Version, October 27, 1995

EVOLUTION OF TRADING STRUCTURES

by

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Abstract

The paper reviews the evolution of trading structures by examining two pertinent strands in the literature on economies with interacting agents, one, works that presume a specified topology of interactions among agents, and two, works that let random mechanisms determine that topology. The paper reviews interactive discrete choice models in isotropic settings and proposes extensions within certain stylized anisotropic settings which are particularly interesting for economists. In particular, circular patterns of interaction highlight the role of money and credit; tree-type settings depict Walrasian interactions. The paper suggests that the random topology of interaction approach, which has employed random graph theory to study evolution of trading structures, may go beyond analyses of sizes of trading groups and thus exploit the full range of possible topological properties of trading structures.

The paper proposes an integration of those approaches which is intended to exploit their natural complementarities. In the simplest possible version, our synthesis involves individual decisions and expectations, randomness, and nature combining to fix an initial "primordial" topology of interaction. The dynamics of interaction move the economy from then on. The evolution of trading structures depends critically upon multiplicity and stability properties of equilibrium configurations of the interaction model. The paper addresses a number of additional topics, including matching models, spatial aspects of the evolution of trading structures and issues of statistical inference.

Presented at the Santa Fe Institute Workshop "The Economy as a Complex Evolving System II," Santa Fe, New Mexico, August, 1995. I thank Buzz Brock, Steven Durlauf, Alan Kirman, Chuck Manski and Douglas North for very useful comments. I also thank Hans Haller for numerous fruitful interactions during the last several years over some of the issues discussed here and Anna Hardman for numerous suggestions and comments on the paper.

1 Introduction

The notion that economic transactions are driven by the gains that individuals and other decision-making units expect to make through exchanging goods and services lies at the heart of economics. Much of the theoretical structure of modern economics assumes a given physical and institutional infrastructure, through which trades take place. At the same time, modern economics has neglected another equally important prerequisite for trade, namely the ways that agents find out about each other's availability and willingness to engage in transactions.

In the most general setting, the present paper addresses a very broad topic. It looks at spatial, temporal and informational aspects of trading structures, that is, when do transactions take place, to what extent do agents who are geographically separated have to come into physical contact in order to transact, and the informational, what information is available to and used by different potential traders?

The casual observer of economic life and activity, whether a historian, geographer, anthropologist, or economist, would notice that transactions take place in fairly well-defined time periods, that some economic activity may be evenly distributed over part of the economically-exploited space, such as agriculture, but that most people live in cities or towns, areas of relatively high population density, and most manufacturing also occurs in high-density areas. Economic activity regularly utilizes space. Electricity is produced in specific sites and transported over long distances by means of physically fixed infrastructure, cereals and minerals are transported in barges and ships. Information moves over space in the form of voice, numerical or visual data. The laws of physics, chemistry, and biology define the technologically feasible choices available to the individual members of the economy at any given point in time. Within these constraints, it is the decisions that individuals make which in turn determine how constraints will evolve in the future. The nature and identity of decision makers can change and the infrastructure that defines the environment of trade structures is endogenous.

The Arrow-Debreu model deals with these problems by defining a set of commodities and by taking as given the corresponding set of markets. It is a feature of the Arrow-Debreu model that commodities, defined in terms of physical characteristics, are indexed in terms of points in time, location in space and contingent events. Evstigneev's extension of the Arrow-Debreu model, which we discuss below, generalizes this notion by endowing the index set with the topology of a directed graph, rather than the tree of events in the Arrow-Debreu model. Many of the questions we raise may be considered as special cases of incomplete markets, where the pertinent contingencies involve the likelihood that agents can or will be connected in order to trade. The paper sets out to exploit the insight that may be gained by considering broad features of economies which consist of large numbers of agents who can be connected in a variety of ways. We use this approach to integrate our knowledge of trade structures, with a fairly broad view of such structures, and address the largely unresolved issue of the endogeneity of markets and institutions which mediate trades. Research under the rubric of economics of information has addressed aspects of strategic release and processing of information, the transmission of information via market signals and the search for trading partners. The emergence of the full topology of the modern economy as consequence of preferences and prevailing communication technologies is however an important unresolved issue in current research. We adopt an approach for most of the paper which is intended to allow us to understand the *emergent* properties of the economy. We abstract from the vast complexity of the problem by restricting ourselves to graph topologies that roughly describe interconnections between agents and markets.

The importance of the topology of interactions may be underscored by reference to Schelling's theory of residential segregation, which evolves from spatial dependence. The following spatial example, also from Schelling (1978), makes the point succinctly: "If everybody needs 100 watts to read by and a neighbor's bulb is equivalent to half one's own, and everybody has a 60-watt bulb, everybody can read as long as he and both his neighbors have their lights on. Arranged on a circle, everybody will keep his lights on if everybody else does (and nobody will if his neighbors do not); arranged in a line, the people at the ends cannot read anyway and turn their lights off, and the whole thing unravels" [*ibid.* p. 214].

The remainder of the paper is organized as follows. Section 2 introduces the basic intuition of our approach and defines the two broad classes of models. Those which involve a prespecified topology of interaction among agents are discussed first in Section 3, and those which do not are discussed in Section 4. Both those sections also contain some original syntheses of the existing literature. Some of those new ideas are pursued further in Section 6, where a prototype equilibrium model of interactions with endogenous links is developed. A number of other topics, notably including spatial models, issues of statistical inference and matching models, are reviewed in Section

5. Section 7 concludes.

2 Economies with Interacting Agents

Modelling trade links is easiest in a setting like with two initially isolated agents where strategic considerations are simple to express [Haller and Ioannides (1992)]. In more complicated settings, intermediation possibilities will affect the likelihood that trade links between two individuals will open up [Kalai, Postlewaite and Roberts (1978)]. A moment's reflection suggests that aggregating individual decisions into a pattern of links in a large economy is an inherently very difficult problem to tackle.

Walrasian equilibrium prices are obtained by solving for the zeroes of a system of Walrasian excess demands. Excess demands are defined as additive with respect to the index of agents participating in each Arrow-Debreu market. We are interested in exploring patterns of dependence across agents which depending upon agents' preferences may imply non-additive aggregation. When aggregation is additive, it is appropriate to invoke powerful statistical theorems to ensure that small risks in large markets may cause uncertainty, in the guise of individual heterogeneity, to have negligible consequences in-the-large. A large number of agents responding to feedback from one another may cause individual, mutually independent, sources of randomness to reinforce and thus result in non-additive aggregation.

The first demonstration that in complex patterns of trade feedback to external shocks may be responsible for non-additive aggregation of individual characteristics is due to B. Jovanovic [Jovanovic (1984; 1985)]. He shows that individual mutually independent shocks affecting a large number of traders who because they are interlinked in the form of a circle are subject to sizable fluctuations in per capita endogenous variables. Such phenomena are, as we shall see, inherent in cyclical patterns of interaction.

The structure of interconnections is crucial for another, relatively unexplored, area of the economics of information, namely how economic news travels in large economies. A prototype investigation here is the work by Allen on the diffusion of information about technological innovation that derives the logistic law from a model based on Markov random fields [Allen (1982a; 1982b)].[1] The appropriate concept for random topology of interac-

[1]For tractable treatments of Markov random fields, see Spitzer (1971), Preston (1974), and Kindermann and Snell (1980).

tion models is that of connectivity, which discuss in Section 4 below.

It may be argued that in an era of nearly ubiquitous communication systems involving telephone, fax, electronic mail, beepers, alarms and superfast communication networks few communication problems should remain. In practice, much ambiguity and uncertainty remain having to do with coordination, resolution of ambiguities, and failure to provide complete information. The other side of this is that contracts have to be incomplete. Problems of imperfect communication have also been addressed by engineers. It is known that decision-making in decentralized environments in the presence of imperfect information, delays, and lack of central control gives rise to complex dynamics. Such dynamics have concerned computer scientists [Huberman (1988); Huberman and Hogg (1988)].[2] It is also known that large communication networks are susceptible to inherent dynamic behavior, as lagged responses tend to build up [Bertsekas (1982)].

Most macroeconomic theorizing in the 1970's and the 1980's rested on some very special models, such as variants of the Samuelson overlapping generations model, which are characterized by stylized restrictions on trading patterns and have been used to motivate the holding of money. The coexistence of a national currency with numerous often geographically restricted credit arrangements (credit cards versus personal credit versus money) provokes further investigation. It is interesting to ponder general results regarding particular patterns of interaction, defined in terms of time-space events, which combine with appropriate endowments and preferences to give rise to a national currency. It is also interesting to study money (and other assets) as emergent properties of an economy.

2.1 Models of Communication in Economies with Interacting Agents

Two distinct approaches have developed in the literature on economies with interacting agents since Fölmer (1974) formally introduced the term.[3] The older approach, proposed by Kirman (1983), involves the use of random graph theory to model the topology of interconnections as random. Kirman

[2]The challenges created by interconnected computer systems have been addressed in the popular press. For example, see Markoff (1990) and Wintsch (1989) for the impact on scientific thinking, and Sanger (1986), for the impact on financial markets.

[3]Kirman (1995) provides a different review of this literature.

et al. (1986), Ioannides (1990) and Durlauf (1994a) are the key contributions in this approach. This literature provides a better understanding of trade among a large number of traders when links among traders are random (and thus the number of links per trader varies). Most of this literature has addressed properties of trade that depend only on the *number* of interlinked agents and ignore the actual *topology*, that is, graph topology,[4] of interconnections. The second, and newer, approach is predicated on a given topology of interconnections. Markov random field theory has been employed as a natural modelling device there.

We review in the paper the accomplishments of both those approaches. We also propose to synthesize them by introducing the notion of a "grand" compound event of trade outcomes. The likelihood of a particular realization of trade outcomes associated with a given topology of interconnections may be obtained in terms of the probability law that describes the evolution of different patterns of interaction. This concept is new, in the context of this literature, and will be used to analyze the endogenous evolution of patterns of interconnections as an outcome of equilibrium interactions. We shall see that certain stylized anositropic topologies of interaction lend themselves to particularly interesting economic applications.

3 Specified Topology of Interaction: Markov Random Fields

The pioneering work in the modern approach to a class of economies with interacting agents is due to Durlauf [Durlauf (1989a,b; 1991a,b; 1993b; 1994a)]. Several of Durlauf's papers share the same analytical core, which involves Markov random field models defined over a countable infinity of agents located in the two dimensional *lattice* of integers Z^2. [5] We first summarize some basic analytical features of the theory and then turn to its applications to modelling interactions in problems with lattices as well as more general graph topologies.

Let $\omega_{i,j}$, $\widetilde{\omega}$, and Ω denote, respectively, the activity of each of a continuum of identical agents located in point (i, j) of the lattice Z^2, the vector denot-

[4]Unless otherwise indicated, the term topology in this paper is meant in the sense of *graph* topology. Other topologies are still pertinent; see Haller (1990; 1994).

