mathbf{\Psi}_t\right\}$ *be independently and identically distributed elements of a stationary stochastic process with positive entries, where* $\tilde{\mathbf{G}}_t$ *are* $I \times I$ *matrices and* $\mathbf{\Psi}_t$ *are* $I-$ *vectors. Under the additional conditions of Theorems A and B, Kesten (1973; 1974) and the assumption of the function* $||m|| = \max_{|y|=1} |ym|$*, where* y *denotes an* I *row vector, and* m *denotes an* $I \times I$ *matrix, as the matrix norm* $||\cdot||$ *for* $I \times I$ *matrices, and* $|\cdot|$ *denotes the Euclidian norm, then:*

Part A. The series of $I-$*vectors*

$$\mathbf{K} \equiv \sum_{t=1}^{\infty} \tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_{t-1})\mathbf{\Psi}_t \tag{B.2}$$

converges w.p. 1, and the distribution of the solution $\tilde{\mathbf{k}}_t$ *of (B.1) converges to that of* $\mathbf{K}$*, independently of* $\tilde{\mathbf{k}}_0$*.*

Part B. For all elements $\mathbf{x}$ *on the unit sphere in* $I\!R^I$*, under certain conditions, there exists a positive constant* κ_1*, and*

$$\lim_{v \to \infty} v^{\kappa_1} Prob\left\{\mathbf{xK} \geq v\right\} \tag{B.3}$$

exists, is finite and for all elements $\mathbf{x}$ *on the unit sphere of* $I\!R^I$ *and for all the elements on the positive orthant of the unit sphere is strictly positive.*

Proposition B.1. Proof

I assume that the pairs $\left\{\tilde{\mathbf{G}}_t, \mathbf{\Psi}_t\right\}$ are independently and identically distributed elements of a stationary stochastic process with positive entries. Adopting as matrix norm $||\cdot||$ for $I \times I$ matrices the function $||m|| = \max_{|y|=1} |ym|$, where y denotes an I row vector, and m

denotes an $I \times I$ matrix, where I adopt the notation $\ln^+$, $\ln^+ x = \min\{\ln x, 0.\}$ If

$$\mathcal{E} \ln^+ ||\tilde{\mathbf{G}}(\mathbf{\Phi}_1)|| < 0,$$

then

$$\text{(B.4)} \qquad \text{Lim}_1 := \lim \left(\ln ||\tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_t)||^{\frac{1}{t}} \right)$$

exists, is constant and finite w.p. 1. If I assume that the $\tilde{\mathbf{G}}$'s are such that $\text{Lim}_1 < 0$, then $||\tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_t)||$ converges to 0 exponentially fast. If $|\Psi_1|^\kappa < \infty$ for some $\kappa > 0$, that is if the starting shock is not too large, with the norm $|\cdot|$ being defined as the Euclidian norm, then the series of the vectors of human capital

$$\mathbf{K} \equiv \sum_{t=1}^{\infty} \tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_{t-1}) \mathbf{\Psi}_t$$

converges w.p. 1, and the distribution of the solution $\tilde{\mathbf{k}}_t$ of (B.1) converges to that of $\mathbf{K}$, independently of $\tilde{\mathbf{k}}_0$. This is simply a rigorous way to establish the limit human capital vector.

In particular, from (B.4), if $\text{Lim}_1 < 0$, then the norm of the product of t successive social interactions matrices, raised to the power of t^{-1}, is positive but less than 1. In that case, Kesten (1973) shows that the distribution of $\mathbf{K}$ can have a thick upper tail. That is, according to Kesten (1973, Theorem A), if in addition to the above conditions there exists a constant $\kappa_0 > 0$, for which

$$\text{(B.5)} \qquad \mathcal{E} \left\{ \frac{1}{I^{\frac{1}{2}}} \min_i \left(\sum_{j=1}^{I} \tilde{\mathbf{G}}_{1_{i,j}} \right) \right\}^{\kappa_0} \geq 1, \text{and } \ \mathcal{E} \left\{ ||\tilde{\mathbf{G}}_1||^{\kappa_0} \ln^+ ||\tilde{\mathbf{G}}_1|| \right\} < \infty,$$

then there exists a $\kappa_1 \in (0, \kappa_0]$ such that

$$\text{(B.6)} \qquad \lim_{v \to \infty} \text{Prob} \left\{ \max_{n \geq 0} |\mathbf{x} \tilde{\mathbf{G}}_1 \cdots \tilde{\mathbf{G}}_n| > v \right\} \sim X(x) v^{-\kappa_1},$$