[5]The development of Markov random fields originated in lattices [Dobrushin (1968a)], but was later extended by Preston (1974) to general graphs.

ing the joint level of activity of agents in all sites, and the set of all possible states. The vector $\widetilde{\omega}$ is typically referred to as a *configuration*. In a typical application, the set S of all possible values of $\omega_{i,j}$ is assumed to be finite. It is assumed, for simplicity, that the choice set is binary, $S = \{0, 1\}$. For example, $\omega_{i,j} = 1$ (0) means the agent produces (does not produce). Alternatively, it may denote that an agent is (is not) informed [*cf.*, Allen, *op. cit.*]. The literature proceeds by making assumptions about local interactions and derives global properties of equilibrium distributions. For example, Durlauf assumes that activity by individual ι located in site (i, j) of Z^2 is affected by activity in all neighboring sites, that is $\{\omega_{i-1,j}, \omega_{i+1,j}, \omega_{i,j-1}, \omega_{i,j+1}\}$.

We now briefly outline the extension of the model that allows for more general patterns of interactions. Let $\mathcal{I}$ be a finite set of individuals, $\iota \in \mathcal{I}$, which are identified with the vertices of a graph. Interactions in an economy are defined in terms of a graph $G(V, E)$, where: V is the set of vertices, $V = \{v_1, v_2, \ldots, v_n\}$, an one-to-one map of the set of individuals $\mathcal{I}$ onto itself – the graph is labelled; $n = |V| = |\mathcal{I}|$ i s the number of vertices (nodes), which is known as the *order* of the graph; E is a subset of the collection of unordered pairs of vertices; $q = |E|$ is the number of edges, which is known as the *size* of the graph. In a typical application in this literature, nodes denote individuals, and edges denote potential interaction between individuals. It will be useful to define the set of *nearest neighbors* of individual $\iota \in \mathcal{I}$, $\nu(\iota) = \{j \in \mathcal{I} | j \neq \iota, \{\iota, j\} \in E\}$. We define an individual ι's *environment* as the state of all other individuals, formally as a mapping $\eta_\iota : \mathcal{I} - \{\iota\} \to \underbrace{\{0, 1\} \times \ldots \times \{0, 1\}}_{n-1}$.

Interactions across individuals are defined in terms of probability distributions for the state of individual ι conditional on her environment, $\pi_\iota(\omega_\iota | \eta_\iota)$. The collection of these distribution functions,$\mathcal{V} = \{\pi_\iota\}_{\iota \in \mathcal{I}}$ is known as a *local specification*. We say that $\mathcal{V}$ is a *nearest neighbor specification* if it implies that the state of individual ι depends only upon the state of her neighbors. A probability measure $\pi_\iota(\dot{})$ is said to define a *Markov random field* if its local specification for ι depends only on knowledge of outcomes for the elements of $\nu(\iota)$, the nearest neighbors of ι. This definition of a *Markov random field* confers a *spatial* property to the underlying stochastic structure, which is more general than the Markov property. We normally impose the assumption that regardless of the state of an individual's neighbors her state is non-trivially random.

Random fields are easier to study in a state of statistical equilibrium.

While in most of our own applications we have a finite number of agents in mind, a countable infinity of agents is also possible [Preston (1974); Allen (1982a,b)]. It is an important fact that for infinite graphs there may be more than one measure with the same local characteristics. When this happens the probabilities relating to a fixed finite set will be affected by the knowledge of outcomes arbitrarily far (or, just quite far, if infinity is construed as an approximation for large but finite sets) from that set. Existence of more than one measure with the same local characteristics is known as a *phase transition.*[6]

We say that a measure μ is a *global phase* for model $(G, S, \mathcal{V})$ if μ is *compatible* with $\mathcal{V}$ in the sense that $\mu\{\omega_\iota = s|\eta_\iota\} = \pi_\iota(s|\eta_\iota),\ s \in S$, where $\pi_\iota(s|\eta_\iota)$ is agent ι's environment. A result from the literature on Markov random fields states that if the set of nodes (individuals) is finite and $\mathcal{V}$ is a strictly positive nearest neighbor specification, then there exists a unique *global phase* which is consistent with the local specification $\mathcal{V}$. The global phase is a Markov random field. The literature has established [Griffeath (1976)] that every Markov random field is equivalent to a *Gibbs state* for some unique *nearest neighbor potential.* The equivalence between Gibbs states and Markov random fields is responsible for an enormous simplification in the formal treatment of Markov random fields. [7] We proceed with definitions for these terms.

A set of vertices κ in an graph $G(V, E)$, $\kappa \subset V$, is a *clique,* or *simplex,* if every pair of vertices in κ are neighbors. A *potential* $\mathcal{D}$ is a way to assign a number $\mathcal{D}_A(\tilde{\omega})$ to every subspecification $\tilde{\omega}_A$ of $\tilde{\omega} = (\omega_1, \ldots, \omega_\iota, \ldots, \omega_n)$. A potential $\mathcal{D}$ is a *nearest neighbor Gibbs potential* if $\mathcal{D}_A(\tilde{\omega}) = 0$, whenever A is not a clique. Let $\Pi(\cdot)$ be the probability measure determined on Ω by a nearest neighbor Gibbs potential $\mathcal{D}$, that is

$$\Pi(\tilde{\omega}) = \frac{1}{\zeta} \exp[\sum_{\kappa \subset V} \mathcal{D}_\kappa(\tilde{\omega})], \tag{1}$$

where the sum is taken over all cliques κ on the graph $G(V, E)$ and ζ is a normalizing constant. It turns out that the probability measure for the state

[6]See Kindermann and Snell (1980), for a simple statement, and Dobrushin (1968a;1968b) and Preston (1974) for elaboration of conditions for the absence of phase transitions. See also Ellis (1985) and Georgii (1988).

[7]In contrast, no such simplification is available if random fields are defined over *directed* graphs, as in the applications pursued by Evstigneev (1988a,b; 1991), Evstigneev and Greenwood (1992), and Evstigneev and Taksar (1992, 1993).

of individual ι conditional on her environment η_ι is given by:

$$\pi_\iota(\omega_\iota|\eta_\iota) = \frac{\exp[\sum_{\kappa \subset V} \mathcal{D}_\kappa(\widetilde{\omega})]}{\sum_{\widetilde{\omega}'} \exp[\sum_{\kappa \subset V} \mathcal{D}_\kappa(\widetilde{\omega}')]}, \tag{2}$$

where $\widetilde{\omega}'$ is any configuration which agrees with $\widetilde{\omega}$ at all vertices except possibly ι.[8]

The integration of spatial with temporal considerations, which Markov random fields make possible, is a powerful modelling tool. In economic applications normally it is the long-run equilibrium properties of the model which are of interest and depend upon the parameters of $(G, S, \mathcal{D})$, or alternatively, of $(G, S, \mathcal{V})$. Unless these parameters are endogenous, the model has equilibrium implications which are quite similar to those of stochastic models indexed on time. For example, planting a rumor may change the system's initial conditions but not its global phase [Allen (1982a)]. The dynamic implications of Markov random field models are rather hard to study, unless particular simplified assumptions are made, such assuming isotropic, i.e. homogeneous settings. Allen (1982a) shows that if $\mathcal{I}$ is assumed to be very large and G is defined as the complete graph – every individual is a neighbor of every other individual, $\nu(\iota) = \mathcal{I} - \{\iota\}, \forall \iota \in \mathcal{I}$, – then the Markov random field model of information diffusion in a homogeneous economy may be treated as a birth-and-death model and implies a logistic growth process for the diffusion of information: the rate of growth of the percentage of people who are informed is proportional to the percentage of people who are uninformed. It is also true that this literature is just starting to deliver completely endogenous equilibria, where the parameters of the the stochastic structure of the random fields are the outcomes of decisions. We return to this below.

The birth-and-death approximation that Allen invokes is a more general property which serves to illuminate the possibilities afforded by the Markov random field model. Preston (1974) proves equivalence between Markov random fields and equilibrium states of time reversible nearest neighbor birth-death semi-groups. A *time reversible nearest neighbor semi-group* on $\mathcal{P}(V)$,

[8] This crucial formula follows readily once one recognizes that for any clique κ that does not contain ι, $\mathcal{D}_\kappa(\widetilde{\omega}\ ) = \mathcal{D}_\kappa(\widetilde{\omega}')$. Thus terms that correspond to cliques that do not contain ι cancel from both the numerator and the denominator of (2) [Kindermann and Snell (1980), p. 27].

the set of subsets of V, may be used to express the dynamic evolution of changes in configurations, which could potentially serve as the basis for modelling general patterns of trade among a number of agents. One could use a very large state space to describe detailed aspects of trade among a finite number of agents. Pricing, market structure and many other issues (and in particular, some of the issues taken up by interesting recent contributions to the literature of economies with interacting agents [Bell (1994); Verbrugge (1994)]) could be better understood as special cases in this more general approach. We are currently working on such an approach.

3.1 Remarks

Modelling interactions in terms of Markov random fields requires that the pattern of potential links are the routes through which dependence across sites is transmitted. It is interesting to think of links between agents as emerging endogenously as outcomes of individuals' decisions. This paper advocates an approach that would help bridge the distance between the random graph and the Markov random field approaches. Such an approach is outlined in Section 6.2 below.

Durlauf has successfully employed the specified topology of interaction approach to a variety of settings. Durlauf (1989a; 1989b) explore the global, or economy-wide consequences of local coordination failures. Durlauf (1993b) considers the role of complementarities in economic growth. He shows that local linkages across industries can create sequential complementarities which build up over time to affect aggregate behavior. Also, industries industries which trade with all other industries can induce takeoff to sustained industrial production and thus may be considered as leading sectors in the sense employed earlier by the economic development literature on the significance of growth through intersectoral linkages. Durlauf's applications involve equilibria which may be non-ergodic, and thus allow for different global probability measures for economic activity to be consistent with the same microeconomic characteristics of agents. Durlauf (1994b; 1994c) deals with an implicit spatial interpretation to study the persistence of income inequality.

As Preston (1974) notes, the equivalence between Gibbs states and the equilibrium states of Markov processes defined over graphs prompts the challenge to construct Markov processes whose equilibrium states are Gibbs states with desirable properties, which may be expressed in terms of appropriate potentials. Such a "reverse" approach especially when combined with the equivalence between Markov random fields and birth-and-death semi-groups suggests that a technique of greater applicability than what we have seen to date. Related to this and virtually unexplored is the economic interpretation of the phenomenon of phase transitions.

3.2 Interactive Discrete Choice

Scholars [9] have been aware of the tantalizing similarity between the Gibbs measure induced by the Gibbs potential, and the logit function, which was used by McFadden (1981) to construct measures for discrete choice models. Blume (1993) and Brock (1993) were the first to exploit the analytical significance of this link. [10] Brock (1993) articulated what had been lacking in the literature so far, namely a Nash equilibrium concept in a model of "Manski-McFadden world of interconnected discrete choosers"[*op. cit.,* p. 20]. Brock's approach provides behavioral foundation for *mean field theory,* whereby the Markov random field model may be simplified by replacing direct interaction linkages between pairs of agents by interactions [11] between agents and the "mean of the field". This is discussed in more detail further below in subsection 3.3. The interactive discrete choice model serves to provide a genuine economic interpretation of the local specification introduced above, that is, the family of distribution functions for $\pi_\iota(\omega_\iota|\eta_\iota)$, under the assumption that it constitutes a nearest neighbor specification. Below we use the interactive discrete choice model in preliminary explorations of a number *anisotropic* settings.