where $0 \leq X(\mathbf{x}) < \infty$, with $X(\mathbf{x}) > 0$, where the (row) vector $\mathbf{x}$ belongs to the positive orthant of the unit sphere of $I\!R^I$, exists and is strictly positive. If, in addition, the components of Ψ_1 satisfy:

$$\text{Prob}\left\{\mathbf{\Psi}_1 = 0\right\} < 1,\ \text{Prob}\left\{\mathbf{\Psi}_1 \geq 0\right\} = 1,\ \mathcal{E}|\mathbf{\Psi}_1|^{\kappa_1} < \infty,$$

then for all elements $\mathbf{x}$ on the unit sphere in $I\!R^I$, then condition (B.3) follows. That is, the upper tail of the distribution of $\mathbf{xK}$,

$$\lim_{v \to \infty} v^{\kappa_1} \text{Prob}\left\{\mathbf{xK} \geq v\right\} \tag{B.7}$$

exists, is finite and for all elements $\mathbf{x}$ in the positive orthant of the unit sphere in $I\!R^I$ is strictly positive.

The intuition of condition (B.5) is that if there exists a positive constant κ_0, for which the expectation of the minimum row sum of the social interactions matrix raised to the power of κ_0, grows with the number of agents I faster than $\sqrt{I}$, roughly speaking, but does not grow too fast so as to blow up, then the contracting effect of the social interactions system does not send human capitals to zero, when the economy starts from an arbitrary initial condition, such as when, for example, all initial human capitals are uniformly distributed. The intuition of condition (B.4) is that the geometric mean of the limit of the sequence of norms of the social interactions matrix is positive but less than 1. Q.E.D.

Discussion of Proposition B.1

Proposition B.1, Part A establishes properties of the limit of the vector of human capitals. Part B relies on these properties to establish a Pareto (power) law for the upper tails of the joint distribution of human capitals, characterized by (B.3). Its significance lies in that a power law is obtained for a sequence of random *vectors*, not just a scalar random variable, as in the previous literature. Its intuition is straightforward. This argument is reminiscent of arguments explaining the emergence of power laws elsewhere in the economics literature. See for the city size distribution case Ioannides (2013), Ch. 8. Given a non-trivial initial value for the cognitive shocks, $\mathbf{\Psi}(1)$, and an arbitrary initial value for human capitals, $\tilde{\mathbf{k}}_0$, the dynamic evolution of human capital according to (B.1) keeps positive the realizations of human capital, while the impact of spillovers is having an overall contracting effect that pushes the realizations and thus the distributions of human capital, too, towards 0. The distribution is prevented from collapsing at 0 by the properties of the contemporaneous cognitive shocks, $\mathbf{\Psi}_t$, and from drifting to infinity by the contracting effect of the spillovers. The contracting effect results from the combination of two key requirements: First, condition (B.5) above requires that there exists a positive constant κ_0, such that the expectation of the minimum row sum of the social interactions matrix raised to the power of κ_0 grows with the number of agents I faster than $\sqrt{I}$, roughly speaking; and second, the geometric mean of the limit of the sequence of norms of the social interactions matrix is positive but less than 1. The convergence in distribution of $\left\{\tilde{\mathbf{G}}(\mathbf{\Phi}_1)\cdots\tilde{\mathbf{G}}(\mathbf{\Phi}_t),\ t\to\infty,\right\}$ to a non-zero matrix is of independent interest and may be ensured under appropriate and not very restrictive conditions. See Kesten and Spitzer (1984). The former condition ensures no collapse at the lower end of the distribution. The latter condition ensures the presence of a contracting effect at the upper end of the distribution.

Thus, the upper tail of the joint distribution of $\mathbf{xK}$ satisfies Prob $\{\mathbf{xK}\geq v\}\propto v^{-\kappa_1}$. That is, for all elements on the unit sphere of $I\!R^I$, the upper tail of $\mathbf{xK}$ is thickened by the combined effect of the contracting spillovers and tends to a power law, $\propto v^{-\kappa_1}$, with a constant exponent $\kappa_1>0$. This result is sufficient for the distribution of human capital in the entire economy to also have a Pareto upper tail. Let f_{k_i} denote the limit distribution of $k_i,\ i=1,\ldots,I$. Then, the economy-wide distribution of human capitals is given by $\sum_i \#\{i\}f_{k_i}(k)$, where $\#\{i\}$ denotes the relative proportion of types i agents. Following Jones (2014) (who deals with the univariate case), one may approximate the value of the Pareto exponent κ_1 in terms of the parameters of the distribution of $\{\tilde{\mathbf{G}},\mathbf{\Psi}_t\}$.