Let agent ι who chooses $\omega_\iota, \omega_\iota \in S$, enjoy utility $U(\omega_\iota; \tilde{\omega}_{\nu(\iota)})$, where $\tilde{\omega}_{\nu(\iota)}$

[9] Including this author; see Haller and Ioannides (1991), p. 17, fn. 1.

[10] We agree with Brock's claim that "the linkage of [mean field theory] and discrete choice theory presented below appears new to this paper." Brock (1993), p. 20.

[11] It appears that Brock (1993) is the first use of the term "interactive" discrete choice. Manski (1993) addresses the inference problem posed by the possibility that average behavior in some group influences the behavior of the individuals that comprise the group (the "reflection" problem). There exists great potential for interactive choice models, as Brock and Durlauf point out, and there also exists great number of potential econometric applications that may disentangle *endogenous effects,* meaning the propensity of an individual to behave in some way varies with the behavior of the group, from exogenous (contextual) effects, where behavior varies with the exogenous characteristics of the group, and correlated effects, where individuals in a group exhibit similar behavior because they have similar individual characteristics or face similar institutional environments. See Ioannides *et al.* (1995) for spatial applications.

denotes the vector containing as elements the decisions made by each of agent ι's nearest neighbors. We assume that an agent's utility function is additively separable in her own decision, ω_ι, and in the impact of her neighbors' decisions upon her own, $\widetilde{\omega}_{\nu(\iota)}$. The specification of an agent's utility function definition of the set of all cliques that include agent ι as a member, $\mathcal{K}(\iota)$,. So we have:

$$U_\iota(\omega_\iota; \widetilde{\omega}_{\nu(\iota)}) \equiv u(\omega_\iota) + \omega_\iota[\sum_{j \in \mathcal{K}(\iota), j \neq \iota} J_{\iota j}\omega_j] + h\omega_\iota + \gamma\epsilon(\omega_\iota), \tag{3}$$

where γ is a parameter and $\epsilon(\omega_\iota)$ a random variable to be specified shortly below. Agent ι's choice is influenced by those of her nearest neighbors, $j \in \nu(\iota)$, via the vector of interaction effects $J_{\iota j}$.

Following Brock (1993), in view of McFadden (1981), and under the additional assumption that $\epsilon(\omega_\iota)$ is independently and identically type I extreme-value distributed[12] across all alternatives and agents $\iota \in \mathcal{I}$, we may write fairly simple expressions for the choice probabilities. Without loss of generality, let S, the range of the choice variable ω_ι, be binary. Then:

$$\pi_\iota(\omega_\iota = 1 | \, \widetilde{\omega}_{\nu(\iota)}) = \text{Prob}\left\{ u(1) - u(0) + \sum_{j \in \mathcal{K}(\iota), j \neq \iota} J_{\iota j}\omega_j] + h \geq -\gamma[\epsilon(1) - \epsilon(0)] \right\}, \tag{4}$$

which may be written in terms of the logistic integral [13]

$$\pi_\iota(\omega_\iota = 1 | \, \widetilde{\omega}_{\nu(\iota)}) = \frac{\exp\left[\beta\left(u(1) - u(0) + \sum_{j \in \mathcal{K}(\iota), j \neq \iota} J_{\iota j}\omega_j] + h\right)\right]}{1 + \exp\left[\beta\left(u(1) - u(0) + \sum_{j \in \mathcal{K}(\iota), j \neq \iota} J_{\iota j}\omega_j] + h\right)\right]}, \tag{6}$$

where β, a positive parameter, is an additional behavioral parameter. $\beta = 0$ implies purely random choice. The higher is β the more concentrated is the distribution. While the extreme value assumption for the ϵ's is made for convenience, there are several substantive arguments in its favor. First, the

[12]If two independent and identically distributed random variables have type I extreme-value distributions, then their difference has a logistic distribution.

[13]When S is not binary then:

$$\pi_\iota(\omega_\iota | \, \widetilde{\omega}_{\nu(\iota)}) = \frac{\exp\left[\beta\left[u(\omega_\iota) + \omega_\iota[\sum_{j \in \mathcal{K}(\iota), j \neq \iota} J_{\iota j}\omega_j] + h\omega_\iota\right]\right]}{\sum_{\omega_\iota \in S} \exp\left[\beta\left[u(\omega_\iota) + \omega_\iota[\sum_{j \in \mathcal{K}(\iota), j \neq \iota} J_{\iota j}\omega_j] + h\omega_\iota\right]\right]}. \tag{5}$$

logistic integral is a fairly good approximation to the normal. Second, and much less known, is the fact that the extreme value distribution is the asymptotic distribution, as $n \to \infty$, for $Y_n = \max_{1\le i\le n}\{X_1,\ldots,X_n\} - \ell n n$, where $X_1,\ldots,X_n$ are independently and identically distributed random variables with zero mean, drawn from a fairly large class of distributions.[14]

The definition of an agent's utility according to (3) along with the discrete choice model implied by the assumption that the ϵ's in (3) are extreme value distributed leads naturally to an interpretation of the underlying Gibbs potential $\mathcal{D}(\widetilde{\omega}) \equiv \sum_{\iota\in\mathcal{I}} u(\omega_\iota) + \sum_{\iota\in\mathcal{I}} \omega_\iota [\sum_{j\in\mathcal{K}(\iota), j\neq\iota} J_{\iota j}\omega_j] + h\sum_{\iota\in\mathcal{I}}\omega_\iota$. as a "social utility function" [*cf.* Brock (1993)]. To this individual choice system, there corresponds an "aggregate" choice mechanism in terms of the probability that the aggregate state $\widetilde{\omega} = (\omega_1,\ldots,\omega_n)$ be chosen. As Brock (1993), p. 19, and Brock and Durlauf (1995) note, choices by the n agents are not necessarily social welfare maximizing. Welfare analysis is facilitated by recognizing that expected social utility, defined as the expectation of the maximum, may play the role of expected indirect utility in a discrete choice setting [Manski and McFadden (1981), 198-272].[15] Unfortunately, the convenience of the equivalence between Gibbs measures and the logit model does not carry over in the case of McFadden's generalized extreme value model [McFadden (1978)].

3.3 Mean Field Theory

Brock (1993) and Brock and Durlauf (1995) attempt to simplify mean field theory and offer some fascinating results. They specify an individual's utility (3) as depending in a number of alternative ways on interaction with others. If it depends on the average decision of all others', then we have: $U_\iota(\omega_\iota; \widetilde{\omega}_{\nu(\iota)}) \equiv u(\omega_\iota) + \omega_\iota J \bar{m}^e_\iota$, where $\bar{m}^e_\iota \equiv (|\mathcal{I}|-1)^{-1} E[\sum_{j\neq\iota}\omega_j]$. In a Nash equilibrium setting, each individual takes others' decisions as given

[14]This class is defined as follows. If $F(x)$ and $f(x)$ denote the probability distribution and probability density functions of the X's, and $\frac{d}{dx}\frac{1-F(x)}{f(x)} \to 0$, as $x \to \infty$, then the standardized variable $\frac{Y_n - a_n}{b_n}$, with $a_n = F^{-1}(1-\frac{1}{n})$, $b_n^{-1} = nf(a_n)$, has an extreme value distribution, i.e., its probability distribution function is given by $\exp[-\exp[-y]]$). It is skewed, with a long upper tail, and a mode at 0; in its standard form, its mean is .57722 and its variance $\frac{\pi^2}{6}$. [Cox and Hinkley (1974), p. 473.] See also appendix 1, Pudney (1989), 293-300, and Lerman and Kern (1983).

[15]This function is a convex function of $u(\cdot)$, and its derivative with respect to u yields the respective conditional choice probability. For a recent application, see Rust (1994), p. 136.

and makes her own decisions subject to randomness that is due to independent and identically distributed draws from an extreme-value distribution. In the simplest possible model, one may take each individual's expectation of the mean to be common across all individuals: $\bar{m}_\iota^e = \bar{m}, \ \forall \iota \in \mathcal{I}$. Once the probability structure has been specified, this condition leads to a fixed point.[16]

Aoki (1994a; 1994b) and Weidlich (1991) are concerned with modelling similar settings where from a single agent's perspective the rest of the environment is construed as a uniform field. In contrast to Brock (1993) and Brock and Durlauf (1995), these approaches rest on weaker behavioral motivations. Aoki typically works with the so-called *master equation*, a version of the Chapman- Kolmogorov equation that keeps track of the time evolution of probability distributions in Markov chains. We eschew further discussion here and refer to Aoki (1996).

3.4 Stylized Asymmetric Patterns of Interaction

We now turn to some globally anisotropic settings, which exhibit some "local" symmetry. Retaining some symmetry not only lends itself naturally to economic interpretation but also confers an advantage in making the model amenable to Nash equilibrium analysis. That assumption is in contrast to the globally symmetric settings underlying mean field theory. We consider, alternatively, the cases of first, of Walrasian star-shaped interaction, and second, of completely circular interaction. The case of complete pairwise interactions may serve as a benchmark case and is examined first.

3.4.1 Complete Pairwise Interaction

The isotropic case of complete pairwise interaction is obtained from the general case of (1) and (2) by specifying that for each ι, $\nu(\iota) = \mathcal{I} - \{\iota\}$, and $J^{\mathrm{CO}} \equiv J_{\iota j}, \ \forall j \in \mathcal{I} - \{\iota\}$. For the discrete choice model generated by (3) we have:

$$U_\iota(\omega_\iota; \omega_{\mathcal{I}-\{\iota\}}) \equiv u(\omega_\iota) + \omega_\iota J^{\mathrm{CO}} \sum_{j \neq \iota} \omega_j + h\omega_\iota + \gamma\epsilon(\omega_\iota); \tag{7}$$

[16] To see that, note that the probability of a social state $\widetilde{\omega}$ is given by the product probability $\mathrm{Prob}(\widetilde{\omega} \,|\bar{m}_\iota^e) = \frac{\exp[\beta(u(\omega_\iota)+\omega_\iota J\bar{m}_\iota^e)]}{\prod_{\iota\in\mathcal{I}} \sum_{\omega_\iota=0,1} \exp[\beta(u(\omega_\iota)+\omega_\iota J\bar{m}_\iota^e)]}$. It follows that the denominator of r.h.s. of this equation may be written as $\sum_{\omega_1=0,1} \cdots \sum_{\omega_n=0,1} \exp[\beta(u(\omega_\iota)+\omega_\iota J\bar{m}_\iota^e)]$,which agrees with (1) and (2) above.

$$\text{Prob}\{\omega_\iota | \, \widetilde{\omega}_{\nu(\iota)}\} = \frac{\exp\left[\beta[u(\omega_\iota) - u(0) + \omega_\iota J^{\text{CO}} \sum_{j\neq\iota} \omega_j + h\omega_\iota]\right]}{1 + \exp\left[\beta[u(1) - u(0) + + J^{\text{CO}} \sum_{j\neq\iota} \omega_j + h]\right]}. \tag{8}$$

The probabilities associated with Nash equilibrium of the interactive discrete choice system may be easier to obtain once we have computed the equilibrium probabilities $\text{Prob}\{\omega_2, \ldots, \omega_n\}$, and so on, of which there 2^{n-1}. Taking advantage of symmetry reduces vastly the number of the relevant unknown probabilities down to n.

Instead of pursuing this further here, we note that the model bears a close resemblance to that of Kauffman (1993), p. 192, the case of the "Grand Ensemble": a given number of nodes n is connected to every other node, and each node is assigned a Boolean function at random from among the maximum possible number of possible logical rules, 2^{2^n}. It is interesting to explore the intuition obtained from the literature related to Kauffman's work, which is discussed in *ibid.* This model may also be considered as an extension of Allen (1982a), which allows us to explore some of the problems Allen takes up in more general settings.

3.5 Walrasian Star-shaped Interaction

This is obtained from the general case above by treating agent 1 as located at the center of a star, so that for $\forall\iota$, $\iota \neq 1$, $\nu(\iota) = \{1\}$, and $J^{\text{W}} \equiv J_{\iota 1}$, and $J^w \equiv J_{\iota j}$, otherwise. Also, $\nu(1) = \mathcal{I} - \{1\}$, $J^{\text{W}} \equiv J_{1\iota}, \forall\iota \neq 1$. For the discrete choice model generated by (3) we have:

$$U_\iota(\omega_1; \omega_2, \ldots, \omega_n) \equiv u(\omega_1) + \omega_1 J^{\text{W}} \sum_{\iota=2}^{n} \omega_\iota + h\omega_1 + \gamma\epsilon(\omega_1); \tag{9}$$

$$U_\iota(\omega_\iota; \omega_1) \equiv u(\omega_\iota) + \omega_\iota J_1 \omega_1 + h\omega_\iota + \gamma\epsilon(\omega_\iota); \ \iota = 2, \ldots, n. \tag{10}$$

This system implies interactive discrete choice probabilities as follows:

$$\text{Prob}\{\omega_1 | \omega_2, \ldots, \omega_n\} = \frac{\exp\left[\beta[u(\omega_1) - u(0) + \omega_1 J^{\text{W}} \sum_{\iota=2}^{n} \omega_\iota + h\omega_1]\right]}{1 + \exp\left[\beta[u(1) - u(0) + + J^{\text{W}} \sum_{\iota=2}^{n} \omega_\iota + h]\right]}. \tag{11}$$

$$\text{Prob}\{\omega_\iota | \omega_1\} = \frac{\exp\left[\beta[u(\omega_\iota) - u(0) + \omega_\iota J_1 \omega_1 + h\omega_\iota]\right]}{1 + \exp\left[\beta[u(1) - u(0) + J_1 \omega_1 + h]\right]}, \ \forall\iota \neq 1. \tag{12}$$

The probabilities associated with Nash equilibrium of the interactive discrete choice system may be obtained in terms of the equilibrium probabilities $\text{Prob}\{\omega_2, \ldots, \omega_n\}$. The general case involves 2^{n-1} such probabilities, but taking advantage of symmetry reduces vastly the number of unknowns down to n.

Walrasian star-shaped interaction is seen here as a prototype for trees, which is one of the graph topologies for which threshold properties have been obtained by the random graphs literature. The model may be augmented to allow for branches with different number of nodes and may also serve as a prototype for hierarchical structures. A variety of economic settings may be explored with this model. The extreme value distribution assumed by the behavioral model fits quite naturally a situation where where agent 1 conducts an auction based on offers by agents $2, \ldots, n$. Alternatively, each of the agents on the periphery may specialize in the production of a differentiated product. The number of agents n may thus reflect the demand for variety, etc.

3.6 Circular Interaction

This is obtained from the general case above by specifying that for each ι, $\nu(\iota) = \{\iota - 1, \iota + 1\}$, and $J_{\iota,\iota-1}, J_{\iota,\iota+1} \equiv J$, where for symmetry, $\{n+1\} = \{1\}, \{1-1\} = \{n\}$. For the discrete choice model generated by (3) we have:

$$U_\iota(\omega_\iota; \omega_{\iota-1}, \omega_{\iota+1}) \equiv u(\omega_\iota) + \omega_\iota J[\omega_{\iota-1} + \omega_{\iota+1}] + h\omega_\iota + \gamma\epsilon(\omega_\iota), \tag{13}$$

which implies interactive discrete choice probabilities as follows:

$$\text{Prob}\{\omega_\iota | \omega_{\iota-1}, \omega_{\iota+1}\} = \frac{\exp[u(\omega_\iota) - u(0) + \omega_\iota J[\omega_{\iota-1} + \omega_{\iota+1}] + h\omega_\iota]}{1 + \exp[u(1) - u(0) + J[\omega_{\iota-1} + \omega_{\iota+1}] + h]}. \tag{14}$$

The probabilities associated with Nash equilibrium of the interactive discrete choice system are easy to obtain once we have computed the equilibrium values of the probabilities $\text{Prob}[\omega_1, \ldots, \omega_n]$. The key is to recognize that symmetry implies that

$$\text{Prob}[\omega_1, \ldots, \omega_n] = \text{Prob}[\omega_2, \ldots, \omega_n, \omega_1] = \ldots = \text{Prob}[\omega_n, \omega_1, \ldots, \omega_{n-1}]. \tag{15}$$

Specifically, for the case of $n = 3$ it suffices, based on (15) to solve for the four unknown probabilities $\text{Prob}[1,1,1]$, $\text{Prob}[1,1,0]$, $\text{Prob}[1,0,0]$, and $\text{Prob}[0,0,0]$. We use the following notation for the respective interactive choice probabilities (14) in terms of the parameters of the model: $\text{Prob}\{1|1,0\} \equiv \frac{\exp[u(1)-u(0)+J+h]}{1+\exp[u(1)-u(0)+J+h]}$; $\text{Prob}\{1|1,1\} \equiv \frac{\exp[u(1)-u(0)+2J+h]}{1+\exp[u(1)-u(0)+2J+h]}$; $\text{Prob}\{0|0,0\} \equiv$

$\frac{1}{1+\exp[u(1)-u(0)+h]}$. Symmetry along with the definition of the interactive choice probabilities (14) imply the following linear system of equations:

$$\text{Prob}[1,1,1] + 3\text{Prob}[1,1,0] + 3\text{Prob}[1,0,0] + \text{Prob}[0,0,0] = 1;$$

$$(\text{Prob}[1,1,1] + \text{Prob}[1,1,0])\text{Prob}\{1|1,1\} = \text{Prob}\{1,1,1\};$$

$$(\text{Prob}[1,0,0] + \text{Prob}[0,0,0])\text{Prob}\{0|0,0\} = \text{Prob}[0,0,0];$$

$$(\text{Prob}[1,1,0] + \text{Prob}[1,0,0])\text{Prob}\{1|0,1\} = \text{Prob}[1,1,0].$$

Let $\mathbf{P}_{ci}(u(1)-u(0), J, h)$, a vector, denote the solution to this system, which is in closed form. The classic example is due to Wicksell (1934) in specializing in different goods. Because of absence of double coincidence of wants agents will avail themselves of the opportunity to trade only if circular sequences of trades may be completed. We pursue this further in Section 6. We note that the circle model has been addressed by the interacting particle systems literature [Ellis (1985), 190-203]. It gives rise to some features which are absent from the Curie-Weiss model, namely a new kind of phase transition described in terms of random waves. This and other features of models of circular interaction are being investigated currently by the author.

3.7 Comparisons

The most important differences among the three stylized topologies that we have examined are associated with the fact that the circular and complete graph topologies allow circular trades to be completed. The results reported by Kauffman (1993) involving autonomous Boolean networks with randomly assigned Boolean functions are indicative of the sort of results one could obtain by different specifications of preferences. For example, the Boolean functions .OR. and .AND. are conceptually similar to substitutability and complementarity, respectively. Analogous interpretations may be obtained for other Boolean functions. The importance of the preference structure in conjunction with the topology of interconnections may be underscored by the example, due to Schelling (1978), of an extreme case of complementarity, preferences for reading and lighting, which we quoted in the Introduction.

The sharp differences that preferences combined with topology make suggest the potential complexity of outcomes, and great richness in dynamic settings. Further study of these models is currently pursued by the author.

3.8 The Arrow-Debreu-Evstigneev Model

This section attempts to address the challenge of extending the Arrow-Debreu model in order to endow it with a general graph topology. It is critical to note that the Arrow-Debreu model depends on time being naturally ordered. The important extension we introduce below is based on the σ-algebras not being necessarily linearly ordered. It is standard to model uncertainty by means of a set $\mathcal{N}$ of states of the world, indexed by $w \in \Omega$, and a finite number of periods, indexed by $t = 0, 1, \ldots, T$. The information in the economy is exogenously specified and is represented by a sequence of partitions of Ω, $\{F_t \mid t = 0, 1, \ldots, T\}$. At time t an agent knows which event has occurred and which cell of F_t contains the true state. It is standard to assume that information increases through time; F_{t+1} is at least as fine as F_t. It does not imply loss of generality to assume that $F_0 = \Omega$, the universe, and $F_T = \{\omega \mid \omega \in \Omega\}$, the discrete partition, the state of the world is revealed by period T. Let us denote by $\mathcal{F}_t$ the σ-field of events, generated by F_t; $F = \{F_t; t \in \{0, 1, \ldots, T\}\}$ is the filtration generated by the sequence of partitions F_t. Consumption goods and endowments in terms of goods may be represented as stochastic processes adapted to $\mathcal{F}_t$.

Evstigneev (1991) extended the Arrow-Debreu model by invoking the index set used[17] to describe communication links in a large economy. This device possesses a multitude of advantages. It implies as special cases graph topologies which describe patterns of trading links that underlie several popular models in economics as special cases. Moreover, it may be considered as a rigorous extension of the Arrow- Debreu model, where the pattern of trading links has the (graph) topology of a tree, the tree of events. We, therefore, refer to this model as the Arrow-Debreu-Evstigneev model, ADE for short.

A key characteristic of the ADE model is that the *index* set T is endowed with the topology of an *oriented*, that is directed, graph.[18] Let $(\Omega, \mathcal{F}, \mathbf{P})$

[17]See also Evstigneev (1988), which solves the problem of optimizing an objective function that roughly corresponds to a theory of teams type problem. The model in Evstigneev (1990), on the other hand, corresponds to a recursive equilibrium-type structure, where economic units may be spatially differentiated and be asymmetrically related to one another.

[18]A generalization to Evstigneev's approach would be to consider $\tilde{T}$ as a *virtual* index set. It would incorporates the set of logical as well technological possibilities that are a prerequisite to actual economic relationships. The actual index set in our setting, T, will be the outcome of individuals' decisions.

be a probability space, as before. It is assumed that to each $t \in T$ there corresponds a set $K(t) \subseteq T$, which contains t and may be interpreted as the set of *descendants* of t, that is the set of agents directly dependent on t. On the graph corresponding to T there is an arrow (arc) leading from t to every element of $K(t)$.

The ADE model explores the notion that the sources of randomness which influence t also influence all agents who descend from her. That is, the $\sigma-$algebras $\mathcal{F}_t$ corresponding to each agent, $\mathcal{F}_t \subseteq \mathcal{F}$, satisfy

$$\mathcal{F}_t \subseteq \mathcal{F}_s, \ \forall s \in K(t).$$

This condition, namely that the σ-algebras $\mathcal{F}_t$ do not decrease as we move along the arrows of the graph means that the random factors which influence agent t also influence all agents in $K(t)$. In case of a cycle, the σ-algebras coincide $\mathcal{F}_s = \mathcal{F}'_s$.

The index set T and the σ-algebras do not have to be linearly ordered. In contrast, the counterpart of our index set in a number of standard economic models has very special, and linearly ordered, structure. In dynamic equilibrium models, it describes the sequential evolution of time. Alternatively, in purely spatial allocation problems T indexes locations in space, sites. Similarly to an agent's descendants we may define the set of of an agent t' *ancestors,* those agents from whom agent t descends. The set of agent's t ancestors is defined as $M(t)$, includes t and consists of the agents $\{s : s \in M(t), \text{if } t \in K(s)\}$.

Evstigneev suggests that time may be introduced explicitly in two alternative ways. One is to think of the elements of the set T as pairs $t = (n, b)$, where $n \in \{1, 2, \ldots, N\}$ denotes discrete-valued time within a finite horizon N, and the set $B(n)$, with generic element $b \in B(n)$, is the set of economic units functioning at time t. With each pair, $t = (n, b)$ $(b \in B(n))$ a set $K(n, b) \subseteq T$ is associated such that $(n', b') \in K(n, b)$ implies $n' \geq n$. A second and perhaps simpler way of introducing time is to assume that an integer-valued function $\nu(t)$, $t \in T$, is given on the graph T, and its values $\nu(t)$ are interpreted as moments of functioning of economic units $t \in T$. It is natural to assume here that $\nu(s) \geq \nu(t)$ for $s \in K(t)$. We define the set of agent's t' *contemporaries* $C(t)$ as all agents that function during the same point in time.

Evstigneev (1991) starts with $Z_t(q)$, the set of preferable programs of agent t under price system $\mathbf{p} : q = (q_s)_{s \in K(t)}$ and proves existence of equilibrium by means of tools from the theory of monotone operators. Evstigneev and Taksar (1992; 1993) offer a sensitivity analysis and an extension for the case of growing economies. Evstigneev's work constitutes a major generalization of the Arrow-Debreu model in a spatial context. Still, like the Arrow-Debreu model, it does not allow for feedback from the anticipation of trade frictions to allocation decisions.

4 Models with Random Topology

The previous section explored interactions across agents, when the topology of interconnections among them is given. Here we study by means of random graph theory how different patterns of interconnections may come about completely randomly. We review the literature that has utilized thius theory in models of trading structures. We end the section by taking up a number of economic issues which may be also be analyzed by random graph but have been paid little attention by the literature.

Random graph models[19] are a natural way to model situations where traders are into contact with other potential trading partners directly as well as indirectly through the partners of their trading partners and so on. There are other good reasons for treating the communication structure of an economy as a random variable. It might not be known who communicates with whom, and any two individuals may attempt to contact one another at random, and such attempts may fail stochastically.

Kirman (1983) was the first to argue in favor of this approach for studying communication in markets. Kirman *et al.* (1986) employ it studying coalition formation in large economies. Ioannides (1990) uses is it in a model of trading uncertainty, where formation of trading groups rather than consummation of bilateral trades is the object of investigation. Durlauf (1994a) uses it to study the role of communication frictions in generating coordination failure which in turn leads to aggregate fluctuations. Bernheim and Bagwell (1985) invoke random graph theory to assess the number of individuals who are linked through interpersonal transfers.

[19]The seminal work by Erdös and Renyi (1960) was followed by a broader set of applications in Erdös and Spencer (1974). See Bollobas (1985) and Palmer (1985) for the latest sources on this vast fascinating literature.

4.1 Random Graph Theory

Random graph theory introduces randomness by means of two alternative models.[20] Model A is defined in terms of graphs of order n and a probability $p = p(n)$ such that an edge exists between any two vertices i and j from among all possible vertices. The sample space consists of all possible labelled graphs of order n, with each of all $\binom{n}{2}$ possible edges being assigned independently with probability p. Model B is defined in terms of graphs of order n and size q, $0 \leq q \leq \binom{n}{2}$. The sample space consists of all possible labelled graphs of order n and size $q = q(n)$, each occurring with equal probability given by $\left(\binom{\binom{n}{2}}{q} \right)^{-1}$. Model A, B will be referred to as $\mathcal{G}^A_{n,p(n)}$, $\mathcal{G}^B_{n,q(n)}$, respectively.

Random graph theory is concerned with properties of graphs, such as connectedness and other, where the likelihood that they prevail satisfies a threshold property as the order of the graph grows. The literature utilizes results from random graph theory when n tends to infinity without specifying the space within which the graph is imbedded. Almost all of the literature works with homogeneous models, but as Erdös and Renyi (1960) speculated, there exists now a bit of literature pertaining to anisotropic models and is discussed below. It is evidence of the richness of this theory that the *emergent*[21] properties of random graphs do appear in homogeneous settings.

In the remainder of this subsection we discuss certain results from random graph theory which are less known and lend themselves neatly to economic applications. We complete the section with a review of the principal economic applications of random graph theory.

4.1.1 Emergence of Cycles

Of particular interest in the section below where we attempt to explore the actual topology of the random graph is conditions for the emergence of cycles. We know from Erdös and Renyi, *op. cit.*, the following facts. The number of *cycles* of order k, $k = 3, 4, \ldots$, contained in $\mathcal{G}^B_{n,cn}$ has a Poisson distribution

[20] Erdös and Renyi (1960) work with Model B but refer to the equivalence between these two models.

[21] I believe that Cohen (1988) and Kauffman (1993) are the first to refer to the properties of random graphs as emergent.

with mean equal to $\frac{(2c)^k}{2k}$. The number of *isolated* cycles of order $k, k = 3, 4, \dots$ contained in $\mathcal{G}^B_{n,cn}$ has a Poisson distribution with mean equal to $\frac{(2ce^{-2c})^k}{2k}$. The number of *components* consisting of $k \geq 3$ points and k edges has a Poisson distribution with mean equal to $\frac{(2ce^{-2c})^k}{2k}\left(1 + k + \frac{k^2}{2!} + \dots \frac{k^{k-3}}{(k-3)!}\right)$. The expected number of *all* cycles contained in $\mathcal{G}^B_{n,cn}$ is equal to $\frac{1}{2}\ell n\left(\frac{1}{1-2c}\right) - c - c^2$, if $c < \frac{1}{2}$; to $\frac{1}{4}\ell nn$, if $c = \frac{1}{2}$. The probability that $\mathcal{G}^B_{n,cn}$ contains at least one cycle is given by $1 - \sqrt{1-2c}e^{c+c^2}$, $\frac{1}{2} \geq c$. The total number of points of $\mathcal{G}^B_{n,cn}$ that belong to some cycle is finite and given by $\frac{4c^3}{1-2c}$, if $c < \frac{1}{2}$. The expected number of points of $\mathcal{G}^B_{n,cn}$ which belong to components containing exactly one cycle is given by $\frac{1}{2}\sum_{k=3}^{\infty}(2ce^{-2c})^k\left(1 + k + \frac{k^2}{2!} + \dots \frac{k^{k-3}}{(k-3)!}\right)$, if $c \neq \frac{1}{2}$, and equal to $\frac{\Gamma(\frac{1}{3})}{12}n^{\frac{2}{3}}$, otherwise. It then follows that for $c < \frac{1}{2}$ all components of $\mathcal{G}^B_{n,cn}$ are with probability tending to 1 either trees or components containing one cycle. In other words, almost all graphs have no components with more than one cycle. Thus, we are not guaranteed a cycle until $c = \frac{1}{2}$, or $p(n) \geq \frac{1}{n}$. Finally, the asymptotic probability of cycles of all orders in $\mathcal{G}^B_{n,n}$ is equal to 1.

4.1.2 Connectivity and Hamiltonicity

A cycle that goes through every node exactly once (a spanning cycle) is known as a *Hamiltonian.* Specifically, we have from Palmer (1985), p. 59, the following result. In the random graph $\mathcal{G}^A_{n,p(n)}$, with $p(n) = \frac{\ell nn + \ell n[\ell nn] + 2c_n}{n}$, the probability of hamiltonicity tends, as $n \to \infty$, to: 0, if $c_n \to -\infty$; $e^{-e^{-2c}}$, if $c_n \to c$; 1, if $c_n \to +\infty$. In contrast, the probability of connectivity in the random graph $\mathcal{G}^A_{n,p(n)}$, with $p(n) = c\frac{\ell nn}{n}$ tends to 1, if $c > 1$. This holds in $\mathcal{G}^B_{n,q(n)}$, with $q(n) = \frac{c}{2}n\ell nn$. In both cases, the random graph is disconnected if $0 < c < 1$. If $p(n) = \frac{\ell nn + 2c}{n}$ the probability of connectivity of $\mathcal{G}^A_{n,p(n)}$ tends to $e^{-e^{-2c}}$. Outside the giant component, there are only isolated agents. Similarly, if $p(n) = \frac{\ell nn + 2c}{n}$ the probability of connectivity of $\mathcal{G}^A_{n,p(n)}$ tends to $e^{-e^{-2c}}$. It is thus clear that in order to make hamiltonicity possible, the probability that any two agents be connected must be larger than the value which ensures connectivity at least by a term $\frac{\ell n[\ell nn]}{n}$, which of course decreases with n.

Connectivity is a useful concept in understanding how news travels through a population of agents. For example, if news can travel through direct and indirect contacts, then the probability of connectivity gives the probability that a given piece of news can reach everyone. Hamiltonicity lends itself readily to modelling the emergence of conditions under which fiat money can finance trades, which we take up in Section 6 below.

4.1.3 Anisotropic Random Graph Models

Kovalenko (1975) provides a rare example of a random graph model where the edge probabilities are not equal. Specifically, Kovalenko considers random graphs where an edge between nodes i and j may occur with the probability p_{ij} independently of whatever other edges exist. He assumes that the probability tends to 0 as $n \to \infty$ that there are no edges leading out of every node and that there are no edges leading into every node. Under some additional limiting assumptions about the probability structure, he shows that in the limit the random graph behaves as follows: there is a subgraph $\mathcal{A}_1$ of *in-isolated* nodes whose order follows asymptotically a Poisson law; there is a subgraph $\mathcal{A}_2$ of *out-isolated* nodes whose order follows asymptotically a Poisson law; all remaining nodes form a connected subgraph. The orders of $\mathcal{A}_1$ and $\mathcal{A}_2$ are asymptotically independent and their parameters are given in terms of the limit of the probability structure.

4.2 Economic Applications of Random Graphs

Here are some examples of how the economics literature has exploited random graph theory. Kirman (1983) works with $\mathcal{G}^A_{n,p(n)}$, where $p(n) \equiv p$, a constant. Then $\mathcal{G}^A_{n,p(n)}$ is strongly connected with probability equal to $\lim_{n\to\infty} : 1-n(1-p)^{n-1}$, which is, of course, equal to zero. Kirman derives a *Probabilistic Limit Theorem for the Core:* the epsilon core is arbitrarily close to a sequence of replica Walrasian economies. This result is strengthened by Kirman, Oddou and Weber (1986), where $p(n)$ is a decreasing function of n, which can go to 0 as fast as $\frac{1}{\sqrt{n}}$, for $n \to \infty$. They also prove a probabilistic theorem that all members of a coalition are required to be in direct contact; in that case, in particular, $p(n)$ should go to 0 slower than $\frac{1}{\ell nn}$. Ioannides (1990) interprets each node as an independently endowed trader, and the isolated components of the random graph as trading groups within which trades are allocated according to Walrasian rules.[22] He then considers the evolution of $\mathcal{G}^B_{n,cn}$, where c is a constant, and associates the emergence of the giant component

[22]Ioannides (1986) considers that interlinked traders form mutual funds.

of the random graph, when $n \to \infty$, with elimination of uncertainty for a certain proportion of the economy. The expected number of components of $\mathcal{G}^B_{n,cn}$ divided by the number of traders n is: $1-c+\frac{O(1)}{n}$, if $c < \frac{1}{2}$; $1-c+\frac{O(\ell nn)}{n}$, if $c = \frac{1}{2}$; and, $\frac{1}{2c}\left(x(c) - \frac{x^2(c)}{2}\right)$, if $c > \frac{1}{2}$, where $x(c)$ is the only solution in $(0, 1)$ of the equation $xe^{-x} = 2ce^{-2c}$. That is, it decreases, as c increases, originally linearly and then slower than linearly with c. The random graph contains a unique giant component of order $[1 - \frac{x(c)}{2c}]n$, and the remaining vertices, whose number is roughly equal to $\frac{x(c)}{2c}]n$, belong to components that are trees of order at most ℓnn. This finding is particularly dramatic if it is presented as a plot of the size of the largest component against c, the ratio of edges to nodes, which has a sigmoidal shape with a sharp rise at $c = \frac{1}{2}$.[23] The random graph $\mathcal{G}^B_{n,cn}$ is of particular interest in that it exhibits an interesting uniformity "in-the-small," in spite of this stark heterogeneity. For any of its points, the probability that it is connected to j other points is given by $\frac{(2c)^j e^{-2c}}{j!}$, $j = 0, 1, \dots$, i.e., it has a Poisson distribution with mean equal to twice the number of edges per trader. Equivalent statements hold for the random graph $\mathcal{G}^A_{n,\frac{2c}{n}}$.

Durlauf (1994a) considers the evolution of a set of industries, each consisting of a large number of costlessly communicating firms. When a set of industries establish contact with one another and form a production coalition, whereby all members of a production coalition must be able to communicate directly through a bilateral link with at least one other member of the coalition, output and profits are improved. Durlauf works with $\mathcal{G}^A_{n,p(n)}$ and assumes that the probability of communication between industries i and j, the probability of an edge between nodes i and j, is a continuous increasing function of industry outputs; he also assumes that conditionally on industry outputs, the event that industry i communicates with industry j is independent of the event that industry i' communicates with industry j', when $\{i, j\}$ and $\{i', j'\}$ are distinct pairs. Durlauf shows that the following conditions are necessary for existence of aggregate cycles: For all finite n, the probability is strictly positive that two isolated industries be connected when they produce their maximum possible output – that is, no all industries are simultaneously isolated; the probability is strictly less than 1 that two different industries that are members of the largest possible coalition of size

[23] Kauffman (1995) uses this model as a metaphor for the origin of an autocatalytic model of chemical reactions that " is ... almost certainly self-sustaining, alive " [*ibid.* p. 58].

n be connected directly when they produce their lowest possible output – not all bilateral links will become permanently established. So long as the probability of each bilateral link is between 0 and 1 for all output levels, the model describing aggregate output will be ergodic. The main theorem in Durlauf (1994) states that under the assumptions made about the sensitivity of the probability of bilateral communication aggregate output fluctuations will be arbitrarily persistent: high output enhances the process of coalition formation and low output impedes it, so that circumstances of high and low aggregate activity are self-reinforcing. All these papers do take advantage of the fascinating properties of random graphs, but they stop short of making as dramatic a use of the *emergent* properties of communication among large numbers of agents as that by mathematical biologists [Cohen (1988); Kauffman (1993; 1995)].

It is most fortunate that random graph theory has obtained such precise results about trees and cycles which, as we argued earlier in subsections 3.5 and 3.6 are of particular relevance for economic applications. Our current research in this area sets out to combine two sets of analytical insights obtained by the two approaches discussed so far, that is, models with specified topology of interaction, discussed in Section 3, and models with random topology, discussed in the present one. Our goal is to obtain an integrated model, a grand synthesis, of the evolution of trading structures. This, as we see it, involves two steps: first, agents' decisions combine with nature to generate a particular structure of interconnections; and second, given a structure of interconnections agents trade. The interplay between those two approaches becomes the basis of the synthesis. Agents' decisions may affect the parameters of the random mechanism of the first step above, and agents' decisions in the second step may affect the further evolution of the economy. Even with static expectations, agents face a tradeoff: heavy investments in establishing a trade infrastructure makes trade more likely but also costs more. With general structures of preferences, some agents may be better off specializing in the production of certain differentiated goods, whereas other agents may be better off providing intermediation services. Circular trading structures facilitate trade but require, in the average, a greater "thickness" of interconnections, as the discussion in subsection 4.1.2 immediately above makes clear. Macro-type properties with multiplier effects are more likely to emanate from models with circular trades. Endogeneity of differentiated product varieties may be modelled in tree-type structures.

5 Other Topics

Spatial aspects of trading structures give rise to issues of spatial complexity and link with the urban economics literature. Our theoretical approach would be complemented by a study of possibilities of statistical inference. Aggregate implications of some of the approaches we have discussed have already been examined extensively by the economies with interacting agents literature. E.g., Durlauf (1989b) derives and tests empirically the model's implications for persistence. Those tests, and other work by Durlauf, work with the aggregate implications of the theory. Techniques of inference that could be aimed at a more "micro" level would be particularly useful. The evolution of trading structures in a spatial context is an area that lends itself to particularly interesting issues of statistical inference. Another would be testing Kauffman's theory of technological change through inference based on the evolution of economic webs [Kauffman (1988)]. The material below demonstrates that it is fairly straightforward to carry out inference in terms of differences across graphs. Two other issues are taken up in the remainder of this section, namely pricing and matching models.

5.1 Spatial Models

The modern economy depends critically on differentiated use of space. Such differentiation emerged during very early stages of development, when settlements appeared to take advantage of economies of scale in trading activities relative to a rural hinterland and to exploit other benefits of close human interactions [Bairoch (1988)]. These processes have been subject of intense investigations by other social scientists but received only scant attention by economists outside the broad area of urban economics, which has almost exclusively focussed on cities [Henderson (1985); 1988)]. Recent work by Paul Krugman on economic geography has revived interest in this area and has given a new impetus by setting a number of new challenges.

Our understanding of the trade infrastructure of the modern economy is unthinkable without an understanding of how cities interact with one another and with their hinterlands. To accomplish this, we must go beyond looking at agents and sites as points in space and bring into the fore their full spatial aspects. However, the graph topology that we have relied upon do far is still useful in understanding a number of key issues.

Modern urban economics has addressed jointly two important questions, namely the role of cities whose primary functions are other than retailing to rural areas and the internal structure of cities. Mills (1967) emphasized the scale economies in industrial production associated with the spatial clustering of workers and firms. These economies derive from ease of communication, benefits from matching of workers and firms, and sharing of fixed costs associated with public transportation infrastructure. Cities produce traded and non-traded goods. There is overwhelming evidence that smaller and medium size cities specialize in the production of groups of different goods. As Henderson (1988) explains it, "separating industries into different cities allows for a greater degree of scale economy exploitation in each industry relative to a given level of commuting costs and city spatial area"[p. 32].

When economic factors such as scale economies due to specialization and/or urbanization interact with geographical features and historical patterns of settlement the outcome is an uneven spatial distribution of economic activity. Researchers have debated a variety of stylized facts characterizing urban systems in different economies and alleged regularities in city size distributions over time in specific economies as well as across different economies [*ibid.* p. 46; Dobkins and Ioannides (1995)]. In the economics literature, however, there has been more emphasis upon the economic functions of cities and less emphasis upon the structural aspects of systems of cities. Empirical results [Henderson, p. 197] suggest that observed differences in urban concentration across economies reflect the composition of national output, with important agriculture and resource-based activities being associated with lower concentration and greater presence of service-type industry being associated with higher concentration and clustering of activity into multi-centered metropolitan areas. Strategic considerations have also been utilized to explain allocation of economic activity over space [Helsley and Strange (1988; 1989)].

Casual observation of the evolution of systems of cities suggests that complexity factors, akin to the ones addressed elsewhere in this paper play an important role. Cities interact in a variety of complex ways, through transportation of goods and movements of people, flows of knowledge, and complex trading arrangements [Krugman (1993; 1994)]. The complexity of real world trading arrangements, especially in their spatial dimensions, closely resembles the theoretical models discussed earlier in the paper. We refrain from pursuing this issue further here, since the paper by Paul Krugman in this volume is dedicated to this topic. We offer an example of how interactions across cities belonging to a system of cities may be highlighted by means of random graph theory.

Let us consider a random graph $\mathcal{G}^B_{n,q(n)}$, where n denotes the number of cities. Let edges represent routes of economic interaction between cities, which in a hypothetical case are placed in the most random way between cities in the system. We are interested in the topology of interconnections across cities as the number of interconnections increases over time. In particular, we are interested in knowing how quickly the isolated parts of the system of cities become integrated into the main interconnected part of the system. If the number of routes per city starts at greater than $\frac{1}{2}$ and increases over time, then the giant component grows by absorbing isolated components. It turns out that the probability is equal to $e^{-2k(c_{t_2}-c_{t_1})}$ that an isolated subsystem of k cities (with the topology of a tree) which is present in the system at time t_1, that is, it belongs to $\mathcal{G}^B_{n,c_{t_1}n}$, should still be isolated at time t_2, that is it belongs to $\mathcal{G}^B_{n,c_{t_2}n},\ c_{t_2} > c_{t_1}$. Thus the life-time of an isolated urban subsystem of order k has approximately an exponential distribution with mean value $\frac{1}{2k}$ and is thus independent of the "age" of the tree.

This simple model, which is a direct outgrowth of random graph theory, suggests that fairly rapid thickening of interconnections commonly associated with systems of cities in growing economies over time could result even from a random placement of interconnections among cities. If the number of new links is endogenized, and perhaps related to incentives as perceived by the members of the subsystem of size k, then depending upon the relationship of the number of links with the number of cities we may be able to make different predictions about the properties of the system of cities. Another result from the random graph literature may be helpful in predicting the minimum number of interconnections, as a function of of the number of cities that renders the system completely connected. Let the number of interconnections between cities be chosen in such a manner that at each stage every interconnection that has not been chosen has the same probability of being chosen as the next and let continue the process until until all cities become interconnected. Then as $n \to \infty$, in model $\mathcal{G}^B_{n,q(n)}$ $q(n)$ is such that $q(n) = \frac{1}{2}n\ell nn + cn$, and $\lim_{n\to\infty}$: Prob $\left[\frac{q(n)-\frac{1}{2}n\ell nn}{n} < c\right] = e^{-e^{-2c}}$.

Many additional issues remain unexplored, especially challenging ones being associated with the spatial as well as functional evolution of urban systems. Explicitly spatial considerations are rather new, however, in the literature.[24] The empirical evidence on the evolution of the US urban system has not been fully explored[25] and some of the analytical material from Arthur (1994), Chs. 3, 6, and 10, may be brought to bear on this issue.

5.2 Statistical Inference

Specifically, one may define a simple metric for graphs, or equivalently, as it turns out for this use, for adjacency matrices corresponding to graphs. Let $\delta_M(g_1, g_2)$ be the symmetric difference metric on graphs, defined by:

$$\delta_M(g_1, g_2) = \frac{1}{2} tr[(G_1 - G_2)^2], \tag{16}$$

which counts the number of edge discrepancies between graphs g_1 and g_2, whose adjacency matrices are G_1 and G_2, respectively. Following Banks and Carley (1992), one may define a probability measure $H(g^*, \sigma)$, over the set $\mathcal{G}_M$ of all graphs on n distinct vertices, where $g^* \in \mathcal{G}_M$ is the central graph and σ a dispersion parameter:

$$P\{g|g^*, \sigma\} = \zeta(\sigma) e^{-\delta_M(g,g^*)}, \ \forall g \in \mathcal{G}_M, \tag{17}$$

where $\zeta(\sigma)$ is a normalizing constant. It is a fortunate fact that the normalizing constant does not depend on g^*, under the above assumptions: $g^* = (1 + e^{-\sigma})^{-\binom{n}{2}}$ [Banks and Carley (1992)]. We are thus free to specify the central graph and thus implicitly test random graph theory. Also, Banks and Carley suggest extensions to the case of digraphs, which could be particularly useful in testing Kauffman's theory of evolution in economic webs [Kauffman (1988)] in terms of long-run changes in input-output relationships.

5.3 Pricing

It is worth mentioning in passing that the dual of whom trades with whom is whose prices are inputs to whom. So, much of what we have been concerned with applies to pricing models. Dynamics of pricing decisions have

[24] See, in particular, Fujita, Krugman and Mori (1994), and Ioannides (1995).
[25] See Dobkins and Ioannides (1995) for a recent inquiry.

been examined by a number of researchers but because of their preoccupation with representative-individual models, which has been eloquently criticized by Kirman (1992), they have not been considered as being central to macroeconomic theorizing. Notable is the work by Blanchard (1983; 1986), who explores the dynamics of pricing decisions in two alternative settings. In Blanchard (1983), final output is produced in a number of discrete stages, each carried out under constant returns to scale by competitive firms. Just as Blanchard allows for the number of stages to change, "leaving the [aggregate] technology unchanged," one can conceive of settings where the number of intermediate goods changes in the Dixit-Stiglitz style [*cf.* Blanchard (1986)]. One may impose patterns of intermediate goods purchases, which may generate interesting dynamics whose steady states may be not unlike the ones explored by Weitzman (1982).

5.4 Matching

Agent matching has been proved useful in a number of areas in economics as a simple way to express a possible dependence between frequency of trades and the number of agents. This literature has typically focussed on actual trades and thus avoided making assumptions about the full pattern of interaction. Matching models have been adopted in macroeconomic applications where the interest was in understanding the impact of frictions [Pissarides (1979; 1990)]. Recently, work in evolutionary game theory has revived interest in matching.

The typical setting in Pissarides (1990) involves a homogeneous model, where matching is construed as an operation in bipartite graphs, where each part groups workers and firms. Let L denote, respectively, the total number of workers, and uL and vL denote unemployed workers and the number of vacant jobs. If only unemployed workers and vacant jobs engage in matching, then the number of job matchings per unit time is given by the matching function: $x = \frac{X(uL,vL)}{L}$. Pissarides invokes empirical evidence to argue that the matching function is increasing in both of its arguments, concave, and homogeneous of degree 1.[26] Homogeneity implies that the rate of job contacts is given as a function of the unemployment and vacancy rates only, $x = x(u,v)$. A number of informational assumptions underlie this construction, i.e., are distinct. workers' and firms' effectiveness in search may be modified

[26]For empirical evidence with US data, see Blanchard and Diamond (1989; 1990).

by search intensity and advertizing, respectively.

Moving to stochastic matching, however, poses some technical problems. For example, Gilboa and Matsui (1990) show that if the populations to be matched and the set of encounters are countably infinite, and purely random matching is imposed (which ensures that everyone is matched), then the distribution of encounters over time equals their common distribution as random variables with probability one: the law of large numbers holds. That is, the population as a whole is not affected by aggregate uncertainty. Boylan (1992) addresses the need for detail in the specification of random matching, which is usually approximated with a deterministic model. Boylan assumes a countably infinite population $\{1, 2, \ldots, n, \ldots\}$ where each individual is to be matched with exactly one other individual. There are m different types of individuals, $S = \{s_1, \ldots, s_m\}$. Matching is such that each individual is matched exactly once and "if John is matched with Paul then Paul is matched with John." Boylan articulates a number of difficulties that random matching schemes face. For example, imposing the condition of equally likely matches leads to a contradiction; the properties of random matching do depend on the assignment of types. He gives conditions under which there exists a random matching scheme which is equivalently handled by a deterministic system. However, for a law of large numbers to hold the random matching scheme must depend on the assignment of types in the population. Boylan (1993) considers a similar set of questions except that the number of matchings per individual in this note is random and converges to a Poisson process as the population grows. He shows that the deterministic process which approximates the random matching process must have a globally stable point.

The literature is concerned primarily with bilateral matching. In contrast, several problems addressed by the literature on economies with interacting agents involve in effect *multilateral* matching. The anisotropic model developed by Kovalenko (1975) may be used to model multilateral matching. Finally, some assignment and network models may actually be considered relevant to the inquiry pursued by this paper, but we shall refrain from discussing them here.

6 Equilibrium Interactions in Anisotropic Settings: A Sketch

This section demonstrates in a preliminary fashion the possibility of equilibrium models in *anisotropic,* that is, non-isotropic, settings. As we see shortly, *controlled* random fields[27] may be used here in a manner that is reminiscent of models of equilibrium search, where market structure is endogenous. In the first subsection below we take up non-additive effects of interaction which serve two functions: one, to motivate our decision to emphasize a circular pattern of interconnections; two, to demonstrate properties of circular trade patterns which are of independent interest. In the second subsection, we offer a preliminary model of equilibrium interactions in a circular setting.

6.1 Non-additive Effects of Interaction

Just as international trade among distinct national economies has complex effects, so does trade among interacting agents, when the patterns of interconnection are complex. A number of papers by Jovanovic [Jovanovic (1984; 1985; 1987)] establish that one may design game structures which may either attenuate or reinforce the effects of independent shocks acting on each of a large number of agents. We interpret such structures as patterns of interaction. We discuss circular interaction in some detail, because it is of particular interest in this paper and is utilized in our example of equilibrium in subsection 6.2 below. We then take up Jovanovic's general results.

6.1.1 Circular Settings

We consider $i = 1, \ldots, n$ distinct sites, each occupied by a continuum of consumers and firms, each of unit measure. Firm i uses input x_i to produce output $\theta_i x_i^\gamma$, $0 < \gamma < 1$, where θ_i and γ are parameters. Firms are owned by the consumers who consume the entirety of their profit incomes on firms' output. In a centralized setting, where own output can be used for both consumption and investment and output may be costlessly transported across all sites, output price everywhere is equal to 1. The profit-maximizing level of output in site i is given by: $\ell n x_i = \frac{\ell n \gamma}{1-\gamma} + \frac{1}{1-\gamma} \ell n \theta_i$. If the set of parameters

[27] In contrast, in Evstigneev' work on controlled random fields, the structure is not endogenous.

$\{\theta_i\}_{i=1}^{i=n}$ are IID random variables, then aggregate output per capita obeys the law of large numbers and is thus equal to: $\frac{\ell n\gamma}{1-\gamma} + \frac{1}{1-\gamma}E\{\ell n\theta_i\}$. While we will discuss below variations of this model, unless the pattern of trade is more complex or the shocks contain an aggregate component, the centralized setting of the Walrasian trading model precludes reinforcement of individual shocks.

Jovanovic (1984) proceeds with the following restrictions on preferences. Consumers must consume goods produced in the site of their residence. Firms on site i must purchase good produced on site $i-1$. Sites are located on a circle. For symmetry, if $i = 1$, $i - 1 = n$. Let prices be $(p_1, p_2, \ldots, p_n)$, with $p_1 = 1$ as the numeraire. Profit maximization requires that $\frac{p_{i-1}}{p_i} = \gamma\theta_i x_i^{\gamma-1}$. The model is closed by imposing equilibrium in the output market. Individuals spend their incomes on consumption. It turns out that firm ownership patterns do not matter and that equilibrium output in sites i and $i-1$ must satisfy $x_i = \gamma\theta_i x_{i-1}^{\gamma}$. It follows that consumption in site $i-1$ is equal to $(1-\gamma)\theta_{i-1}x_{i-1}^{\gamma}$. The model closes with the "initial" condition $x_{n+1} = x_1$, with which the solution follows:

$$\ell n x_i = \frac{\gamma\ell n\gamma}{1-\gamma} + \sum_{j=1}^{n} \frac{\gamma^{n-j}}{1-\gamma^n}\ell n\theta_{i+j}, \tag{18}$$

where for $i + j > n$, $\theta_{i+j} \equiv \theta_{i+j-n}$. By the change of variable $\epsilon_i \equiv (1-\gamma^n)^{-1}\ell n\theta_i$, (18) becomes $\ell n x_i = \frac{\gamma\ell n\gamma}{1-\gamma} + \sum_{j=1}^{n}\gamma^{n-j}\epsilon_{i+j}$. We now see a pattern of dependence [28] across outputs in all sites: the x_i's contain a common component. Some further manipulation makes that dependence even clearer. We retain the ϵ's as given parameters and vary the θ_i's as $\gamma \to 1$. This yields the result that $\ell n x_i = 1 + \sum_{j=1}^{n}\epsilon_j$, so that the x_i's are perfectly correlated. Furthermore, as the limit is taken the uncertainty at the individual level becomes negligible, as the variance of θ_i tends to 0. In contrast, the centralized

[28] Jovanovic (1984) does point out the fact that basic patterns of circular dependence have been studied in the context of *circular* serial correlation. Actually, the dynamics of a system exhibiting circular serial correlation of order T involve the eigenvalues and eigenvectors of a matrix $\mathbf{B}$, whose only nonzero elements are $b_{i,i+1} = 1,\ i = 1, \ldots, T-1, b_{T,1} = 1$. The eigenvalues of $\mathbf{B}$, are the Tth roots of 1 and thus naturally involve the trigonometric functions and so do the corresponding eigenvectors [Anderson (1971), 278-284.] Periodic dynamics associated with this system are thus inherent in it and thus particularly interesting. In fact, the circular interaction model that we are investigating is, as Jovanovic notes, a natural way to motivate the circular serial correlation model.

setting when pursued in the same manner yields: $\ell n\tilde{x}_i = 1 + n\epsilon_i$, i.e., outputs are IID across sites.

Jovanovic does emphasize that the reinforcement of local shocks is achieved by a complete absence of a coincidence of wants between any two neighbors. He concludes by expressing that "there is hope, therefore, of analyzing the role of money in such structures"[*ibid.* p. 21], but seems to be unaware of the fact that a similar set of questions have been originally addressed by Cass and Yaari (1966) in a deterministic setting.

6.1.2 General Settings

Jovanovic (1987), p. 399, proves a general theorem which if suitably interpreted allows us to assess the likelihood of similar results for other general patterns of interaction. Consider that $\{\theta_i\}_{i=1}^{i=n}$ are independent random variables, with θ_i being the "micro" shock to agent i' reaction function (more generally, correspondence), which may be written as $\phi(x_{\{\mathcal{I}-i\}}, \theta_i)$. A *Nash equilibrium,* is a solution to the system of structural relations $x_i \in \phi(x_{\{\mathcal{I}-i\}}, \theta_i)$. Under fairly mild conditions on the θ_i' s, any reduced-form $h(\theta_1, \ldots, \theta_n)$, which relates the endogenous x's to the exogenous θ_i's, can be generated by a family of games, with agents acting with common knowledge of all shocks. Thus, restricting attention to stochastically independent agents imposes, in general, no restriction on the distribution of observed outcomes. From the perspective of the present paper, restrictions that are associated with specific topological structures of interactions are of particular interest.

6.2 Equilibrium in Circular Settings

Let us consider now that an original random matching results in n agents' ending up in a circle. We invoke a setting very much like that of subsection 3.6, where agent ι has as neighbors agents $\iota - 1$ and $\iota + 1$. An agent may be active or inactive, a state that is determined as the outcome of a comparison of expected benefits and costs, in a way which will be precisely specified below. That is, the interactive discrete choice model of subsection 3.2 is utilized here as a model of market participation.

In the context of the present discussion, the solution in subsection 6.1 applies only if all agents are active. The expression for x_ι is symmetric with respect to location on the circle. If we now allow for agents to be active or inactive, we quickly recognize that we must account for all possible outcomes.

However, even if only one agent on the circle is inactive, the equilibrium outcome for all agents is $x_\iota = 0$, in which case consumption is also equal to 0 and $u(0) = 0$.

We can go further by restricting attention to the case of three agents, without loss of generality, as it turns out. Such a setting contains key features of the model. For that case, we have already derived in subsection 3.6 the equilibrium probabilities. A typical agent, say agent 1, in this setting may be found active or inactive. The probabilities of these events may be computed. Agent 1 is active with probability given by

$$\text{Prob}\{1\} = \text{Prob}[1,1,1] + 2\text{Prob}[1,1,0] + \text{Prob}[1,0,0]. \tag{19}$$

We may take $u(1)$ to be expected utility conditional on an agent's being active, which we may compute from (18) as a function of parameters, denoted by $\bar{u}$. By multiplying with $\frac{\text{Prob}[1,1,1]}{\text{Prob}\{1\}}$ we obtain expected utility conditional on an agent's being active. It is now clear that a simultaneity is involved in computing the equilibrium state probabilities associated with model (13). The equilibrium state probabilities depend upon $u(1)$, expected utility conditional on an agent's being active, and the latter in turn depends upon equilibrium state probabilities. Let the equilibrium values of the probabilities be denoted by *'s. They satisfy:

$$\mathbf{P}^*_{ci} = \mathbf{P}_{ci}(\bar{u}\frac{\text{Prob}^*[1,1,1]}{\text{Prob}^*\{1\}}, J, h), \tag{20}$$

We may now use these results as ingredients for a dynamic model, very much along the lines of Brock and Durlauf (1995). That is, we use the above model to predict the motion of an entire economy. The economy consists of agents in three sites, who are assumed to act with static expectations and whose states are determined according to the interactive discrete choice model. The first step in the study of these dynamics is to consider that utility values $u(\omega_\iota)$ are exogenous. This will allow us to separate the effects upon the dynamics of the interactive discrete choice model from those of the endogeneity of utility associated with the state of being active. We may then proceed with the case of endogenous utilities, by assigning to those terms the values of expected utilities implied by the model. The model exhibits a fundamental simultaneity. The Gibbs state is defined in terms of the interactive choice probabilities, and the interactive choice probabilities depend upon the Gibbs state through the calculation of expected utilities.

The Gibbs state for the model is defined as fixed point. As an extension of this basic framework, one could also introduce a mean field effect along with neighbor interactions.[29]

6.3 A Research Agenda

The fact that Nash equilibrium places few restrictions on our ability to recover reaction functions (and thus, according to Jovanovic's Lemma 1, *ibid.*, p. 398, agents' preferences as well) allows us to speculate that it should be possible to obtain fairly general results for equilibrium interactions. The example of subsection 6.2 demonstrates that it is possible to go beyond mean field theory and characterize equilibrium interactions in anisotropic settings as long as one restricts oneself to a given topology of interactions. The more challenging problem is, of course, to let preferences, costs of interaction and costs of trading determine the topology of interactions at equilibrium.

Random graph theory allows us to make predictions about topologies which are most likely to prevail under different assumptions about the dependence upon the number of agents n of $p(n)$, the probability of an edge, or of $q(n)$, the number of edges. It remains to link these probabilities, which we see as determining the original topology of potential interaction, with the fundamentals of the problem, such as preference and cost parameters. Even before that is accomplished we speculate that a number of interesting results may be obtained; those results would rest on multiplicity and stability properties of equilibrium. Under the assumption that preferences allow a wide variety of potential interactions, the original random matching provides the boundary conditions for the interactive choice model of subsection 3.2. If the unique equilibrium configuration is globally stable, then the boundary conditions do not matter. If, on the other hand, there exist multiple equilibria, it matters whether or not a particular initial potential topology of interaction lies in the domain of attraction of a stable or of an unstable equilibrium. Furthermore, the original matching may not perfectly random; it might reflect specific, nonhomogeneous, features of a particular setting.

We speculate that such an approach may lead to investigations in a number of directions which we are currently pursuing. We outline a few of them here. One direction could be a model for an old problem: the appearance

[29] This is particularly interesting from a macroeconomic point of view in the sense that one could potentially distinguish between true "macro" shocks and aggregate shocks which result from aggregation of individual shocks.

of money as an emergent property of an economic system. When links per trader are few, then cycles are rather unlikely and trade is restricted to bilateral or autarkic. With more links per trader, cycles become likely, and circular sequences of bilateral trades can develop. We think of such cycles as different subeconomies, each of which is equivalent to a centralized setting through the use of IOU-type money in the style of Wicksell. With even more links per trader, a hamiltonian cycle is likely, which would be associated with the emergence of a single currency. A second direction could be a model of the spatial differentiation of the economy, which may make it conducive to spontaneous growth in certain circumstances and not in others [Kelly (1994)]. Such an approach would provide a new dimension to the growing endogenous growth literature. A third direction could be a new look at the fundamentals of the process of technological change, perhaps along the lines of Kauffman's ideas on the evolution of economic webs. A fourth direction could be to link formal theories of trade infrastructures and of relational constraints in general equilibrium [Haller (1994); Gilles, Haller and Ryus (1994)]. The circle model on its own gives rise to particularly interesting dynamics. The results obtained for the deterministic case by the mathematical biology literature and for the stochastic case by the interacting particle systems literature are particularly promising. Finally, we also note the potential afforded by modelling trade based on evolving networks of interconnected traders, which we discussed in Section 3 above.

7 Summary and Conclusion

Our review of the evolution of trading structures has attributed a central role to the literature of economies with interacting agents. It identifies two main strands in that literature, namely works that presume a given topology of interactions among agents and those that let random mechanisms determine that topology. Both are recent and very fast-growing bodies of knowledge that shares methodological tools with other social sciences.

The paper addresses two key weaknesses of these two fundamental approaches to the study of trading structures. The specified topology of interaction approach has concerned itself primarily with isotropic settings. However, some anisotropic settings are particularly interesting for economists. For example, circular patterns of interaction highlight the role of money and credit; tree-type of interactions may be seen as depicting Walrasian interactions. The random topology of interaction approach has concerned itself primarily with the sizes of trading groups, and thus has not exploited the entire range of topological properties of trading structures which may emerge.

The paper proposes an integration of those approaches which is intended to exploit their natural complementarities. In the simplest possible version, our synthesis involves individual decisions and expectations, randomness, and nature combining to fix an initial "primordial" topology of interaction. The dynamics of interaction move the economy from then on. The evolution of trading structures depends critically upon multiplicity and stability properties of equilibrium configurations of the interaction model.

The paper has also pointed to links with spatial economics and as well as with processes of growth and technological change. We hope that those suggestions will become fruitful avenues for specific applications of further research on the evolution of trading structures.

We think that ultimately it should be possible to address the problem of emergent market structures by means of mathematical tools involving controlled random fields.[30] Our analytical intuition is that just as models of controlled stochastic processes (such as search models) lead to equilibrium descriptions of the economy, it should be possible to apply the same intuition to models that generalize the notion of dependence into spatial *cum* temporal contexts. Individuals may sample over space, in a timeless context, or over space and time. The modelling problem which must be solved would require conditioning each individual's state on her neighbor's state, rather than on links between neighbors being operative or not. This intuition is very similar to the one underlying the use by search theory of sequential statistical decision theory. Just as equilibrium in search models takes the form of invariant distributions for the variables of interest, equilibrium in models involving random fields involves probability distributions associated with the active or inactive state of subsets of relevant populations.

[30]For mathematical theories that lend themselves to such applications, see Krengel and Sucheston (1981), Mandelbaum and Vanderbei (1981) and Evstigneev (1988a,b).

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